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# ON MODELS OF $U_q(sl(2))$ AND $q$-EULER INTEGRAL TRANSFORMATION

VIVEK SAHAI

*Department of Mathematics & Astronomy, Lucknow University, Lucknow* 226 007 *(E-mail: lutl @ lw1.vsnl.net.in)*

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We discuss few models of the quantum universal enveloping algebra of $sl(2)$ from the special function point of view, using the $q$-Euler integral transformation. Some interesting identities are obtained.

**Key Words : Integral Transformation – $U_q(sl(2))$, $q$-Integral Transformation**

## 1. INTRODUCTION

In recent papers, the author[1&2] has constructed models of irreps of classical $sl(2)$ in terms of difference-differential operators using Euler integral transformation. These models have culminated in various recurrence relations and identities involving ordinary hypergeometric series. To have an extension of this to $q$-series, we introduce the idea of $q$-Euler transformation, which is $q$-analogue of the Euler integral transformation, to construct models of $U_q(sl(2))$, the quantum universal enveloping algebra of $sl(2)$, in terms of difference-dilation operators. These models reduce to the ones invlving difference-differential operators in the limit $q \to 1$. Section-wise treatment is as follows.

In Section 2, we list few definitions and results needed for our discussion. In Section 3, we discuss models of $U_q(sl(2))$ and give its two variable models in terms of dilation operators in which ${}_1\phi_0$ functions appear as the basis functions. In Section 4, $q$-Euler transformation is defined and transforms of certain expressions are obtained. Using a theorem, we upgrade the models of Section 3 into new models of $U_q(sl(2))$ in terms of difference-dilation operators in which the basis functions involve the basic hypergeometric functions ${}_2\phi_1$. These models, in section 5, are exploited for identities, which are believed to be new.

## 2. PRELIMINARIES

The generalized basic (or $q$-) hypergeometric functions are defined as

$$ {}_r\phi_s\left(\begin{matrix} a_1, \ldots, a_r \\ b_1, \ldots, b_s \end{matrix} ; \begin{matrix} x \\ q^k \end{matrix}\right) = \sum_{n=0}^{\infty} \frac{(a_1, \ldots, a_r; q)_n}{(b_1, \ldots, b_s, q; q)_n} q^{kn(n-1)/2} x^n, \quad \ldots (1) $$

where $(a; q)_n$ is the $q$-shifted factorial and

$$ (a_1, \ldots, a_r; q)_n = (a_1; q)_n \ldots (a_r; q)_n = \prod_{i=1}^{r} (a_i; q)_n. \quad \ldots (2) $$

The $q$-analogues of the exponential function are

$$ e_q(x) = \sum_{n=0}^{\infty} \frac{x^n}{(q; q)_n} = \frac{1}{(x; q)_\infty}, \quad |x| < 1 \quad \ldots (3) $$

and

$$ E_q(x) = \sum_{n=0}^{\infty} (-1)^n q^{n(n-1)/2} \frac{x^n}{(q; q)_n} = (x; q)_\infty. \quad \ldots (4) $$

$E_q(x)$ converges for all $x$.

The $q$-derivative operator is defined as

$$ \Delta_z f(z) = \frac{(1 - T_z) f(z)}{(1 - q)z}, \quad \ldots (5) $$

where $T_z$ is the $q$-dilation operator defined by $T_z f(z) = f(qz)$. Indeed, as $q \to 1$, the $q$-derivative operator becomes ordinary differential operator, provided $f$ is differentiable at $z$.

Also we need the $q$-analogue of Kampé de Fériet function[3] $F_{D; E; F}^{A: B; C}$ as

$$ \phi_{D: E; F}^{A: B; C}\left(\begin{matrix} (a_i) : (b_j) ; (c_k) \\ (d_l) : (e_m) ; (f_n) \end{matrix} ; \begin{matrix} x, y \\ q^k, q^l \end{matrix}\right) $$

$$ = \sum_{p, r=0}^{\infty} \frac{((a_i); q)_{p+r} ((b_j); q)_p ((c_k); q)_r}{((d_l); q)_{p+r} ((e_m), q; q)_p ((f_n), q; q)_r} q^{kp(p-1/2} x^p q^{lr(r-1)/2} y^r, \quad \ldots (6) $$

where $(a_i)$ abbreviates parameters $a_1, \ldots, a_A$ and $((a_i); q)_{p+r}$ denotes $(a_1, \ldots, a_A; q)_{p+r}$ etc.

## 3. MODELS OF $U_q(sl(2))$

The Lie algebra $sl(2)$ consists of all $2 \times 2$ complex matrices with trace zero. It has a basis

$$\mathcal{J}^+ = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}, \mathcal{J}^- = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \mathcal{J}^0 = \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}, \quad \ldots (7)$$

satisfying the commutation relations $[\mathcal{J}^0, \mathcal{J}^\pm] = \pm \mathcal{J}^\pm, [\mathcal{J}^+, \mathcal{J}^-] = 2\mathcal{J}^0$.

Let $V_q$ be a complex vector space consisting of $q$-special functions with a basis $\{\phi_\lambda \mid \lambda \in S\}$ such that the functions $\{f_\lambda = \lim_{q \to 1} \phi_\lambda \mid \lambda \in S\}$ form a basis for the vector space, say $V$. Let $A(V_q)$ be the associative algebra of all linear operators on $V_q$ over a complex field. Then a representation $\rho_q$ of $sl(2)$ on $V_q$ is defined as a mapping $\rho_q : sl(2) \to A(V_q)$ satisfying :

(i) $\rho_q(ax + by) = a\rho_q(x) + b\rho_q(y)$

(ii) there exists a Lie algebra representation $\rho$ of $sl(2)$ on $V$ such that

$$\lim_{q \to 1} \rho_q(x)\, \phi_\lambda = \rho(x) f_\lambda, \text{ for all } x, y \in sl(2) \text{ and } a, b \in C.$$

This representation $\rho_q$ of $sl(2)$ is said to be irreducible if there is no proper subspace $W_q$ of $V_q$ which is invariant under $\rho_q$. Define

$$J_q^+ = \rho_q(\mathcal{J}^+), J_q^- = \rho_q(\mathcal{J}^-), J_q^0 = \rho_q(\mathcal{J}^0), \quad \ldots (8)$$

where $J_q^+, J_q^-, J_q^0 \in A(V_q)$. On his work on irreps of $sl(2)$, Manocha[4] has defined the following commutator rules satisfied by $J_q$-operators

$$\begin{aligned} &J_q^0 J_q^+ - q J_q^+ J_q^0 = J_q^+, \\ &q J_q^0 J_q^- - J_q^- J_q^0 = -J_q^-, \\ &q J_q^+ J_q^- - J_q^- J_q^+ = 2J_q^0 - (1-q) J_q^0 J_q^0. \end{aligned} \quad \ldots (9)$$

These commutator rules were later generalized in Sahai[5] for the 4-dimensional complex Lie algebra $\mathcal{G}(a, b), a, b \in C$. Note that $\mathcal{G}(a, b) \tilde{=} \mathcal{G}(1, 0) \tilde{=} sl(2) \oplus (\varepsilon)$ if $a \neq 0$, where $(\varepsilon)$ is the one dimensional Lie algebra generated by $\varepsilon$.

Further, the Casimir operator $C_q$ on $V_q$ is defined by $C_q = qJ_q^+ J_q^- + q^{2u} J_q^0 J_q^0 - q^{2u} J_q^0$, satisfying

$$qJ_q^+ C_q = C_q J_q^+, J_q^- C_q = qC_q J_q^-, J_q^0 C_q = C_q J_q^0. \quad \ldots (10)$$

Indeed, as $q \to 1$, $\rho_q$ reduces to a Lie algebra representation $\rho$ of $s(2)$. Following the analysis as in Manocha and Miller[6], we have the following theorem:

***Theorem*** 3.1 — *Every representation $\rho_q$ of $sl(2)$ is isomorphic to a representation in the following list:*

*(i) The representation $D_q(\alpha, u), \alpha \in C \setminus \{0\}, 0 \leq Re\, \alpha < 1$, such that $\alpha - 2u$ is not an integer.*

$$S = \{\alpha + n \mid n = 0, \pm 1, \pm 2, \ldots\}.$$

*(ii)* *The representation* $\uparrow_q(u), u \in C$ *and* $2u$ *is not a non-negative integer.* $S = \{0, 1, 2, \ldots\}$.

*(iii)* *The representation* $D_q(2u)$ *where* $2u$ *is a non-negative integer.* $S = \{0, 1, 2, \ldots, 2u\}$.

*For each of these representations there is a basis of* $V_q$ *consisting of vectors* $\{f_\lambda\}$, *defined for each* $\lambda \in S$, *such that*

$$J_q^0 f_\lambda = \frac{1-q^{\lambda-u}}{1-q} f_\lambda, J_q^+ f_\lambda = \frac{q^{2u}-q^\lambda}{1-q} f_{\lambda+1}, J_q^- f_\lambda = -\frac{1-q^\lambda}{1-q} f_{\lambda-1}. \quad \ldots (11)$$

Guided by the above theorem, we give below two-variable models of representation $D_q(\alpha)$ in the particular case when $u = 0$.

***Model I***

$$J_q^+ = t(zT_t \Delta_z + t\Delta_t),$$

$$J_q^- = t^{-1} T_z^{-1} [z(1 - zT_t)\Delta_z - t\Delta_t],$$

$$J_q^0 = t\Delta_t \quad \ldots (12)$$

and

$$f_\lambda(z, t) = \frac{(q^\lambda z; q)_\infty}{(z; q)_\infty} t^\lambda = {}_1\phi_0 \left( \begin{matrix} q^\lambda \\ - \end{matrix} ; z \right) t^\lambda, \lambda \in S. \quad \ldots (13)$$

***Model II***

$$J_q^+ = t[z(1 - z) T_t \Delta_z + t\Delta t]$$

$$J_q^- = t^{-1} T_z^{-1} (z\Delta_z - t\Delta_t),$$

$$J_q^0 = t\Delta_t \quad \ldots (14)$$

and

$$f_\lambda(z, t) = \frac{(z; q)_\infty}{(q^\lambda z; q)_\infty} t^\lambda = {}_1\phi_0 \left( \begin{matrix} q^{-\lambda} \\ - \end{matrix} ; q^\lambda z \right) t^\lambda, \lambda \in S, \quad \ldots (15)$$

where $S = \{\alpha + n \mid \alpha \in C \setminus \{0\}, 0 \le Re\,\alpha < 1, n = 0, \pm 1, \pm 2 \ldots\}$.

The dilation models given above satisfy (9) as well as (11) with $u = 0$. A model of representation $\uparrow_q$ is same as Model II above with $S = \{0, 1, 2, \ldots\}$.

We remark that the commutation relations given by eqs. (9) are equivalent to the ones introduced by Jimbo[7]. Jimbo has shown that the commutation relations among the generators $e$, $f$, $h$, given by

$$[h, e] = 2e, \quad [h, f] = -2f, \quad [e, f] = \frac{q^{h/2} - q^{-h/2}}{q^{1/2} - q^{-1/2}}, \qquad \ldots (16)$$

define an associative algebra $U_q(sl(2))$, a $q$-deformation of universal enveloping algebra of $sl(2)$. Now the equivalence of (9) and (16) follows from the fact that if the operators $e$, $f$, $h$ satisfy (16) then $J_q$-operators defined by

$$J_q^+ = e, \quad J_q^- = q^{2u + \frac{h-1}{2}} f, \quad J_q^0 = \frac{1 - q^{h/2}}{1 - q} \qquad \ldots (17)$$

satisfy (9). Indeed, as the parameter $q \to 1^-$, the relations (16) reduce to familiar commutator rules for $sl(2)$.

## 4. TRANSFORMED MODELS

In this section, we introduce a $q$-integral transformation, based on $q$-analogue of Euler integral transformation[8] and then compute transforms of certain operator expressions. We shall make use of these operator transformations to obtain new models from the ones given in Section 3 in which the basis functions will be in terms of ${}_2\phi_1$, the basic hypergeometric function.

Define

$$h(\beta, \gamma, x) = I[f(ux)] = \Gamma_q \begin{pmatrix} \gamma \\ \beta, \gamma - \beta \end{pmatrix} \int_0^1 u^{\beta - 1} \frac{(qu; q)_\infty}{(q^{\gamma - \beta} u; q)_\infty} f(ux)\, d_q u, \qquad \ldots (18)$$

$$Re\, \gamma > Re\, \beta > 0.$$

Defining the difference operators $E_\beta(h(\beta, \gamma)) = h(\beta + 1, \gamma)$, $L_\beta(h(\beta, \gamma)) = h(\beta - 1, \gamma)$ and $E_{\beta\gamma} = E_\beta E_\gamma$ etc., we have the following transforms of operator expressions.

$$I[u \Delta_u f] = x \Delta_x h,$$

$$I[u^2 \Delta_u f] = \frac{1 - q^\gamma}{1 - q^\gamma} x \Delta_x E_{\beta\gamma} h,$$

$$I[\Delta_u f] = \frac{1 - q^{\gamma - 1}}{1 - q^{\beta - 1}} x \Delta_x L_{\beta\gamma} h,$$

$$I[uf] = \frac{1 - q^\beta}{1 - q^\gamma} E_{\beta\gamma} h. \qquad \ldots (19)$$

To obtain models of $U_q(sl(2))$ in terms of difference-dilation operators in which the basis functions are in terms of ${}_2\phi_1$ functions, we need the following theorem.

***Theorem*** 4.1 — *Let $\rho_q$ be a representation of $sl(2)$ in terms of $\{J_q^+, J_q^-, J_q^0\}$ with basis functions $\{f_\lambda \mid \lambda \in S\}$. Then $\rho_q$ is also a representation of $sl(2)$ in terms of $\{K_q^+, K_q^-, K_q^0\}$ with basis functions $\{h_\lambda \mid \lambda \in S\}$ w'h*

$$K_q^+ = I J_q^+ I^{-1}, K_q^- = I J_q^- I^{-1}, K_q^0 = I J_q^0 I^{-1}, h_\lambda = I f_\lambda. \quad \ldots (20)$$

To transform the two-variable dilation models to difference-dilation ones, we put $z = ux$ in eqs. (12) and (13) (that is, Model I). This substitution gives $z\Delta_z = u\Delta_u$ and $T_z f(z) = f(qz) = T_u f(ux)$. Using Theorem 4.1, Model I takes the shape

***Model IA***

$$K_q^+ = t\,[x\,\Delta_x\,T_t + t\Delta_t],$$

$$K_q^- = t^{-1}\,T_x^{-1}\left[x\,\Delta_x - xT_t\left(\frac{1-q^\beta}{1-q^\gamma}\,x\Delta_x\,E_{\beta\gamma}\right) - t\,\Delta t\right],$$

$$K_q^0 = t\,\Delta_t \quad \ldots (21)$$

and

$$h_\lambda(x, t) = {}_2\phi_1\left(\begin{matrix} q^\lambda, b \\ c \end{matrix}; x.\right) t^\lambda, \quad \lambda \in S, \quad \ldots (22)$$

where $b = q^\beta$ and $c = q^\gamma$.

Likewise, Model II is transformed to the following

***Model IIA***

$$K_q^+ = t\left[x\,\Delta_x\,T_t - xT_t\left(\frac{1-q^\beta}{1-q^\gamma}\,x\Delta_x\,E_{\beta\gamma}\right) + t\,\Delta t\right],$$

$$K_q^- = t^{-1}\,T_x^{-1}\,(x\,\Delta_x - t\,\Delta_t),$$

$$K_q^0 = t\,\Delta_t \quad \ldots (23)$$

and

$$h_\lambda(x, t) = {}_2\phi_1\left(\begin{matrix} q^{-\lambda}, b \\ c \end{matrix}; q^\lambda x\right) t^\lambda, \quad \lambda \in S. \quad \ldots (24)$$

The $K$-operators given above satisfy the following relations

$$K_q^0\,h_\lambda = \frac{1-q^\lambda}{1-q}\,h_\lambda,\ K_q^+\,h_\lambda = \frac{1-q^\lambda}{1-q}\,h_{\lambda+1},\ K_q^-\,h_\lambda = -\frac{1-q^\lambda}{1-q}\,h_{\lambda-1},$$

$$C_q h_\lambda = 0 \text{ where } C_q = qK_q^+\,K_q^- + K_q^0\,K_q^0 - K_q^0 \quad \ldots (25)$$

as well as the commutation relations

$$K_q^0\,K_q^+ - qK_q^+\,K_q^0 = K_q^+,$$

$$qK_q^0\,K_q^- - K_q^-\,K_q^0 = -K_q^-$$

$$qK_q^+ K_q^- - K_q^- K_q^+ = 2K_q^0 - (1-q) K_q^0 K_q^0 \qquad \ldots (26)$$

and as such give difference-dilation models of $U_q(sl(2))$. Further

$$qK_q^+ C_q = C_q K_q^+, K_q^- C_q = qC_q K_q^-, K_q^0 C_q = C_q K_q^0. \qquad \ldots (27)$$

Eqs. (27) will become instrumental in obtaining identities.

## 5. IDENTITIES

### 5.1. *Based on Model IA*

We have shown in eq. (25) that

$$h_\lambda(x, t) = {}_2\phi_1\left(\begin{matrix} q^\lambda, b \\ c \end{matrix}; x\right) t^\lambda, \quad \lambda \in S, \qquad \ldots (28)$$

is a solution of $C_q h_\lambda(x, t) = 0$ where $C_q = qK_q^+ K_q^- + K_q^0 K_q^0 - K_q^0$. This suggests that

$$u(x, t) = \sum_{n=0}^{\infty} \frac{(a, b_1, \ldots, b_k; q)_n}{(c_1, \ldots, c_k, q; q)_n} {}_2\phi_1\left(\begin{matrix} aq^n, b \\ c \end{matrix}; x\right) t^{\alpha+n} \qquad \ldots (29)$$

is a solution of $C_q u(x, t) = 0$. Using the fact that $qK_q^+ C_q = C_q K_q^+$, we have $C_q[e_q(sK_q^+)u](x, t) = 0$, where

$$[e_q(sK_q^+)u](x, t) = \frac{\left(\frac{ast}{1-q}; q\right)_\infty}{\left(\frac{st}{1-q}; q\right)_\infty} \phi_{1:1:k}^{1:1:k}\left(\begin{matrix} a:b; b_1, \ldots. b_k \\ \frac{ast}{1-q} : c; c_1, \ldots., c_k \end{matrix}; x, t\right) t^\alpha \qquad \ldots (31)$$

where $A_n$ is found by putting $x = 0$. Rescaling suitably, we eventually arrive at

$$\frac{(at; q)_\infty}{(t; q)_\infty} \phi_{1:1:k}^{1:1:k}\left(\begin{matrix} a:b; b_1, \ldots. b_k \\ at : c; c_1, \ldots., c_k \end{matrix}; x, -\omega t\right)$$

$$= \sum_{n=0}^{\infty} \frac{(a; q)_n}{(q; q)_n} {}_2\phi_2\left(\begin{matrix} aq^n, b \\ c \end{matrix}; x\right) {}_{k+1}\phi_k\left(\begin{matrix} q^{-n}, b_1, \ldots, b_k & q_n \omega \\ c_1, \ldots, c_k & q^{-1} \end{matrix}; t^n\right) \qquad \ldots (32)$$

where $\omega = \frac{-s}{1-q}$ and $\phi_{1:1:k}^{1:1:k}$ is a particular case of $q$-Kampé de Fériet function given in (6).

*5.2 Based on Model IIA*

We follow exactly the same approach as given in section 5.1 with the change that we use the other $q$-exponential $E_q$ instead of $e_q$.

We have shown that $C_q\, h_\lambda = 0$, where $h_\lambda$ is given by eq. (24). It therefore follows that

$$u(x,t) = \sum_{n=0}^{\infty} \frac{(a, b_1, \ldots, b_k; q)_n}{(q, c_1, \ldots, c_k; q)_n}\, {}_2\phi_1\left( \begin{matrix} a^{-1}q^{-n}, b \\ c \end{matrix} ; aq^n x \right) t^{\alpha+n} \qquad \ldots (33)$$

is a solution of $C_q\, u(x,t) = 0$. Considering that $K_q^- C_q = {}_qC_q\, K_q^-$, we have $C_q\, [E_q(sK_q^-)u]\,(x,t) = 0$.

By Weisner's expansion, we have

$$[E_q\,(sK_q^-)u]\,(x,t) = \sum_{n=-\infty}^{\infty} B_n\, h_{\alpha+n}\,(x,t), \qquad \ldots (34)$$

where $B_n$ is obtained by putting $x = 0$. This leads to

$$\frac{\left(\frac{\omega}{t}; q\right)_\infty}{\left(\frac{a\omega}{t}; q\right)_\infty} \sum_{n=0}^{\infty} \frac{\left(a, b_1, \ldots, b_k, \frac{a\omega}{t}; q\right)_n}{(c_1, \ldots, c_k, q; q)_n}\, {}_2\phi_2 \left( \begin{matrix} a^{-1}q^{-n}, b \\ \frac{q^{1-n}t}{a\omega}, c \end{matrix} ; \begin{matrix} \frac{-qxt}{\omega} \\ q \end{matrix} \right) t^n$$

$$= \sum_{n=-\infty}^{\infty} \frac{\Gamma_q(\alpha+n) \prod_{i=1}^{k} \Gamma_q(\beta_i+n) \prod_{i=1}^{k} \Gamma_q(\gamma_i)}{\Gamma_q(\alpha) \prod_{i=1}^{k} \Gamma_q(\beta_i) \prod_{i=1}^{k} \Gamma_q(\gamma_i+n)\, \Gamma_q(q+1)}$$

$$\times {}_{k+2}\phi_{k+1} \left( \begin{matrix} aq^n, aq^{n+1}, b_1q^n, \ldots. b_kq^n \\ q^{n+1}, c_1^n, \ldots., c_kq^n \end{matrix} ; \begin{matrix} -\omega \\ q \end{matrix} \right) {}_2\phi_1 \left( \begin{matrix} a^{-1}, q^{-n}, b \\ c \end{matrix} ; aq^n x \right) t^n$$

$$\ldots (35)$$

where $\omega = \frac{-s}{1-q}$, $b_i = q^{\beta_i}$ and $c_i = q^{\gamma_i}$ for $i = 1, \ldots, k$.

We have discussed above identities corresponding to models of representation $D_q(\alpha)$. We now consider the Model IIA for the representation $\uparrow_q$ (that is, eqs. (23) and (24) with $\lambda \in S = \{n = 0, 1, 2, \ldots\}$). It can be verified that

$$u(x,t)=\sum_{n=0}^{\infty}\frac{(a,b_1,\ldots,b_k;q)_n}{(q,c_1,\ldots,c_k;q)_n}\,{}_2\phi_1\left(\begin{matrix}q^{-n},b\\ c\end{matrix};q^n x\right)t^n \qquad \ldots (36)$$

satisfies $C_q u(x,t)=0$. Using the relation $\frac{1}{q}K_q^- C_q = C_q K_q^-$, we have $C_q\,[E_q(sK_q^-)u]\,(x,t)=0$.

Using Weisner's expansion and proceeding as in the above two cases, we get the following identity:

$$\phi^{k+1:1;1}_{l:1;}\left(\begin{matrix}a,b_1,\ldots.,b_k:b;\frac{\omega}{t};&-xt,t\\ c_1,\ldots.,c_k:c;-;&q,1\end{matrix}\right)$$

$$=\sum_{n=0}^{\infty}\frac{(a,b_1,\ldots..,b_k;q)_n}{(q,c_1,\ldots..,c_k;q)_n}\,{}_2\phi_1\left(\begin{matrix}q^{-n},b\\ c\end{matrix};q^n x\right){}_{k+1}\phi_k\left(\begin{matrix}aq^n,b_1q^n,\ldots,b_kq^n&-\omega\\ c_1q^n,\ldots..,c_kq^n&q\end{matrix};t^n\right)$$

$$\ldots (37)$$

where $\omega=\dfrac{-s}{1-q}$.

The particular case of (37) corresponding to $k = 1$ corrects as identity appearing in Manocha[4].

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