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# A CUBIC DIOPHANTINE EQUATION AND A RELATED DIOPHANTINE CHAIN

AJAI CHOUDHARY

*High Commissioner, High Commission of India, Bandar Seri Begawan, Brunei Darussalam, C/o Ministry of External Affairs, South Block, New Delhi* 110 011
*e-mail: ajaic203@yahoo.com*

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This paper gives a complete solution of the cubic diophantine equation $f(x, y) = f(u, v)$ where $f(x, y)$ is an arbitrary cubic form in two variables. This leads to a complete solution of $x^3 + y^3 = u^3 + v^3$ not hitherto obtained. Other methods of solving the equation $f(x, y) = f(u, v)$ have also been described. Finally, a method of generating an arbitrarily long diophantine chain $f(\alpha_1, \beta_1) = f(\alpha_2, \beta_2) = f(\alpha_3, \beta_3) = \ldots = f(\alpha_n, \beta_n)$ has been given.

**Key Words : Cubic Diophantine Equation; Diophantine Chain**

We are concerned in this study with the cubic diophantine equation

$$f(x, y) = f(u, v) \qquad \ldots (1)$$

where $$f(x, y) = ax^3 + bx^2y + cxy^2 + dy^2, \qquad \ldots (2)$$

$a$, $b$, $c$, $d$ being arbitrary integers. We will assume that at least two of the coefficients $a$, $b$, $c$, $d$ are non-zero as otherwise (1) reduces to a very simple equation which is readily solvable. We give below a four-parameter solution of (1) which will be shown to be complete. We also give additional solutions of (1) which are simpler, though not necessarily complete. Finally, given any positive integer $n$, howsoever larger, we show that it is possible to obtain $n$ pairs of integers $(\alpha_i, \beta_i)$, $i = 1, 2, \ldots, n$ such that

$$f(\alpha_1, \beta_1) = f(\alpha_2, \beta_2) = f(\alpha_3, \beta_3) = \ldots = f(\alpha_n, \beta_n). \qquad \ldots (3)$$

As eq. (1) is homogeneous, any integer solution $(x, y, u, v)$ of (1) leads to another integer solution $(kx, ky, ku, kv)$ where $k \neq 0$ is an integer. There is, therefore, no loss of generality in considering eq. (1) with $(x, y, u, v) = 1$. Moreover, any rational solution of (1), or of (3), may be multiplied throughout by a suitable constant to yield a solution in integers.

***Theorem*** 1 — *The complete solution of the diophantine equation*

$$ax^3 + bx^2y + cxy^2 + dy^3 = au^3 + bu^2v + cuv^2 + dv^3 \qquad \ldots (4)$$

*with* $(x, y, u, v) = 1$ *is given by*

$$\left.\begin{array}{l}\rho x = pm + gn,\ \rho u = (q-p)m - pn,\\ \rho y = rm + sn,\ \rho v = (s-r)m - rn,\end{array}\right\} \quad \dots (5)$$

*where p, q, r, s are arbitrary parameters and m, n, ρ are defined as follows :*

*(i) Generally, we take*

$$\left.\begin{array}{l}m = \phi_1 (p, q, r, s)\\ n = \phi_2 (p, q, r, s)\end{array}\right\} \quad \dots (6)$$

*where* $\phi_1 (p, q, r, s) =$

$$- \{a(p+q)(p^2 - pq + q^2) + b(p^{2'}r + q^2 s) + c(pr^2 + qs^2) + d(r+s)(r^2 - rs + s^2) \quad \dots (7)$$

*and* $\phi_2 (p, q, r, s) = a(2p-q)(p^2 - pq + q^2) + b\ \{p^2 (2r-s) - 2pq(r-s) + q^2 (r-s)\}$

$$+ c\ \{p(2r^2 - 2rs + s^2) - q(r-s)^2\} + d(2r-s)(r^2 - rs + s^2). \quad \dots (8)$$

If, however, for any specific values of $p$, $q$, $r$, $s$, we have both

$$\phi_1 (p, q, r, s) = 0 \quad \dots (9)$$

as well as

$$\phi_2 (p, q, r, s) = 0, \quad \dots (10)$$

then $m$ and $n$ may be taken as arbitrary parameters.

(ii) $\rho$ is a non-zero integer so chosen that $(x, y, u, v) = 1$.

PROOF : To solve eq. (4), we write

$$\left.\begin{array}{ll}x = pm + qn, & u = (q-p)m - pn,\\ y = rm + sn, & v = (s-r)m - rn.\end{array}\right\} \quad \dots (11)$$

Substituting these values of $x$, $y$, $u$, $v$ in (4), we get on factorisation,

$$(m^2 + mn + n^2)\ \{m\, \phi_2 (p, q, r, s) - n\, \phi_1 (p, q, r, s)\} = 0. \quad \dots (12)$$

Thus, taking $m$ and $n$ as in (6), the relation (12) is satisfied.

If, however, for any specific values of $p$, $q$, $r$, $s$, we find that both the relations (9) and (10) are satisfied, then relation (12) is identically satisfied for all values of $m$ and $n$ and we may accordingly take $m$ and $n$ as arbitrary parameters. Thus, we obtain the solution stated in the theorem.

We now show that the above solution is complete, Let $(X, Y, U, V)$ be a given solution of eq. (4) so that

$$aX^3 + bX^2y + cXY^2 + dY^3 = aU^3 + bU^2V + cUV^2 + dV^3. \quad \dots (13)$$

We choose the parameters $p$, $q$, $r$, $s$ are follows :

$$p = X - U, q = U + 2X, r = Y - V, s = V + 2Y. \quad \dots (14)$$

With these values of $p$, $q$, $r$, $s$, we find that

$$\phi_1(p, q, r, s) - \phi_2(p, q, r, s) = -9\ \{aX^3 + bX^2y + cXY^2 + dY^3$$

$$- (aU^3 + bU^2V + cUV^2 + dV^3)\}$$

$$= 0. \qquad \text{... (15)}$$

If $\phi_1(p, q, r, s) \neq 0$, then in view of (15), we also have $\phi_2(p, q, r, s) \neq 0$. We now have to choose $m$ and $n$ according to Theorem 1. It follows from (6) and (15) that

$$m = \phi_1(p, q, r, s) = \phi_2 = (p, q, r, s) = n$$

and $n \neq 0$. The solution given by (5) now is as follows :

$$\rho x = 3Xn, \rho y = 3Yn, \rho u + 3Un, \rho v = 3Vn.$$

We now take $\rho = 3n$, and it is clear that the parameters as chosen in (14) generate the given solution $(X, Y, U, V)$.

If, however, $\phi_1(p, q, r, s) = 0$, then it follows from (15) that we will also have $\phi_2(p, q, r, s) = 0$ and, in accordance with Theorem 1, we may take $m = n = 1$, so that the solution given by (5) is

$$\rho x = 3X, \rho y = 3Y, \rho u = 2U, \rho v = 3V.$$

Taking $\rho = 3$, we find that in this case also the parameters $p$, $q$, $r$, $s$ as chosen in (14) generate the given solution. This proves that the solution given by the theorem is complete.

We now apply Theorem 1 to the problem of equal sums of two cubes for which we get the new solution given below. In view of the historical importance of the problem, the result is stated as a theorem.

***Theorem*** 2 — *The complete solution of the diophantine equation*

$$x^3 + y^3 = u^3 + v^3 \qquad \text{... (16)}$$

*with* $(x, y; u, v) = 1$ *is given by the trivial solution* $x = -y$, $u = -v$ *and the solution*

$$\left.\begin{aligned} \rho x &= (p^2 - pq + q^2)^2 + (p(r+s) - q(2r-s))(r^2 - rs + s^2) \\ \rho y &= (p(r-2s) + q(r+s))(p^2 - pq + q^2) + (r^2 - rs + s^2)^2 \\ \rho u &= (p^2 - pq + q^2)^2 + (p(r-2s) + q(r+s))(r^2 - rs + s^2) \\ \rho v &= (p(r+s) - q(2r-s))(p^2 - pq + q^2) + (r^2 - rs + s^2)^2 \end{aligned}\right\} \qquad \text{... (17)}$$

*where* $p$, $q$, $r$, $s$ *are arbitrary parameters and* $\rho$ *is a non-zero integer so chosen that* $(x, y, u, v) = 1$.

PROOF : Substituting $a = d = 1$, $b = c = 0$ in eq. (4), we get eq. (16) and with these values of $a$, $b$, $c$, $d$ we get

$$\phi_1(p, q, r, s) = -\{(p+q)(p^2 - pq + q^2) + (r+s)(r^2 - rs + s^2)\}$$

$$= -(p^3 + q^3 + r^3 + s^3) \qquad \dots (18)$$

and

$$\phi_2(p, q, r, s) = (2p - q)(p^2 - pq + q^2) + (2r - s)(r^2 - rs + s^2)$$

$$= p^3 + (p-q)^3 + r^3 + (r-s)^3. \qquad \dots (19)$$

Assuming that $\phi_1(p, q, r, s)$ and $\phi_2(p, q, r, s)$ are not both zero, we may take, according to Theorem 1, $m = \phi_1(p, q, r, s)$ and $n = \phi_2(p, q, r, s)$, and we thus get, from (5), the solution (17) given above (on changing the sign of the variables). As this solution has been derived from the complete solution of eq. (4) provided by Theorem 1, it yields all solutions of eq. (16) other than those corresponding to the exceptional case of Theorem 1 when $p, q, r, s$ are such that $\phi_1(p, q, r, s)$ and $\phi_2(p, q, r, s)$ are both zero.

We now consider the exceptional situation when $p, q, r$ and $s$ are such that $\phi_1(p, q, r, s)$ and $\phi_2(p, q, r, s)$ are both zero i.e.

$$(p+q)(p^2 - pq + q^2) + (r+s)(r^2 - rs + s^2) = p^3 + q^3 + r^3 + s^3 = 0 \qquad \dots (20)$$

and

$$(2p - q)(p^2 - pq + q^2) + (2r - s)(r^2 - rs + s^2) = p^3 + (p-q)^3 + r^3 + (r-s)^3 = 0. \qquad \dots (21)$$

It follows from (20) and (21) that

$$(p+q)(p^2 - pq + q^2) = -(r+s)(r^2 - rs + s^2) \qquad \dots (22)$$

and

$$(2p - q)(p^2 - pq + q^2) = -(2r - s)(r^2 - rs + s^2), \qquad \dots (23)$$

so that

$$(p^2 - pq + wq^2)(r^2 - rs + s^2)(p+q)(2r - s)$$

$$= (p^2 - pq + q^2)(r^2 - rs + s^2)(2p - q)(r+s). \qquad \dots (24)$$

There are now two possibilities :

(i) If

$$(p^2 - pq + q^2)(r^2 - rs + s^2) \neq 0 \qquad \dots (25)$$

then it follows from (24) that

$$(p+q)(2r - s) = (2p - q)(r+s)$$

or,

$$ps = qr. \qquad \dots (26)$$

Now, if $r \neq 0$, we may take $p = kr$, $q = ks$, and it follows from (20) and (21) that

$$(k^3 + 1)(r^3 + s^3) = 0 \qquad \dots (27)$$

and

$$(k^3 + 1)\{r^3 + (r-s)^3\} = 0. \qquad \dots (28)$$

If $k \neq -1$, it follows from (27) that $s = -r$, and so from (28). we get $9(k^3+1)r^3 = 0$ which is a contradiction since $r \neq 0$. Thus, we must have $k = -1$ i.e. $p = -r$ and $q = -s$. If, however, $r = 0$, it follows from (25) and (26) that $p = 0$, and using (20), we get $q = -s$. Thus, we again get $p = -r$ and $q = -s$.

Applying Theorem 1 to eq. (16) by taking $p = -r$ and $q = -s$, we get $m$ and $n$ as arbitrary (since $\phi_1(p, q, r, s)$ and $\phi_2(p, q, r, s)$ are now both zero), and hence it follows from (5) that this solution is $x = -y$, $u = -v$. This solution $x = -y$, $u = -v$ of eq. (16) is not generated by the solution (17) of eq. (16).

(ii) If $(p^2-pq+q^2)(r^2-rs+s^2) = 0$, we must either have $(p^2-pq+q^2) = 0$ i.e. $p = 0$ and $q = 0$ in which case (20) and (21) together imply $r = 0$ and $s = 0$, or we must have $(r^2-rs+s^2) = 0$ i.e. $r = 0$ and $s = 0$ in which case (20) and (21) together imply $p = 0$ and $q = 0$. Thus, in this case, we must have $p = q = r = s = 0$.

This corresponds to the trivial solution (0, 0, 0, 0) of eq. (16).

This trivial solution is readily obtained from solution (17).

Thus, all solutions of eq. (16) are given by (17) together with the trivial solution $x = -y$, $u = -v$. This solution of eq. (16) has apparently not been obtained earlier.

While Theorem 1 gives a complete solution of eq. (1), the solution is, in general, of degree four in terms of the four parameters $p, q, r, s$. It may sometimes be advantageous to have a simpler solution even though it may not be necessarily complete. A solution of degree two in terms of two parameters is obtained by the following method. To solve (1), we write

$$\left.\begin{array}{l} x = p\,\theta + \alpha,\ u = r\,\theta = \alpha, \\ y = q\,\theta + \beta,\ v = \beta \end{array}\right\}. \qquad \ldots(29)$$

Substituting these values of $x, y, u, v$ in (1), we get

$$f(p\theta + \alpha, q\theta + \beta) - f(r\,\theta + \alpha, \beta) = 0$$

which, using Taylor's expansion, may be written as,

$$\theta^3\ \{f(p, q) - f(r, 0)\} + \frac{\theta^2}{2}\left[p^2 f_{xx}(\alpha, \beta) + 2pq f_{xy}(\alpha, \beta) + q^2 f_{yy}(\alpha, \beta) - r^2 f_{xx}(\alpha, \beta)\right]$$

$$+\ \theta\,[p f_x(\alpha, \beta) + q f_y(\alpha, \beta) - r f_x(\alpha, \beta)] = 0. \qquad \ldots (30)$$

Since $\alpha, \beta$ are arbitrary, and at least two of the integers $a, b, c, d$ are non-zero, $f_x(\alpha, \beta)$ is not identically zero, and so we may choose

$$r = \left\{ p f_x(\alpha, \beta) + q f_y(\alpha, \beta) \right\} \{f_x(\alpha, \beta)\}^{-1} \qquad \ldots (31)$$

so that the coefficient of $\theta$ in (30) becomes zero. With this value of $r$, we can solve eq. (30) for $\theta$ to obtain

$$\theta = -\frac{p^2 f_{xx}(\alpha, \beta) + 2pq f_{xy}(\alpha, \beta) + q^2 f_{yy}(\alpha, \beta) - r^2 f_{xx}(\alpha, \beta)}{2\,(f(p, q) - f(r, 0))} \qquad \ldots (32)$$

We note that $q$ is a factor of both the numerator and the denominator of the right-hand side of (32) and can be cancelled out. The values of $r$ and $\theta$ given by (31) and (32) respectively when substituted in (29), yield, on clearing denominators, a solution of degree two in terms of the parameters $p$ and $q$. In this solution, $\alpha$ and $\beta$ are also arbitrary and may be taken as constants.

As an example, we will solve by this method the equation

$$x^3 = x^2y + xy^2 + y^3 = u^3 + u^2v + uv^2 + v^3. \qquad \ldots (33)$$

Substituting the values of $x$, $y$, $u$ and $v$ given by (29) in eq. (33), we get

$$(p^3 + p^2q + pq^2 + q^3)\, \theta^3 + \{(3\,\alpha + \beta)\, p^2 + 2(\alpha + \beta)\, pq + (\alpha + 3\,\beta)\, q^2 - (3\,\alpha + \beta)\, r^2\}\theta^2$$

$$+ \ \{(3\,\alpha^2 + 2\,\alpha\beta + \beta^2)\, p + (\alpha^2 + 2\,\alpha\beta + 3\,\beta^2)\, q - (3\,\alpha^2 + 2\,\alpha\beta + \beta^2)\, r\} \ \theta = 0. \ \ldots (34)$$

Equating the coefficient of $\theta$ in (34) to zero, we get

$$r = \frac{(3\,\alpha^2 + 2\,\alpha\beta + \beta^2)\, p + (\alpha^2 + 2\,\alpha\beta + 3\,\beta^2)\, q}{3\,\alpha^2 + 2\,\alpha\beta + \beta^2}.$$

While $\alpha$ and $\beta$ could be retained as arbitrary parameters, we take for simplicity, $\alpha = 1$ and $\beta = 2$, so that

$$r = \frac{11p + 17q}{11}. \qquad \ldots (35)$$

With these values of $\alpha, \beta$ and $r$, eq. (34) gives

$$\theta = -\frac{143\,(44p + 23q)}{2420p^2 + 4103pq + 1791q^2}. \qquad \ldots (36)$$

Substituting the values of $\alpha, \beta, r$ and $\theta$ in (29), we get, on clearing denominators, the following solution of degree of two in terms of the parameters $p$ and $q$ :

$$\left.\begin{aligned} x &= -3872\,p^2 + 814\,pq + 1791\,q^2, \ u = -3872\,p^2 - 8910\,pq - 3292\,q^2 \\ y &= 4840p^2 + 1914\,pq + 293\,q^2, \ v = 4840p^2 + 8206\,pq + 3582\,q^2. \end{aligned}\right\} \qquad \ldots (37)$$

Next, we note that when $f(x, y)$ is a reducible cubic form, we may obtain a linear solution as indicated below. Let

$$f(x, y) = (Ax + By)\,(Px^2 + Qxy + Ry^2)$$

where we may assume, without loss of generality, that $A \neq 0$. To solve (1), we will solve the simultaneous equations

$$Ax + By = Au + Bv \qquad \ldots (38)$$

and

$$Px^2 + Qxy + Ry^2 = Pu^2 + Quv + Rv^2 \qquad \ldots (39)$$

Substituting the value of $x$ obtained from eq. (38) in eq. (39), and removing the factor $(y - v)$, we get the linear equation

$$(A^2R - ABQ + B^2P)\,y + A^2\,Q - 2ABP)\,u + (A^2R - B^2P)v = 0. \qquad \ldots (40)$$

Thus, we get two linear eqs. (38) and (40) in terms of four variables $x$, $y$, $u$ and $v$. These two equations can readily be solved to obtain a solution of (1) in terms of two linear parameters. However, we note that when $f\,(x, y)$ is a symmetric function of $x$ and $y$, the solution obtained by this method is $x = v$, $y = u$ which, in this case, is a trivial solution.

As an example of this method, when

$$f\,(x, y) = (x + 2y)\,(x^2 + 5xy + 7y^2),$$

a solution of the equation

$$(x + 2y)\,(x^2 + 5xy + 7y^2) = (u + 2v)\,(u^2 + 5uv + 7v^2)$$

is given by

$$x = 3u + 8v,\ y = -u - 3v.$$

Finally we will generate the diophantine chain (3) where $f\,(x, y)$ is given by (2). For this purpose, we shall use the identity

$$f\,(\alpha, \beta) = f\,(\alpha_1, \beta_1) \qquad \ldots (41)$$

where

$$\left.\begin{aligned} \alpha_1 &= (\alpha\,(2ap^3 + bp^2q - dq^3) + \beta\,(bp^3 + 2cp^2q + 3dpq^2))\,(f\,(p, q))^{-1}, \\ \beta_1 &= -\,(\alpha\,(3ap^2q + 2bpq^2 + cq^3) + \beta\,(-\,ap^3 + cpq^2 + 2dq^3))\,(f\,(p, q))^{-1}, \end{aligned}\right\} \qquad \ldots (42)$$

the values of $p$ and $q$ being given by

$$\left.\begin{aligned} p &= -f_y\,(\alpha, \beta) = -\,(b\,\alpha^2 + 2c\,\alpha\,\beta + 3\,d\,\beta^2), \\ q &= f_x\,(\alpha, \beta) = 3\,a\,\alpha^2 + 2\,b\,\alpha\,\beta + c\,\beta^2. \end{aligned}\right\} \qquad \ldots (43)$$

To establish the establish the identity (41), we substitute the values of $x$, $y$, $u$, $v$ given by (29) in eq. (1) and, as before, obtain eq. (30). We now choose $p$ and $q$ as in (43) and take $r = 0$, so that in eq. (30), the coefficient of $\theta$ becomes zero and the equation can be solved for $\theta$ to obtain

$$\theta = -\frac{p^2 f_{xx}\,(\alpha, \beta) + 2pqf_{xy}\,(\alpha, \beta) + q^2 f_{yy}\,(\alpha, \beta)}{2f\,(p, q)}$$

$$= -\frac{p^2\,(3a\,\alpha + b\,\beta) + 2pq\,(b\,\alpha + c\,\beta) + q^2\,(c\,\alpha + 3\,d\,\beta)}{f\,(p, q)}. \qquad \ldots (44)$$

Substituting $r = 0$ and the above value of $\theta$ in (29) leads to the identity (41). Thus, given any two rational numbers $\alpha, \beta$, we may obtain two other rational numbers $\alpha_1, \beta_1$ such that the relation (41) holds. We may now start with the two rational numbers $\alpha_2, \beta_2$ such that

$$f(\alpha_1, \beta_1) = f(\alpha_2, \beta_2).$$

Given any positive integer $n$, howsoever large, this process can be continued to obtain $n$ pairs $(\alpha_i, \beta_i)$, $i = 1, 2, ..., n$ such that, for each $i$ the $\alpha_i, \beta_i$ are rational functions of $\alpha, \beta$ and

$$f(\alpha_1, \beta_1) = f(\alpha_2, \beta_2) = f(\alpha_3, \beta_3) = \ldots = f(\alpha_n \, \beta_n). \qquad \ldots (45)$$

We will now show that there exist rational numbers $\alpha, \beta$ such that the process can indeed be continued $n$ times and that no two pairs $(\alpha_j, \beta_j)$ and $(\alpha_k, \beta_k)$ are identical for $j \neq k$. The process will stop at some stage if for a certain pair $(\alpha_i, \beta_i)$, $1 \leq i \leq n$, the corresponding values of $p$, $q$ given by

$$p = -(b\,\alpha_i^2 = 2c\alpha_i\,\beta_i + 3d\beta_i^2),\ q = 3a\,\alpha_i^2 + 3b\,\alpha_i\,\beta_i + c\,\beta_i^2$$

are such that either the numerator or the denominator of the fraction on the right-hjand side of (44) becomes zero i.e. either

$$p^2\,(3a\,\alpha + b\,\beta) + 2pq\,(b\,\alpha + c\,\beta) + q^2\,(c\,\alpha + 3d\,\beta) = 0 \qquad \ldots (46)$$

or

$$f\,(p,\ q) = 0. \qquad \ldots (47)$$

Moreover, two pairs $(\alpha_j, \beta_j)$, $1 \leq j \leq n$ and $(\alpha_k, \beta_k)$, $1 \leq k \leq n$ will be identical if

$$\alpha_j = \alpha_k \qquad \ldots (48)$$

and

$$\beta_j = \beta_k. \qquad \ldots (49)$$

Now $\alpha_i, \beta_i$ are functions of $\alpha, \beta$ for $1 \leq i \leq n$ and so any relation of the type (46), (47), (48) or (49) may be considered as an equation in $\alpha$ by treating $\beta$ as a constant (say, $\beta = 1$). Each such equation has only a finite number of roots. Moreover, for a given $n$, there are only a finite number of such equations. Thus, by taking $\beta = 1$ and choosing a rational value for $\alpha$ so as to avoid the finite number of values that lead to a relation of the type (46), (47), (48) or (49), it is ensured that the process can be repeated $n$ times to yield distinct pairs of rational numbers $(\alpha_j, \beta_j)$, $j = 1, 2, ..., n$ such that the relation (3) is satisfied.

A solution in integers is readily obtained by multiplying by a suitable constant.

As a numerical example, when

$$f(x, y) = x^3 + x^2y + xy^2 + y^3,$$

taking $\alpha = 1$, $\beta = 2$, we get the diophantine chain

$$f(33202159324,\ 66404318648) = f(-\ 235249445942,\ 240108298526)$$

$$= f(128213056201,\ -\ 108795903913).$$