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OSCILLATION OF FIRST ORDER NEUTRAL NONLINEAR DIFFERENTIAL EQUATIONS WITH VARIABLE COEFFICIENTS*

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The first order neutral nonlinear differential equation with variable coefficients is considered

$$\frac{d}{dt}(x(t)-R(t)x(t-r))+P(t)\prod_{i=1}^{m}|x(t-\tau_i)|^{\alpha_i}\operatorname{sgn}x(t-\tau_i)=0,$$

where $P, R \in C([t_0,\infty), R^+)$, $r \in (0,\infty)$ and $\tau_1, \tau_2, ..., \tau_m$ are nonnegative numbers. We will establish a necessary and sufficient condition for the oscillation of all solutions. This will lead to several new criteria for the oscillation of the above equation without the restriction

$$\int_{t_0}^{\infty} P(s)ds=\infty \quad \text{or} \quad \int_{t_0}^{\infty} sP(s)\int_{s}^{\infty} P(u)duds=\infty,$$

commonly used in the literature.

Key Words : Neutral Nonlinear Equation; Oscillation; Comparison Theorem.

1. INTRODUCTION

This work is motivated by several recent investigations[1&2], in which comparison and oscillation theorems are given for linear neutral differential equations. Here we consider a more general nonlinear differential equation of the form

$$\frac{d}{dt}(x(t)-R(t)x(t-r))+P(t)\prod_{i=1}^{m}|x(t-\tau_i)|^{\alpha_i}\operatorname{sgn}x(t-\tau_i)=0, \qquad \ldots (1)$$

where $P, R \in C([t_0,\infty), R^+)$, $r \in (0,\infty)$ and $\tau_1, \tau_2, \ldots, \tau_m$ are nonnegative numbers, $R(t) \geq 0$ and $P(t) \geq 0$ for $t \geq t_0$ such that $P(t)$ is not identically zero for all large t, and each α_i is a positive number for $i = 1, 2, ..., m$ such that $\alpha_1 + \alpha_2 + \ldots + \alpha_m = 1$.

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When $m = 1$, eq. (1) reduces to the linear equation

$$\frac{d}{dt}(x(t) - R(t)x(t-r)) + P(t)x(t-\tau) = 0, \quad \ldots (2)$$

that has already been studied[1-15] and elsewhere.

Let $T_0 = \max\{r, \tau_1, \ldots, \tau_m\}$. By a solution of eq. (1) we means a function $x \in C([t_0 - T_0, \infty), R)$ such that $x(t) - R(t)x(t-r)$ is continuously differentiable and satisfies (1) on $[t_0, \infty)$.

Let $\phi \in C([t_0 - T_0, t_0], R)$ be a given initial function. One can easily see by the method of steps that eq. (1) has unique solution $x \in C([t_0 - T_0, \infty), R)$ such that $x(t) = \phi(t)$ for $t_0 - T_0 \le t \le t_0$. A solution of eq. (1) is said to oscillate if it has arbitrarily large zeros. Otherwise the solution is called monooscillatory. In the sequel, unless otherwise specified, when we write a functional inequality we shall assume that it holds for all sufficiently large values.

The oscillation of eq. (2) has recently been investigated by a number of researchers and many sufficient conditions for the oscillation of all solutions have been obtained[1-12]. In these papers except [1 & 2], the hypothesis

$$\int_{t_0}^{\infty} P(s)ds = \infty \quad \ldots (3)$$

plays an essential role. It is then interesting to ask if the condition (3) can be replaced by others. This question was addressed by Yu Wang and Qian[7], in order to answer the open problem proposed by Qian and Ladas[3]. In 1994, by using results of [1], Chen, Yu and Huang[2] established some new sufficient conditions for the oscillation of all solutions of eq. (2), which assume the condition

$$\int_{t_0}^{\infty} sP(s) \int_{s}^{\infty} P(u)duds = \infty \quad \ldots (4)$$

to hold. However, if $R(t) \equiv 1, P(t) = t^{-\alpha}\left(\frac{3}{2} < \alpha < 2\right), m = 1$, then eq. (1) reduces to the equation

$$\frac{d}{dt}(x(t) - x(t-r)) + t^{-\alpha}x(t-\tau) = 0, \quad \frac{3}{2} < \alpha < 2. \quad \ldots (5)$$

Since

$$\int_{t_0}^{\infty} sP(s) \int_{s}^{\infty} P(u)duds = \int_{t_0}^{\infty} s^{-\alpha+1} \int_{s}^{\infty} u^{\alpha} duds$$

$$= \frac{1}{\alpha - 1} \int_{t_0}^{\infty} s^{2-2\alpha} ds = \frac{1}{(\alpha-1)(2\alpha-3)} t_0^{3-2\alpha} < \infty.$$

Thus, the above mentioned criteria[1&2] cannot be applied to this equation.

In this paper, we will establish several further oscillation results for (1) when conditions (3)

and (4) do not satisfy. This is done by Lemmas 1 and 2 given in sections 2 and 3. These lemmas are interesting in their own right.

2. A COMPARISON THEOREM

The following preparatory result is known[1], the proof of which, for the sake of completeness, will be included here.

Lemma 1 — Assume that there exists $t^* \geq t_0$ such that

$$R(t^* + nr) \leq 1 \text{ for } n = 0, 1, \ldots \qquad \ldots (6)$$

Then for any eventually positive solution $x(t)$ of eq. (1), the function $y(t)$ defined by

$$y(t) = x(t) - R(t)x(t-r) \qquad \ldots (7)$$

will satisfy

PROOF : It is clear from (1) that $y'(t) \leq 0$ and is not identically zero for all large t. Thus $y(t)$ is eventually positive or eventually negative. Suppose that $x(t) > 0, y'(t) \leq 0$ and $y(t) < 0$ for $t \geq T$, then $y(t) \leq y(T) < 0$ for $t > T$. By choosing n so large that $t^* + nr \geq T$, we see from (7) that

$$\begin{aligned} x(t^* + nr + jr) &= y(t^* + nr + jr) + R(t^* + nr + jr)x(t^* + nr + jr) \\ &\leq y(T) + x(t^* + nr + (j-1)r) \\ &= y(T) + y(t^* + nr + (j-1)r) + R(t^* + nr + (j-1)r)x(t^* + nr + (j-1)r) \\ &\leq \ldots \leq (j+1)y(T) + x(T + nr), \quad j \geq 0. \end{aligned}$$

By letting j tend to infinity, we see that the right hand side diverges to $-\infty$, which is contrary to our assumption that $x(t) > 0$. The proof is complete.

The following result extends Theorem 1 in [2] to the nonlinear neutral equation (1) and also improves Theorem 1 in[16] by relaxing the restriction $R(t) \equiv r \in (0, 1)$.

Theorem 1 — *Assume that (6) holds and that either*

$$R(t) + P(t))r > 0 \qquad \ldots (8)$$

or $r > 0$ *and* $P(s) \not\equiv 0$, *for* $s \in [t, t+r]$. $\qquad \ldots (9)$

Then every solution of eq. (1) oscillates if and only if the corresponding differential inequality

$$\frac{d}{dt}(x(t) - R(t)x(t-r)) + P(t) \prod_{i=1}^{m} |x(t-\tau_i)|^{\alpha_i} \operatorname{sgn} x(t-\tau_i) \leq 0 \qquad \ldots (10)$$

has no eventually positive solutions.

PROOF : The sufficiency is obvious. To prove the necessarity, we assume that $x(t)$ is an eventually positive solution of (10). Set $y(t)$ as in (7). Then by Lemma 1, we find

$$y(t) \geq \int_t^{\infty} P(s) \prod_{i=1}^{m} x(s-\tau_i)^{\alpha_i} ds. \qquad \ldots (11)$$

That is,

$$x(t) \geq R(t)\, x(t-r) + \int_t^{\infty} P(s) \prod_{i=1}^{m} x(s-\tau_i)^{\alpha_i}\, ds. \qquad \ldots (12)$$

Let $T > t_0$ be fixed so that (12) holds for all $t \geq T$. Now we consider the set of functions:

$$E = \{u \in C([T-T_0, \infty), R^+) : 0 \leq u(t) \leq 1, \text{ for } t \geq T-T_0\},$$

and define a mapping F on E as

$$(Fu)(t) = \begin{cases} \dfrac{1}{x(t)} \left[R(t)\, u(t-r)\, x(t-r) + \displaystyle\int_t^{\infty} P(s) \prod_{i=1}^{m} x(s-\tau_i)^{\alpha_i}\, ds\right], & t \geq T, \\ \dfrac{t-T+T_0}{T_0}\,(Fu)(T) + \left(1 - \dfrac{t-T+T_0}{T_0}\right), & T-T_0 \leq t \leq T. \end{cases}$$

It is easy to see, using (12), that F maps E into itself. Moreover, for any $u \in E$ we have $(Fu)(t) > 0$ for $T-T_0 \leq t < T$. Next we define the sequence $u_k(t)$ in E by

$$u_0(t) = 1, \quad t \geq T - T_0,$$

$$u_{k+1}(t) = (Fu_k)(t), \qquad \text{for} \quad k = 0, 1, \ldots .$$

Then, by using (12) and a simple induction, we can easily see that

$$0 \leq u_{k+1} \leq u_k(t) \leq 1 \quad \text{for} \quad t \geq T - T_0, \quad k = 0, 1, \ldots .$$

Set

$$u(t) = \lim_{k \to \infty} u_k(t), \; t \geq T - T_0.$$

Then it follows from Lebesgue's Dominated Convergence Theorem that $u(t)$ satisfies

$$u(t) = \frac{1}{x(t)} \left[R(t)u(t-r)x(t-r) + \int_t^{\infty} P(s) \prod_{i=1}^{m} u(s-\tau_i)^{\alpha_i}\, x(s-\tau_i)^{\alpha_i}\, ds\right], \; t \geq T,$$

and

$$u(t) = \frac{t-T+T_0}{T_0}\,(Fu)(T) + \left(1 - \frac{t-T+T_0}{T_0}\right), \qquad T-T_0 \leq t \leq T.$$

Again set

$$w(t) = u(t)x(t).$$

Then $w(t)$ satisfies $w(t) > 0$ for $T-T_0 \leq t < T$ and

$$w(t) = R(t)w(t-r) + \int_t^\infty P(s) \prod_{i=1}^m w(s-\tau_i)^{\alpha_i} ds, \quad t \geq T. \qquad \ldots (13)$$

Clearly, $w(t)$ is continuous on $[T-T_0, \infty)$.

It remains to show that $w(t)$ is positive for all $t \geq T-T_0$. Assume that there exists $t^* \geq T-T_0$ such that $w(t) > 0$ for $T-T_0 \leq t < t^*$ and $w(t^*) = 0$. Then $t^* \geq T$ and by (13) we get

$$0 = w(t^*) = R(t^*)w(t^*-r) + \int_{t^*}^\infty P(s) \prod_{i=1}^m w(s-\tau_i)^{\alpha_i} ds,$$

which implies

$$R(t^*) \equiv 0, \text{ and } P(t) \prod_{i=1}^m w(t-\tau_i)^{\alpha_i} \equiv 0, \quad \text{for all } t \geq t^*.$$

This contradicts (8) or (9). Therefore, $w(t)$ is positive on $[T-T_0, \infty)$. It is easy to see that $w(t)$ is a positive solution of eq. (1), which implies that inequality (10) having no eventually positive solution is a necessary condition for the oscillation of all solutions of eq. (1). The proof is complete.

We now compare eq. (1) with the equation

$$\frac{d}{dt}[x(t) - R^*(t)x(t-r)] + P^*(t) \prod_{i=1}^m |x(t-\tau_i)|^{\alpha_i} \operatorname{sgn} x(t-\tau_i) = 0 \qquad \ldots (14)$$

to obtain the following comparison theorem:

Theorem 2 — *Assume that (6) holds and that (8) or (9) holds for $P^*(t)$ and $Q^*(t)$. Further assume that*

$$R(t) \geq R^*(t) \text{ and } P(t) \geq P^*(t). \qquad \ldots (15)$$

Then the fact that every solution of eq. (14) oscillates implies that the oscillation of all solutions of eq. (1).

PROOF : Suppose the contrary and let $x(t)$ be an eventually positive solution of eq. (1). Set $y(t)$ as in (7). Then by Lemma 1 we have

$$y'(t) \leq 0 \text{ and } y(t) > 0.$$

Thus, by integrating eq. (1) from t to ∞, we obtain

$$y(t) = y(\infty) + \int_t^\infty P(s) \prod_{i=1}^m x(s-\tau_i)^{\alpha_i} ds.$$

Noting $y(\infty) = \lim_{t \to \infty} y(t) \geq 0$ and (15), we obtain

$$x(t) \geq R^*(t)x(t-r) + \int_t^\infty P^*(s) \prod_{i=1}^m x(s-\tau_i)^{\alpha_i} ds,$$

which, using the method similar to that in the proof of Theorem 1, yields that eq. (15) has also an eventually positive solution. This is a contradiction and the proof is complete.

We remark that Theorem 2 can be regarded as an extension of Theorem 2 in [2] and an improvement of Theorem 2 in [16].

3. OSCILLATION CRITERIA

We now turn to the question as to when eq. (1) is oscillatory. This question is important if we want to apply Theorem 2.

Lemma 2 — Assume that $R(t) \geq 1$ for $t \geq t_0$ and

$$\int_{t_0}^{\infty} P(s) \exp\left[\frac{1}{r}\int_{t_0}^{s} uP(u)du\right] ds = \infty. \qquad \ldots (16)$$

Let $x(t)$ be an eventually positive solution of eq. (1) and $y(t)$ defined by (7). Then eventually

$$y'(t) \leq 0 \text{ and } y(t) < 0. \qquad \ldots (17)$$

PROOF : From (1) and (7), we have

$$y'(t) = -P(t) \prod_{i=1}^{m} x(t-\tau_i)^{\alpha_i} \leq 0. \qquad \ldots (18)$$

Therefore, if (17) does not hold, then eventually $y(t) > 0$, that is,

$$x(t) \geq R(t)x(t-r) \geq x(t-r). \qquad \ldots (19)$$

Let $t_1 \geq t_0$ be such that $x(t-r) > 0$ for $t \geq t_1$ and such that (12) holds for $t \geq t_1$. Define

$$m = \min\{x(t) : t \in [t_1 - r, t_1]\}.$$

Then $x(t) \geq m$ for $t \geq t_1$. Set $\tau^* = \max\{r, \tau_1, \ldots, \tau_m\}$, and we have

$$x(t) \geq m \quad \text{for} \quad t \geq t_1 + \tau^* = t_2.$$

For convenience, we denote

$$N(t) = \left[\frac{t-t_2}{r}\right], \qquad \ldots (20)$$

where $\left[\frac{t-t_2}{r}\right]$ is the greatest integer parts of $\frac{t-t_2}{r}$. Then,

$$x(t) \geq y(t) + x(t-r)$$

$$\geq y(t) + y(t-r) + \ldots + y(t-(N(t)-1)r) + x(t-N(t)r), \quad t \geq t_2. \qquad \ldots (21)$$

Noting $y(t)$ is decreasing and $x(t - N(t)r) \geq m$ for $t \geq t_2$. Thus, by (14) we obtain

$$x(t) \geq N(t)\, y(t) + m, \qquad t \geq t_2. \qquad \ldots (22)$$

Substituting this into (1), we have

$$y'(t) + P(t) \prod_{i=1}^{m} [N(t - \tau_i)\, y(t - \tau_i) + m]^{\alpha_i} \leq 0, \qquad t \geq t_2 + \tau_0 = t_3,$$

where $\tau_0 = \max \{\tau_1, \ldots, \tau_m\}$. By Holder Inequality [14, p20], we have

$$\prod_{i=1}^{m} [N(t - \tau_i)\, y(t - \tau_i) + m]^{\alpha_i} \geq \prod_{i=1}^{m} N(t - \tau_i)^{\alpha_i} \prod_{i=1}^{m} N(t - \tau_i)^{\alpha_i} + m.$$

Thus, we have

$$y'(t) + P(t) \prod_{i=1}^{m} N(t - \tau_i)^{\alpha_i}\, y(t) + P(t)m \leq 0, \qquad t \geq t_3.$$

Then

$$\left[y(t) \exp\left(\int_{t_3}^{t} P(s) \prod_{i=1}^{m} N(s - \tau_i)^{\alpha_i}\, ds \right) \right]^{t}$$

$$+ mP(t) \exp\left(\int_{t_3}^{t} P(s) \prod_{i=1}^{m} N(s - \tau_i)^{\alpha_i}\, ds \right) \leq 0, \qquad t \geq t_3. \qquad \ldots (23)$$

Integrating (23) from t_3 to $t \geq t_3$ we have

$$y(t) \exp\left(\int_{t_3}^{t} P(s) \prod_{i=1}^{m} N(s - \tau_i)^{\alpha_i}\, ds \right) - y(t_3)$$

$$+ m \int_{t_3}^{t} P(s) \exp\left(\int_{t_3}^{t} P(s) \prod_{i=1}^{m} N(u - \tau_i)^{\alpha_i}\, du \right) ds \leq 0, \qquad t \geq t_3. \qquad \ldots (24)$$

If the condition

$$\int_{t_0}^{\infty} P(s) ds = \infty \qquad \ldots (25)$$

is satisfied, then every solution of eq. (1) oscillates[3]. Hence, we assume

$$\int_{t_0}^{\infty} P(s)ds < \infty. \qquad \ldots (26)$$

Noting $\prod_{i=1}^{m} N(t-\tau_i)^{\alpha_i}/t \to \frac{1}{r}\ (t \to \infty)$, it is easy to see that

$$\int_{t_3}^{\infty} P(s)\left[\frac{s}{r} - \prod_{i=1}^{m} N(s-\tau_i)^{\alpha_i}\right]ds$$

is absolutely convergent and

$$\lim_{s\to\infty} \frac{\exp\left(\int_{t_3}^{s} P(u)\right)\prod_{i=1}^{m} N(u-\tau_i)^{\alpha_i}\,du}{\exp\left(\frac{1}{r}\int_{y_3}^{s} uP(u)du\right)}$$

exists. By condition (16) we obtain

$$\int_{t_3}^{\infty} P(s)\exp\left(\int_{t_3}^{s} P(u)\right)\prod_{i=1}^{m} N(u-\tau_i)^{\alpha_i}\,du)\,ds = \infty.$$

Letting $t \to \infty$ in (24) we obtain a contradiction and the proof is complete.

In view of Lemmas 1 and 2, it is now easy to see the following result.

Theorem 3 — *Assume that (16) holds and*

$$R(t) \equiv 1. \qquad \ldots (27)$$

Then every solution of eq. (1) oscillates.

Example 1 — Consider neutral delay differential equation

$$\frac{d}{dt}(x(t) - x(t-r)) + t^{-\alpha}x(t-\tau) = 0, \qquad \ldots (5)$$

where $r > 0$, $\tau > 0$, $1 < \alpha < 2$, which satisfies all conditions of Theorem 3. Hence all solutions of the equations oscillate. But condition (4) is not satisfied when $\frac{3}{2} < \alpha < 2$. On the other hand, by Theorem 1 in [13], we see that eq. (5) has a bounded nonoscillatory solution if and only if $\alpha > 2$. Therefore, it remains an open problem to determine the oscillation of all solutions of eq. (5) with $\alpha = 2$.

Theorem 4 — *Assume that (6) and (16) hold, and that*

$$\prod_{i=1}^{m} R(t-\tau_i)^{\alpha_i} P(t) \geq P(t-r), \qquad \ldots (28)$$

Then every solution of eq. (1) oscillates.

PROOF : Otherwise, eq. (1) would have an eventually positive solution $x(t)$. Let $y(t)$ be defined by (7). Then by Lemma 1 we have

$$y'(t) < 0 \text{ and } y(t) > 0.$$

From, eqs. (1), (7) and (28), we have

$$y'(t) = -P(t) \prod_{i=1}^{m} x(t-\tau_i)^{\alpha_i}$$

$$= -P(t) \prod_{i=1}^{m} [y(t-\tau_i) + R(t-\tau_i)x(t-r-\tau_i)]^{\alpha_i}$$

$$\leq -P(t) \prod_{i=1}^{m} y(t-\tau_i)^{\alpha_i} + y'(t-r),$$

where we have used the Holder inequality [17, p 20] to obtain the last equality. This implies that $y(t)$ is a positive solution of the inequality

$$y'(t) - y'(t-r) + P(t) \prod_{i=1}^{m} y(t-\tau_i)^{\alpha_i} \leq 0, \qquad \ldots (29)$$

which satisfies all conditions of Lemma 1, hence $u(t) = y(t) - y(t-r) > 0$ eventually. On the other hand, since (29) satisfies all conditions of Lemma 2, $u(t) = y(t) - y(t-r) < 0$ eventually, which is a contradiction and the proof is complete.

In case the assumption (28) is not satisfied, we may check to see if there is some number $c \in [0, 1)$ such that $cP(t-r) \leq P(t) \prod_{i=1}^{m} R(t-\tau_i)^{\alpha_i}$ for all large t.

Theorem 5 — *Assume that (6) and (16) hold and that there is some number* $C \in [0, 1)$ *such that*

$$cP(t-r) \leq P(t) \prod_{i=1}^{m} R(t-\tau_i)^{\alpha_i}$$

for all large t. Then (1) is oscillatory provided that the following inequality

$$z'(t) + \frac{c}{1-c} P(t) z(t-r-\tau) \leq 0, \quad r = \min \{\tau_1, \ldots, \tau_m\} \qquad \ldots (30)$$

does not have an eventually positive solution.

PROOF : Suppose to the contrary that $x(t)$ is an eventually positive solution of (1). Then by means of Lemma 1, the function $y(t)$ defined by (7) will satisfy $y(t) > 0$ and $y'(t) \leq 0$ for all large t. Then, as in the proof of Theorem 4, we see that

$$y'(t) - cy'(t-r) + P(t) \prod_{i=1}^{m} y(t-\tau_i)^{\alpha_i} \leq 0$$

for all large t. Since $y(t)$ is nonincreasing, we have

$$y'(t) - cy'(t-r) + P(t)y(t-\tau) \leq y'(t) - cy'(t-r) + P(t) \prod_{i=1}^{m} y(t-\tau_i)^{\alpha_i} \leq 0$$

for all large t, where $\tau = \min\{\tau_1, \ldots, \tau_m\}$. Let $z(t) = y(t) - cy(t-r)$ for $t \geq t_0$. By means of Lemma 1, it is clear that $z'(t) \leq 0$ and $z(t) > 0$ for t greater than or equal to some number T Without loss of generality, we may also assume that $y(t) > 0$ for $t \geq T$. Now

$$\begin{aligned} y(t) &= z(t) + cy(t-r) \\ &\geq z(t) + cz(t-r) + \ldots + c^j z(t-jr) + c^{j+1} y(t-(j+1)r) \\ &> (c + c^2 + \ldots + c^{j+1}) z(t-r) = \frac{c(1-c)}{1-c} z(t-r) \end{aligned}$$

for $t > (j+1)r + T + r$. Thus

$$z'(t) + \frac{c}{1-c} P(t) z(t-r-\tau) \leq 0$$

for all large t, which is contrary to our hypothesis. The proof is complete.

We remark that several explicit conditions ensuring that (30) cannot have an eventually positive solution have already been established. A sample of these conditions can be found in the recent book by Gyori and Ladas[14].

There is an oscillation criterion [15, Theorem 2.1] the proof of which can be extended to fit our equation (1). But our results drop the condition (2.4) in [15] and relax the restriction $R(t) = R_0 \in [0, 1)$.

Theorem 6 — *Suppose* $P(t) \geq p > 0, R(t) \geq 0$ *for* $t \geq t_0$ *such that (6) holds. Suppose further that* $\tau_i \in (0, \infty)$, i = *1, 2, .., m and*

$$\inf_{t \geq t_0, \lambda > 0} \left\{ \frac{P(t)}{P(t-r)} \prod_{i=1}^{m} R(t-\tau_i)^{\alpha_i} e^{r\lambda} + \frac{1}{l\lambda} e^{\lambda} \sum_{i=1}^{m} \alpha_i r_i \int_{t}^{t+l} P(s)ds \right\} > 1. \qquad \ldots (31)$$

Then (1) is oscillatory, where $l = r, r_1, \ldots, \tau_m$.

PROOF : Suppose to the contrary that $x(t)$ is an eventually positive solution of (1). Then arguments similar to those used in the proof of Theorem 4, the function $y(t)$ defined by (7) is positive, nonincreasing and satisfies

$$y'(t) - \frac{P(t)}{P(t-r)} \prod_{i=1}^{m} R(t-\tau_i)^{\alpha_i} y'(t-r) + P(t) \prod_{i=1}^{m} y(t-\tau_i)^{\alpha_i} \le 0, \qquad \ldots (32)$$

say $t \ge t_0$. For the sake of convenience, let us write

$$Q(t) = \frac{P(t)}{P(t-r)} \prod_{i=1}^{m} R(t-\tau_i)^{\alpha_i}, \ t \ge t_0.$$

Let $w(t) = -\dfrac{y'(t)}{y(t)}$ for $t \ge t_0$. Then $w(t) > 0$ for $t \ge t_0$ and

$$w(t) \ge Q(t)w(t-r) \exp\left(\int_{t-r}^{t} w(s)ds \right) + P(t) \prod_{i=1}^{m} \left[\exp\left(\int_{t-r_i}^{t} w(s)ds \right) \right]^{\alpha_i}$$

for $t \ge t_0 + \mu$, where $\mu = max \{r, \tau_1, \ldots, \tau_m\}$.

The rest of the proof of Theorem 6 is similar to that of Theorem 2.1 in [15] we will omit it here. The proof is complete.

As noted in [15], condition (31) is sharp when $m = 1$, and $R(t) \equiv R > 0$, $P(t) \equiv P$ for $t \ge t_0$.

REFERENCES

1. J. S. Yu, Z. C. Wang and C. Qian, *Bull. Austral. Math. Soc.* **45** (1992) 195-220.
2. M. P. Chen, J. S. Yu and L. H. Huang, *J. math. Anal. Appl.* **185** (1994) 288-301.
3. Q. Chuanxi and G. Ladas, *Appl. Anal.* **32** (1989) 215-228.
4. W. Chuanzi and G. Ladas, *Rad. Math.* **5**(1989) 321-31.
5. Q. Chuanxi, G. Ladas, B. G. Zhang and T. Zhao, *Appl. Anal.* **35** (1990) 187-94.
6. M. K. Grammatikopoulos, E. A. Grove and G. Ladas, *J. math. Anal. Appl.* **120** (1986) 510-20.
7. D. A. Georgiou and C. Qian, *J. math. math Sci.* **14** (1991) 689-96.
8. E. A. Grove, G. Ladas and J. Schinas, *Canad. math. Bull.* **31** (1988) 459-66.
9. E. A. Grove, M. R. S. Kulenovic, and G. Ladas, *J. differential Eq.* **68** (1987) 373-82.
10. K. Gopalsamy and B. G. Zhang, *J. math. Anal. Appl.* **151** (1990) 42-57.
11. B. S. Lalli and B. G. Zhang, *Appl. Anal.* **39** (1990) 265-74.
12. Y. H. Yu, *Acta Appl. Math. Sinica* **14** (1991) 404-10.
13. B. G. Zhang and J. S. Yu, *Scientia Sinica (Series A)* **8** (1992) 785-90.
14. I. Gyori and G. Ladas, Oscillatory Theory of Delay Differential Equations with Applications, Oxford Univ. Press, New York, (1991).
15. L. H. Erbe and Q. K. Kong, *Can. J. math.* **46(2)** (1994) 284-97.