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GLOBAL ATTRACTIVITY IN IMPULSIVE FUNCTIONAL DIFFERENTIAL EQUATIONS*

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In this paper, the asymptotic behaviour of solutions of the following impulsive delay differential equation

$$\begin{cases} \dot{x}(t) = f(t, x(t-\tau_1(t)), x(t-\tau_2(t)), \ldots, x(t-\tau_m(t))), t \geq t_0\ t \neq t_k, t-\tau_i(t) \neq t_k \\ x(t_k^+) - x(t_k) = I_k(x(t_k) - x^*) \quad i = 1, 2, \ldots, m, \qquad k=1,2,\ldots \end{cases}$$

is discussed. The conditions that guarantee the attractivity of its equilibrium x^* are given. Our results apply to a large variety of nonlinear delay differential equations with impulses.

Key Words : Delay Differential Equation; Impulse.

1. INTRODUCTION

Recently, the attractivity in impulsive delay differential equations has been studied (see the references [1-10]). In [4] and [10], the attractivity of the zero solution of impulsive linear delay differential equations is discussed. In [9] and [2], sufficient conditions for the attractivity of the trivial solution of impulsive nonlinear differential equations with one constant delay are given. The objective of this paper is to investigate the attractivity of the following nonlinear delay differential equation with impulses.

$$\left.\begin{cases} \dot{x}(t) = f(t, x(t-\tau_1(t)), x(t-\tau_2(t)), \ldots, x(t-\tau_m(t))), t \geq t_0\ t \neq t_k, t-\tau_i(t) \neq t_k \\ x(t_k^+) - x(t_k) = I_k(x(t_k) - x^*) \quad i = 1, 2, \ldots, m, \qquad k=1,2,\ldots \end{cases}\right\} \quad \ldots (1)$$

Throughout this paper, we always assume that

(H1) $f \in C[[t_0, \infty) \times L, \infty)^m, R]$ for some $L \geq -\infty$ and for every $t \geq t_0$, $f(t, u_1, u_2, \ldots, u_m)$ is nonincreasing in each of its arguments $u_1, u_2, \ldots, u_m$.

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(H2) there exists an $x^* > L$ such that $f(t, x^*, \ldots, x^*) \equiv 0$ for all $t \geq t_0$.

(H3) $\tau_i \in C[[t_0, \infty), [0, \infty)]$ and $t - \tau_i(t)$ is increasing and $\lim_{t \to \infty} [t - \tau_i(t)] = \infty$ for $i = 1, 2, \ldots, m$.

(H4) $t_0 < t_1 < \ldots < t_k < \ldots$, $\lim_{k \to \infty} t_k = \infty$. $I_k : (L - x_*, \infty) \to R$ is continuous and $I_k(0) = 0$.

It is clear that x^* is an equilibrium of eq. (1). We will give sufficient conditions under which all the solutions of (1) tend to x^*. Our results generalize the results in [8]. We will use the results to deal with the following impulsive delay differential equations.

$$\left.\begin{aligned} \dot{x}(t) &= r(t)\left[1 - \frac{cx(t)x^p(t - \tau(t))}{1 + x^p(t - \tau(t))}\right] \quad t > 0,\ t \neq t_k,\ t - \tau(t) \neq t_k \\ x(t_k^+) - x(t_k) &= I_k(x(t_k) - K), \qquad k = 1, 2, \ldots \end{aligned}\right\} \quad \ldots (2)$$

where $c, p \in (0, \infty)$ $r, \tau \in C[[0, \infty), [0, \infty)]$, $\lim_{t \to \infty} [t - \tau(t)] = \infty$, for some $\tau' > 0$, $\inf_{t \geq 0} \tau(t) \geq \tau'$, and K is the equilibrium of eq. (2), and

$$\left.\begin{aligned} \dot{N}(t) = r(t)\,\dot{N}(t) &= r(t)\,N(t)\left[\frac{N^* - \sum_{i=1}^{m} a_i N^p(t - \tau_i(t))}{N^* + \sum_{i=1}^{m} s_i(t)\,N^p(t - \tau_i(t))}\right], \quad t > 0,\ t \neq t_k,\ t - \tau_i(t) \neq t_k \\ x(t_k^+) - x(t_k) &= I_k(x(t_k) - N^*)\ i = 1, 2, \ldots, m,\ k = 1, 2, \ldots, \end{aligned}\right\} \quad \ldots (3)$$

where $K, p \in (0, \infty)$, $a_i \in [0, \infty)$, $N^* = \left(K / \sum_{i=1}^{m} a_i\right)^{\frac{1}{p}}$, $\tau_i \in C[[0, \infty), (0, \infty)]$, $s_i \in C[[0, \infty), [0, \infty)]$, $\lim_{t \to \infty} [t - \tau_i(t)] = \infty$, for $i = 1, 2, \ldots, m$. When $I_k \equiv 0$, the attractively of eq. (1) has been studied in [8], and as applications sufficient conditions for the asymptotic property of eqs. (2) and (3) with $I_k \equiv 0, k = 1, 2, \ldots$, have been given in [8]. In this paper we investigate the attractivity of eq. (1), and our results will be used to discuss eq. (2), (3), and the following impulsive delay differential equation.

$$\left.\begin{aligned} \dot{x}(t) &= \frac{b}{1 + \sum_{i=1}^{m} a_i x^n (t - \tau_i)} - cx(t), \; t > 0, \; t, \; t - \tau_i \neq t_k \\ x(t_k^+) - x(t_k) &= I_k(x(t_k) - x^*), \; i = 1, 2, \ldots, m, \; k = 1, 2, \ldots, \end{aligned}\right] \quad \ldots (4)$$

where b, c and $a_i \in (0, \infty)$ $i = 1, 2, \ldots, m$, and x^* is the only equilibrium of eq. (4). When $I_k \equiv 0$, $m = 1$, and $a_1 = 1$, eq. (4) is reduced to the following equation

$$\dot{x}(t) = \frac{b}{1 + [x(t - \tau)]} - cx(t), \; t \geq 0. \quad \ldots (5)$$

Eq. (5) has been studied as a model of hematopoisis in [5] and [7]. Our results, when restricted on (5), improve the results in [5].

Definition 1 — A solution of eq. (1) is called oscillatory about x^* if $x(t) - x^*$ is neither eventually positive nor eventually negative. Otherwise, it is called nonoscillatory about x^*.

2. Main Results

In this section, we will state our main results and some applications.

Theorem 1 — *If*

$$\left| \int_{t_0}^{\infty} f(s, u, u, \ldots, u)\, ds \right| = \infty \text{ for all } u \in (L, \infty), \; u \neq x^* \quad \ldots (6)$$

and there exist $\{b_k\}_{k=1,2,\ldots}$ *such that* $|I_k(u)| \leq b_k |u|$ *for* $u \in (L - x^*, \infty)$ *and* $k = 1, 2, \ldots,$ *and* $\sum_{k=1}^{\infty} b_k < \infty$, *then every solution of eq. (1) that is nonoscillatory about* x^* *tends to* x^* *as* t *tends to* ∞.

Proof : Let $x(t)$ be a solution of (1) that is nonoscillatory about x^*, and without loss of generality we assume that $x(t) > x^*$ for $t \geq t_0$. From (H3), there exists $T' > t_0$, and a positive integer N such that $t - \tau(t) > t_0$ for $t > T'$ and $t > T'$ for $k \geq N$. Then

$$\begin{aligned} \dot{x}(t) &= f(t, x(t - \tau_1(t)), \ldots, x(t - \tau_2(t))) \\ &\leq f(t, x^*, \ldots, x^*) \\ &= 0, \text{ for } t > t_N, \; t \neq t_k, \; t - \tau_i(t) \neq t_k, \; i = 1, \ldots, m, \; k = 1, 2, \ldots . \end{aligned}$$

So $x(t)$ is nonincreasing in (t_k, t_{k+1}) for $k \geq N$. We claim that $\{x(t_k)\}_{k = 1,2, \ldots}$ is bounded. Indeed, for $k > N$,

$$x(t_k) - x^* \leq x(t_{k-1}^+) - x^* \leq x(t_{k-1}) - x^* + I_k(x(t_{k-1}) - x^*)$$

$$\leq (1 + b_{k-1})\,(x(t_{k-2}) - x^*)$$

$$\leq (1 + b_{k-1})\,(x(t_{k-2}) - x^* + I_{k-2}\,(x(t_{k-2}) - x^*)$$

$$\leq (1 + b_{k-1})\,(1 + b_{k-2})\,(x(t_{k-2}) - x^*)$$

...

$$\leq \left[\prod_{i=N}^{k-1} (1 + b_i)\right] (x(t_N) - x^*),$$

which, combining with $\prod_{i=1}^{\infty} (1 + b_k) < \infty$, leads to the boundedness of $\{x(t_k)\}_{k = 1, 2, \ldots}$.

Set

$$y(t) = x(t) - \sum_{t_0 \leq t_k < t} I_k(x(t_k) - x^*). \qquad \ldots (7)$$

We will show that $y(t)$ is nonincreasing on $[t_N, +\infty)$. For any $e_1, e_2 \in [t_N, \infty)$, $e_1 < e_2$, $e_1 \in (t_k, t_{k+1}]$, $e_2 \in (t_j, t_{j+1}]$, $j \geq k \geq N$,

$$x(e_2) - x(e_1) \leq x(t_j^+) - x(e_1) \leq x(t_j) + I_j\,((x(t_j) - x^*) - x(e_1)$$

...

$$\leq x(t_{k+1}) + \sum_{e_1 \leq t_k < e_2} I_k\,(x(t_k) - x^*) - x(e_1)$$

$$\leq \sum_{e_1 \leq t_k < e_2} b_k\,(x(t_k) - x^*),$$

and then $y(e_2) - y(e_1) \leq x(e_1) - \sum_{e_1 \leq t_k < e_2} b_k(x(t_k) - x^*) \leq 0$, so $y(t)$ is nonincreasing on $[t_N, \infty)$. Noticing the boundedness of $\{x(t_k)\}$ and (7), we have $\lim_{t \to \infty} y(t) = L$ for some $L \in R$, and then $\lim_{t \to \infty} x(t) = l \geq x^*$ exists. If $l > x^*$, then for sufficiently large t, $x(t - \tau_i(t)) > \dfrac{l + x^*}{2} = l_1 > x^*$, $i = 1, 2, \ldots, m$, and

$$l - x(t) = \int_t^{\infty} f(s, x(s - \tau_1)\,(s)), \ldots, x(s - \tau_m\,(t))) + \sum_{t_k > t} I_k(x(t_k) - x_*)$$

$$\leq \int_{t}^{\infty} f(s, l_1, \ldots, l_1)ds + \sum_{t_k > t} b_k(x(t_k) - x_*).$$

This, in view of (6), is a contradiction, so $l = x^*$. The proof is complete.

Theorem 2 — *Assume that there exists nonincreasing function* $F \in C[(L, \infty), R]$ *satisfying the following conditions :*

$$(h1) \quad \left| \int_{t_0}^{\infty} f(s, u, \ldots, u)\, ds \right| = \infty \text{ for } u \in (L, \infty), u \neq x^*;$$

$$(h2) \quad F(L+) \in R \text{ and } x^* + f(x^* + F(L+)) > L;$$

$$(h3) \quad \int_{t-\tau(t)}^{t} f(s, u, \ldots, u)ds \leq F(u) \text{ for } L < u < x^*$$

and

$$\int_{t-\tau(t)}^{t} f(s, u, \ldots, u)ds \geq F(u) \text{ for } x^* < u < \infty$$

where $\tau(t) = \max_{1 \leq i \leq m} \{\tau_i(t)\}$;

$$(h4) \quad \text{There exist } b_k, k = 1, 2, \ldots \text{ such that } |I_k(u)| \leq b_k |u| \text{ for } u \in L - x^*, \infty) \text{ and } \sum_{k=1}^{\infty} b_k < \infty;$$

(h5) The solution of the IVP

$$y_{n+1} = x^* + F(y_n)$$

$$y_0 = x^* + F(L+)$$

tends to x^* *as* n *tends to* ∞. *Then every solution of eq. (1) tends to* x^* *as* t *tends to* ∞.

PROOF : From Theorem 1, Theorem 2 is true for the nonoscillatory solutions, so we only need to complete the proof for the oscillatory solutions. Let $x(t)$ be a solution of eq. (1) that is oscillatory about x^*. Noticing that for sufficiently large k, $(x(t_k^+) - x^*)(x(t_k) - x^*) \geq (1 - b_k)(x(t_k) - x^*)^2 \geq 0$, we know that $x(t)$ must have arbitrary large zeros, and between every two of the adjacent zeros $x(t) - x^*$ remains nonnegative or nonpositive. So we can choose $\{c_n\}_{n = 1, 2, \ldots}$ satisfying $t_0 < c_1 < c_2 < \ldots < c_n, \ldots,$ and $\lim_{n \to \infty} c_n = \infty$, such that for $t \in [c_n, c_{n+1}]$ and $s \in [c_{n+1}, c_{n+2}]$, $(x(t) - x^*)(x(s) - x^*) \leq 0$ and there exists a $t_n' \in (c_n, c_{n+1})$ such that $x(t_n') \neq x^*, n = 1, 2, \ldots$.

Set

$$Q^+ = \{[c_n, c_{n+1}] \mid x(t) \geq x^*\} \quad Q^- = \{[c_n, c_{n+1}] \mid x(t) \leq x^*\},$$

$$x^+(t) = \max\{x(t), x(t^+)\} \quad x^-(t) = \min\{x(t), x(t^+)\}$$

and for $a \leq b \leq \infty$

$$S(a, b) = \sum_{a \leq t_k < b} b_k \mid x(t_k) - x^* \mid, \quad S_0(a, b) = \sum_{a \leq t_k < b} b_k$$

where $a = \sum_{i=1}^{m} a_i$;

and (h3) holds for $L = 0$.

Then every solution of eq. (4) tends to x^* as t tends to ∞.

Set

$$f(u_1, u_2, \ldots, u_{m+1}) = \frac{b}{1 + \sum_{i=1}^{m} a_i u_i^n} - c u_{m+1}$$

and $$F(u) = \tau\left[\frac{b}{1 + au^n} - cu\right], \quad h(u) = x^* + F(u).$$

It is clear that (h1)-(h4) are satisfied, so it remains to show that (h5) is satisfied. From the Lemma 1, to this end we only need to show that

$$h'(u)\, h'(h(u)) < 1. \qquad \ldots (14)$$

Observe that

$$h'(u) = \tau\left[-abn \frac{u^{n+1}}{(1 + au^n)^2} - c\right],$$

and set

$$g(s) = abn \frac{s^{n-1}}{(1 + as^n)^2} + c,$$

then we have

$$h'(u)h'(h(u)) = \tau^2\, g(u)\, g(h(u)),$$

and

$$g'(s) = abn \frac{[(n-1)\ (n+1)\ as^n]\, s^{n-2}}{(1 + as^n)^3}.$$

In the case (i), $0 < n \le 1$, so $g'(s) < 0$ for $s > 0$. Observe that when $0 < u < x^*$, $h(u) > h(x^*) = x^*$, $g(u) < g(0) = c$,

$$g(h(u)) < g(x^*) = adn \frac{x^{*(n-1)}}{(1+ax^{*n})^2} + c.$$

We have

$$h'(u)\, h'(h(u)) < \tau^2 c\left(abn \frac{x^{*(n-1)}}{(1+ax^{*n})} + c \right) \le 1.$$

Similarly, when $x^* < u < \infty$, we also have

$$h'(u)\, h'(h(u)) < 1$$

Now consider the case (ii). Let

$$s_0 = a^{-\frac{1}{n}} \left(\frac{n-1}{n+1} \right)^{\frac{1}{n}},$$

then $g'(s_0) = 0$. It is easy to see that $g(s)$ is strictly increasing on $(0, s_0)$, and strictly decreasing on (s_0, ∞). Observe that $0, u < x^*$ implies $h(u) > x_*$ and that $x^* < u < \infty$ implies $h(u) < x^*$. We have

$$g(u)\, g(h(u)) < g(s_0)\, g(x^*)$$

$$= \left(\frac{1}{4n} a^{\frac{1}{n}} b(n+1)^{1+\frac{1}{n}} (n-1)^{1-\frac{1}{n}} + c \right) (abnx^{*(n-1)} (1+ax^{*n})^{-2} + c) \le 1.$$

So the proof of Example 1 is completed.

Remark 1 : When $m = 1$, $a = 1$ and $I_k \equiv 0$, eq. (4) reduces to eq. (5). From the Corollary 4.8.1 and 4.8.2 [5], each of the following three conditions

(*i*) $n \le 1$ and $b < 2cx^*$;

(*ii*) $n > 1$, $\frac{1}{4n} b(n+1)^{1+\frac{1}{n}} (n-1)^{1-\frac{1}{n}} < c$; and

(*iii*) $n > 1$, $\tau\left[\frac{1}{4n} b(n+1)^{1+\frac{1}{n}} (n-1)^{1-\frac{1}{n}} + c \right] < 1$

will guarantee the asymptotic constancy of the positive solution of eq. (5). Noticing that $bnx^{*(n-1)} [1+x^{*n}]^{-2} + c = g(x^*) \le g(s_0) = \frac{b}{4n} (n-1)^{1-\frac{1}{n}} (n+1)^{1-\frac{1}{n}}$, we see that our results improve the results in [5].

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