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A CRITERION FOR A-STATISTICAL CONVERGENCE

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The sequence x is A-statistically convergent to L provided that for each $\varepsilon > 0$, the characteristic sequence of the set $K(\varepsilon) := \{k \in N : |x_k - L| \geq \varepsilon\}$ is A-summable to zero where A is a non-negative regular matrix. The case in which $A = C_1$, the Cesáro mean of order one, yields the statistical convergence. In [Shoenberg[1]; Theorem 4] Schoenberg proved a criterion for statistical convergence. In the present paper, we give an analogue of Schoenberg's criteria for A-statistical convergence. We also extend Lemma 3.2 of Salat[2] for A-statistical convergence.

Key Words : Statistical Convergence; A-Statistical Convergence; A-Density

1. INTRODUCTION

If K is a set of positive integers, K_n will denote the set $\{k \in K : k \leq n\}$ and $|K_n|$ will denote the cardinality of K_n. The natural density of K is given [3] by $\delta(K) := \lim_n \frac{1}{n} |K_n|$, if it exists. The real number sequence $x = (x_k)$ is statistically convergent to L provided that for every $\varepsilon > 0$ the set $K := K(\varepsilon) := \{k \in N : |x_k - L| \geq \varepsilon\}$ has natural density zero[4]. In this case we write $\delta\text{-lim } x = L$. Hence, x is statistically convergent to L iff $(C_1 \chi_{K(\varepsilon)})_n \to 0$, (as $n \to \infty$, for every $\varepsilon > 0$), where C_1 is the Cesáro mean of order one and χ_K is the characteristic function of the set K. Some results on statistically convergence may be found [5-7]. Schoenberg[1] called it D-convergence and gave the following criteria : $\delta\text{-lim } x = L$ if and only if $\lim_n \frac{1}{n} \sum_{k=1}^{n} e^{itx_k} = e^{itL}$ for every real t.

For a non-negative regular matrix $A = (a_{nk})$, following Freedman and Sember[8], a set K will be said to have A-density if $\delta_A := \lim_n (A_{\chi_K})_n$ exists. The real number sequence $x = (x_k)$ is A-statistically convergent to L provided that for every $\varepsilon > 0$ the set $K(\varepsilon)$ has A-density zero[6]. In this case we write $\delta_A - \lim x = L$. The case in which $A = C_1$, the Cesáro mean of order one, yields the statistical convergence.

Lacunary statistical convergence,[9-10] can be regarded as special case of A-statistical convergence. A lacunary sequence is an increasing sequence of integers $\theta = (k_r)$ such that $k_0 = 0$ and $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$. Set $I_r = (k_{r-1}, k_r]$.

For a lacunary sequence θ, a sequence x is $S_\theta - \lim x = L$ provided that for every $\varepsilon > 0$,

$$\lim_r \frac{1}{h_r} |\{k \in I_r : |x_k - l| < \varepsilon\}| = 1.$$

This definition is included by the definition of A-statistical convergence above, where the matrix C_θ associated with a given lacunary sequence θ is determined by

$$C_\theta = (c_{nk}) = \begin{cases} \frac{1}{h_n}, & \text{if } k \in I_n \\ 0, & \text{otherwise.} \end{cases}$$

In the present paper, we give an analogue of Schoenberg's criteria[1] for A-statistical convergence. We also extend Lemma 3.2 of Salat [2] for A-statistical convergence.

2. THE MAIN RESULTS

To be able to give the promised criteria we first need a lemma and a known result.

Lemma 1 — If $\delta_A - \lim x = L$ and g, defined for all real y, is continuous at $y = L$, then $\delta_A - \lim g(x) = g(L)$.

PROOF : Let $\delta_A - \lim x = L$. Since g is continuous at $y = L$, for a given $\varepsilon > 0$ there is a $\delta > 0$ such that

$$|y - L| < \delta \qquad \text{implies} \qquad |g(y) - g(L)| < \varepsilon.$$

Hence,

$$|g(y) - g(L)| \geq \varepsilon \qquad \text{implies} \qquad |y - L| \geq \delta$$

and in particular

$$|g(x_k) - g(L)| \geq \varepsilon \qquad \text{implies} \qquad |x_k - L| \geq \delta.$$

Thus

$$K := \{k \in N : |g(x_k) - g(L)| \geq \varepsilon\} \subset K' := \{k \in N : |x_k - L| \geq \delta\}.$$

Since $\delta_A - \lim x = L$, we have that $\lim_n (A_{\chi_{K'}})_n = 0$ and therefore $\lim_n (A_{\chi_K})_n = 0$ which completes the proof.

Theorem 1 — *If $x \in l_\infty$, the set of all bounded sequences, and $\delta_A - \lim x = L$, then x is A-summable to L*[11].

Now we are ready to establish the promised criteria for A-statistical convergence.

Theorem 2 — *The number sequence $x = (x_k)$ is A-statistically convergent to the number L if and only if for each number t we have*

$$\lim_n \sum_{k=1}^{\infty} a_{nk} e^{itx_k} = e^{itL}. \qquad \ldots (1)$$

PROOF : Let $\delta_A - \lim x = L$. For a fixed value of t, $g(x) = e^{itx}$ is a continuous function

of x. Hence, Lemma 1 implies that $\delta_A - \lim e^{itx_k} = e^{itL}$. Since $\left\{ e^{itx_k} \right\}_{k=1}^{\infty}$ is a bounded sequence, by Theorem 1, we have $\lim_n \sum_{k=1}^{\infty} a_{nk} e^{itx_k} = e^{itL}$, this completes the proof of necessity.

Conversely, suppose that $\lim_n \sum_{k=1}^{\infty} a_{nk} e^{itx_k} = e^{itL}$. We now show that

$$\delta_A - \lim x = L. \qquad \ldots (2)$$

In order to do so, we first define a continuous function M by

$$y_k := \begin{cases} 0, & \text{if } y \le -1, \\ 1+y, & \text{if } -1 < y < 0, \\ 1-y, & \text{if } 0 \le y < 1, \\ 0, & \text{if } y \ge 1. \end{cases}$$

By the inverse Fourier transformation, we get

$$M(y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\frac{\sin \frac{t}{2}}{\frac{t}{2}} \right)^2 e^{ity}\, dt, \qquad (-\infty < y < \infty). \qquad \ldots (3)$$

It is enough to prove (2) for the case in which $L = 0$. Let $\varepsilon > 0$ and let $K := K(\varepsilon) := \{k \in N : |x_k| \ge \varepsilon\}$. By an appropriate change of variables, (3) becomes

$$M\left(\frac{y}{\varepsilon}\right) = \frac{\varepsilon}{2\pi} \int_{-\infty}^{\infty} \left(\frac{\sin \frac{\varepsilon t}{2}}{\frac{\varepsilon t}{2}} \right)^2 e^{ity}\, dt$$

whence

$$\sum_{k=1}^{\infty} a_{nk} M\left(\frac{x_k}{\varepsilon}\right) = \frac{\varepsilon}{2\pi} \int_{-\infty}^{\infty} \left(\frac{\sin \frac{\varepsilon t}{2}}{\frac{\varepsilon t}{2}} \right)^2 \left(\sum_{k=1}^{\infty} a_{nk} e^{itx_k} \right) dt.$$

Observe now that (3) is an absolutely convergent integral, i.e., the relation is valid with the integral taken in the sense of Lebesgue. By the Lebesgue's Dominated Convergence Theorem we have

$$\lim_n \sum_{k=1}^{\infty} a_{nk} M\left(\frac{x_k}{\varepsilon}\right) = \frac{\varepsilon}{2\pi} \int_{-\infty}^{\infty} \left(\frac{\sin\frac{\varepsilon t}{2}}{\frac{\varepsilon t}{2}}\right)^2 \left(\lim_n \sum_{k=1}^{\infty} a_{nk} e^{itx_k}\right) dt$$

$$= \frac{\varepsilon}{2\pi} \int_{-\infty}^{\infty} \left(\frac{\sin\frac{\varepsilon t}{2}}{\frac{\varepsilon t}{2}}\right)^2 dt$$

$$= M(0) = 1.$$

Considering the function M,we get

$$\sum_{k=1}^{\infty} a_{nk} M\left(\frac{x_k}{\varepsilon}\right) = \sum_{1 < \frac{x_k}{\varepsilon} < 0} a_{nk} M\left(\frac{x_k}{\varepsilon}\right) + \sum_{0 \le \frac{x_k}{\varepsilon} < 1} a_{nk} M\left(\frac{x_k}{\varepsilon}\right)$$

$$\le \sum_k a_{nk} \chi_N(k) - \sum_k a_{nk} \chi_K(k).$$

Since A is regular and $\lim_n \sum_{k=1}^{\infty} a_{nk} M\left(\frac{x_k}{\varepsilon}\right) = 1$, we conclude $\lim_n \sum_{k=1}^{\infty} a_{nk} \chi_K(k) = 0$, hence the proof is completed.

Taking the matrix $\boldsymbol{A}$ to be the $\boldsymbol{C}_1$matrix we immediately get the following result of Schoenber[1]; theorem 4]:

Corollary 1 : $\delta - \lim x = L$ if and only if $\lim_n \frac{1}{n} \sum_{k=1}^{n} e^{itx_k} = e^{itL}$ for every real t.

Again, taking the matrix $\boldsymbol{A}$ to be the $\boldsymbol{C}_\theta$ matrix we get the following:

Corollary 2 : $S_\theta - \lim x = L$ if and only if $\lim_r \frac{1}{h_r} \sum_{k \in I_r} e^{itx_k} = e^{itL}$ for every real t.

In the next theorem we show, for a sequence x belonging to the space $S^* := \left\{ x : \left\{ \sum_{k=1}^{\infty} a_{nk} |x_k| \right\}_{n=1}^{\infty} \in l_\infty \right\}$, that (1) holds for every rational number t if and only if x is A-statistically convergent. This also extends a result due to Salat [2; Lemma 3.2].

Theorem 3 — *Let $x = (x_k) \in S^*$. Then x is A-statistically convergent to the real number L if and only if for each rational number t we have*

$$\lim_n \sum_{k=1}^{\infty} a_{nk} e^{itx_k} = e^{itL}.$$

PROOF : Necessity. Let $x \in S^*$ and $\delta_A - \lim x = L$. Using Theorem 2, we immediately conclude that for each rational number t, $\lim_n \sum_{k=1}^{\infty} a_{nk} e^{itx_k} = e^{itL}$.

Sufficieny — Let $\lim_n \sum_{k=1}^{\infty} a_{nk} e^{itx_k} = e^{itL}$ for each rational number t. Let t_0 be an arbitrary real number. We shall complete the proof by showing that $\lim_n \sum_{k=1}^{\infty} a_{nk} e^{it_0 x_k} = e^{it_0 L}$. For a rational number t, write

$$C_n(t_0, t) := \sum_{k=1}^{\infty} a_{nk} e^{it_0 x_k} - \sum_{k=1}^{\infty} a_{nk} e^{itx_k}.$$

Since $e^{i\theta L} = \cos \theta L + i \sin \theta L$, $(\theta \in R)$, we have

$$| C_n(t_0,t) | \leq \sum_{k=1}^{\infty} a_{nk} \sqrt{(\cos t_0 x_k - \cos t x_k)^2 + (\sin t_0 x_k - \sin t x_k)^2}.$$

It follows from the mean value theorem that

$$| C_n(t_0, t) | \leq | t - t_0 | \sum_{k=1}^{\infty} a_{nk} | x_k |. \qquad \ldots (4)$$

Since $x \in S^*$, there exists $M > 0$ such that for all n

$$\sum_{k=1}^{\infty} a_{nk} | x_k | \leq M. \qquad \ldots (5)$$

Now (4) and (5) imply that

$$| C_n(t_0, t) | \leq M | t - t_0 |. \qquad \ldots (6)$$

Moreover, we have

$$\sum_{k=1}^{\infty} a_{nk} e^{it_0 x_k} = \sum_{k=1}^{\infty} a_{nk} e^{itx_k} + C_n(t_0, t), \qquad \ldots (7)$$

from which we also get

$$\left| \sum_{k=1}^{\infty} a_{nk} e^{it_0 x_k} - e^{it_0 L} \right| \leq \left| \sum_{k=1}^{\infty} a_{nk} e^{itx_k} - e^{itL} \right| + | e^{itL} - e^{it_0 L} | + | C_n(t_0, t) | . \quad \ldots (8)$$

Let $\varepsilon > 0$. Since $g(x) = e^{ixL}$ is a continuous for $x \in R$, there exists a rational number t such that

$$| e^{itL} - e^{it_0 L} | < \frac{\varepsilon}{3} . \quad \ldots (9)$$

It also follows from (6) that

$$| C_n(t_0, t) | < \frac{\varepsilon}{3} , \qquad (n = 1, 2, \ldots). \quad \ldots (10)$$

By hypothesis there exists an n_0 such that for each $n \geq n_0$

$$\left| \sum_{k=1}^{\infty} a_{nk} e^{itx_k} - e^{itL} \right| < \frac{\varepsilon}{3} \quad \ldots (11)$$

holds. Now (8), (9), (10) and (11) imply for each $n > n_0$, that

$$\left| \sum_{k=1}^{\infty} a_{nk} e^{it_0 x_k} - e^{it_0 L} \right| \leq \varepsilon$$

whence the proof is completed.

In conclusion, we note that the case in which $A = C_1$, reduces to Lemma 3.2 of Salat[2].

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