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INTERPOLATORY QUADRATURE RULE FOR EVALUATION OF CERTAIN HYPERSINGULAR INTEGRALS

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In this paper, we introduce a quadrature rule for the numerical evaluation of certain hypersingular integrals. The rule is obtained by using Hermite interpolation polynomial, and error bound is made.

Key Words : Interpolatory Quadrature; Hypersingular Integrals; Hermite Interpolation Polynomial

1. INTRODUCTION

We consider Hadamard finite-part (f.p) integrals of the form

$$I(f; s) = \oint_{-1}^{1} \frac{f(x)}{(x-s)^2}\, dx,\ s \in (-1, 1), f(-1) = f(1) = 0. \qquad \ldots (1.1)$$

Several boundary problems are formulated as singular integral equations involving such integrals. In order to ensure the existence of $I(f; s)$ as a f.p. integral it is assumed that f is differentiable in a neighbourhood of s with its derivative satisfying a Lipschitz condition.

The f.p. integral $I(f; s)$ exists in the sense of the Hadamard finite part; that is

$$I(f; s) = \lim_{\varepsilon \to 0} \left[\left(\int_{-1}^{s-\varepsilon} + \int_{s+\varepsilon}^{1} \right) \frac{f(x)}{(x-s)^2}\, dx - 2\frac{f(s)}{\varepsilon} \right]. \qquad \ldots (1.2)$$

Alternatively, we may define $I(f, s)$ to be the Cauchy principal value (C.p.v.) integral:

$$I(f; s) = \oint_{-1}^{1} \frac{f'(x)}{(x-s)}\, dx. \qquad \ldots (1.3)$$

In this paper, we propose to obtain the approximation of the integral without using rules which require the evaluation of the derivatives of f, we consider formulas of interpolatory type.

2. BASIC DEFINITIONS AND PROPERTIES

The concept of finite part integral seems to have been first introduced by Hadamard[1] in 1923. Although in 1951 Mangler[4] used it in problems of aerodynamics of wings.

Definition 2.1 —

$$I(f; x) = \int_{-1}^{1} \frac{\left[f(t) - \sum_{k=0}^{1} \frac{f^{(k)}(x)}{k!} (t-x)^k \right]}{(t-x)^2} dt + \sum_{k=0}^{1} \frac{f^{(k)}(x)}{k!} \int_{-1}^{1} \frac{dt}{(t-x)^{2-k}}, \quad -1 < x < 1. \qquad \ldots (2.1)$$

Property 2.1 —

$$I(f; s) = \frac{d}{ds} \int_{-1}^{1} \frac{f(t)}{t-s} dt, \qquad \ldots (2.2)$$

Evaluation of Certain Finite-Part Integrals

In our problem the following expressions are quite useful :

$$\oint_{1}^{1} \frac{U_n(t)\sqrt{1-t^2}}{t-x} dt = -\pi T_{n+1}(x) \qquad (n \geq 0). \qquad \ldots (2.3)$$

$$\oint_{1}^{1} \frac{U_n(t)\sqrt{1-t^2}}{(t-x)^2} dt = -\pi(n+1)U_n(x) \qquad (n \geq 0). \qquad \ldots (2.4)$$

$$\oint_{-1}^{1} \frac{T_n(t)}{(t-x)\sqrt{1-t^2}} dt = \begin{cases} 0 & (n=0) \\ \pi U_{n-1}(x) & (n \geq 1). \end{cases} \qquad \ldots (2.5)$$

$$\oint_{-1}^{1} \frac{T_n(t)}{(t-x)^2\sqrt{1-t^2}} dt = \begin{cases} 0 & (n=0, 1) \\ \dfrac{\pi}{1-x^2}\left[-\dfrac{n-1}{2} U_n(x) + \dfrac{n+1}{2} U_{n-2}(x) \right] & (n \geq 2), \end{cases} \qquad \ldots (2.6)$$

where T_n and U_n are Chebyshev polynomials of the first and second kind respectively.

3. QUADRATURE FORMULAS

In 1981, Paget[5], starting from the Gaussian formula

$$S(f) = \int\!\!\!\!\!\!-_a^b w(t)\frac{f(t)}{t-x}\,dt = \sum_{i=1}^{n} h_i \frac{f(t_i)-f(x)}{t_i - x} + q_0(x)f(x), \quad a < x < b, x \neq t_i,$$

where $q_0(x) = \int\!\!\!\!\!\!-_a^b \frac{w(t)}{t-x}\,dt$ and $\{t_i\}$, $\{h_i\}$ denote nodes and weights of the ordinary Gaussian rule associated with the weight function $w(t)$, and using relation (2.2), derives a formula for the corresponding f.p. integral. He also gives a contour integral representation for the remainder term when f is analytic.

Ioakimidis and Theocaris[3] in 1979 and Ioakimidis[2] in 1985 consider the Gaussian formula for the case of (2.1) with a weight function $w(t)$. By applying the ordinary Gaussian rule to the corresponding ordinary integral.

Now we introduce certain interpolatory quadrature rule, which is benefit for our problem.

Theorem 3.1 — *Let $Q_N = (\{w_k\}, \{t_k\}, k = 1, \ldots, N)$ be an N point polynomial interpolatory quadrature rule for $\int_a^b w(t)f(t)dt$. Then $S(f)$ can be approximated by $S_N(f)$, where*

$$S_N(f) = \sum_{k=1}^{N} \frac{w_k f(t_k)}{t_k - x} + \frac{q_N(x)f(x)}{r_N(x)}, \quad t_k \neq x, \quad k = 1, 2, \ldots, N, \qquad \ldots (3.1)$$

and $S_N(f) = S(f_N)$, where $f_N(t)$ is the unique polynomial which interpolates to

$$f(t) \text{ at } \{t_1, t_2, \ldots, t_N, x\}, \; r_N(t) = \prod_{k=1}^{N} (t - t_k), \text{ and } q_N = S(r_N).$$

Moreover, if Q_N is Gaussian rule, then S_N is Gaussian in the sense that it integrates exactly all polynomials of degree $\leq 2N$.

PROOF : We construct $f_N(t)$ as

$$f_N(t) = \sum_{k=1}^{N} l_k(t)f(t_k) + l_{N+1}f(x), \text{ where}$$

$$l_k(t) = r_N(t)\,(t-x)/r'_N(t_k)\,(t-t_k)\,(t_k-x), \; k = 1, 2, \ldots, N, \text{ and}$$

$$l_{N+1}(t) = r_N(t)/r_N(x).$$

Thus,

$$S_N(f) = S(f_N)$$

$$= \sum_{k=1}^{N} \int_{a}^{b} f(t_k)\,[w(t) r_N(t)/r'_N(t_k)\,(t-t_k)\,(t_k-x)]\,dt$$

$$+ q_N(x) f(x)/r_N(x),\ x \neq t_k,\ k = 1, 2, \ldots, N.$$

Now $w_k = \int_a^b w(t)\, r_N(t)/r'_N(t_k)\,(t-t_k)dt$, so that (3.1).

To prove that S_N is Gaussian if Q_N, let $p(t)$ be a polynomial $\leq 2N$, then

$$S(p) - p(x)\, S(1) = \sum_{k=1}^{N} \frac{w_k p(t_k)}{t_k - x} - p(x) \sum_{k=1}^{N} \frac{w_k}{t_k - x}. \qquad \ldots (3.2)$$

But $$\sum_{k=1}^{N} \frac{w_k}{t_k - x} = S(1) - \frac{q_N(x)}{r_N(x)},$$

and substituting into (3.2) gives

$$S(p) = \sum_{k=1}^{N} \frac{w_k\, p(t_k)}{t_k - x} + \frac{q_N(x) p(x)}{r_N(x)},\ t_k \neq x, k = 1, 2, \ldots, N, \qquad \ldots (3.3)$$

as required. □

Now we introduce the main result

Theorem 3.2 — *Let* $x_k = \cos\frac{(2k-1)}{2(M+1)}\pi$, $k = 1, 2, \ldots, M+1$, $z_m = \cos\frac{\pi m}{M+1}$, $m = 1, 2, \ldots, M$.

Then the integral $I(u) = \int_{-1}^{1} \frac{\sqrt{1-x^2}\,u(x)}{(x-z_m)^2}\,dx$ *can be approximated by* $I_M(u)$ *where*

$$I_M(u) = \frac{\pi}{M+1} \sum_{k=1}^{M+1} \frac{(1-x_k^2)\,u(x_k) - (1-z_m^2)\,u(z_m)}{(x_k - z_m)^2}. \qquad \ldots (3.4)$$

Moreover,

$$| I(u) - I_M(u) | \leq \frac{B}{2^M (2M+2)!} \int_{-1}^{1} \sqrt{1-x^2} \prod_{k=2}^{M} | x - x_k |\, dx, \qquad \ldots (3.5)$$

$$B = \max | u^{(2M+2)}(x) |,\ x \in (-1, 1).$$

PROOF : We construct the polynomial $H(x)$ of lowest degree, satisfying :

$$H(x_k) = u(x_k) \quad k = 1, 2, \ldots, M+1, \; H'(x_k) = u'(x_k) \quad k = 2, 3, \ldots, M,$$

and $$H(z_m) = u(z_m), \; H'(z_m) = u'(z_m).$$

This form of interpolation was investigated by Hermite. This problem has a solution which is furthermore unique.

In fact, $H(x) = L(x) + w_{M+2}(x)\, H_{M-1}(x)$, $w_{M+2}(x) = \prod_{k=1}^{M+1} (x - x_k)\,(x - z_m)$, $L(x)$ is the unique polynomial which interpolates $u(x)$ at $\{x_1, x_2, x_{M+1}, z_m\}$,

$$H_{M-1}(x) = \sum_{i=2}^{M} \frac{u'(x_i) - L'(x_i)}{w'_{M+2}(x_i)} \prod_{\substack{2 \le k \le M \\ k \ne i}} \frac{(x - x_k)}{(x_i - x_k)} \frac{(x - z_m)}{(x_i - z_m)} + \frac{u'(z_m) - L'(z_m)}{w'_{M+2}(z_m)} \prod_{k=2}^{M} \frac{x - x_k}{z_m - x_k}.$$

It is clear that the degree of $H(x)$ will be not higher than $2M + 1$. Then the Quadrature rule (3.4) will be defined as

$$I_M(u) = I(H) = I(L) + I(w_{M+2}\, H_{m-1}). \qquad \ldots (3.6)$$

Using (2.4), we can calculate

$$I(L) = \sum_{i=1}^{M+1} \frac{u(x_i)}{(x_i - z_m)^2\, T'_{M+1}(x_i)} \left[\frac{\pi}{M+1} T'_{M+1}(x_i)\,(1 - x_i^2) - \frac{\pi}{4}\,(2U_M(z_m) - U_{M-2}(z_m) - U_{M+2}(z_m)) \right]$$

$$+ \frac{u(z_m)}{T_{M+1}(z_m)} \left[-(M+1)\,\pi\, T_{M+1}(z_m) - \frac{\pi}{2}\,(U_{M+1}(z_m) + U_{M-1}(z_m)) \right]; \qquad \ldots (3.7)$$

also

$$I(w_{M+2}\, H_{M-1}) = \frac{\pi}{4} \frac{u'(z_m) - L'(z_m)}{w'_{M+2}(z_m)} [2U_M(z_m) - U_{M-2}(z_m) - U_{M+2}(z_m)]. \qquad \ldots (3.8)$$

Since

$$U_M(z_m) = 0, \; U_{M-2}(z_m) + U_{M+2}(z_m) = 0, \; U_{M-1}(z_m) + U_{M+1}(z_m) = 0, \qquad \ldots (3.9)$$

we deduce that $I(w_{M+2}\, H_{M-1}) = 0$.

Now, we apply formula (3.3) for the case $w(t) = \frac{1}{\sqrt{1 - t^2}}$, $p(t) = 1, t \in (-1, 1)$, using also (2.6), one obtains

$$\frac{\pi}{M+1}\sum_{k=1}^{M+1}\frac{1}{(x_k-z_m)^2}+\pi\frac{U'_M(z_m)}{T_{M+1}(z_m)}=0,$$

since $(1-z_m^2)\,U'_M(z_m)=-(M+1)\,T_{M+1}(z_m)$,

so

$$\frac{1}{M+1}\sum_{k=1}^{M+1}\frac{(1-z_m^2)}{(x_k-z_m)^2}=M+1. \qquad \ldots (3.10)$$

Substituting (3.9) and (3.10) into (3.7), we have $I_M(u)$ in the required formula (3.4).

Finally it is known that

$$u(x)-H(x)=\frac{u^{(2M+2)}(\zeta)}{(2M+2)!}(x-z_m)^2\prod_{k=2}^{M}(x-x_k)\,\frac{T_{M+1}(x)}{2^M},\ \zeta\in(-1,1)$$

so formula (3.5) is obvious. □

Remark : If M is replaced by $M_r=r(M+1)-1$, so z_m is defined as $z_m=\cos\left(\frac{\pi m}{M_r+1}\right)$, $m=1,2,\ldots,M_r$. This gives a new formula with degree of precision at least $2r(M+1)-1$

NUMERICAL EXAMPLE

In the following example we tabulate the results for $u(t)=-(t^2+6t-0.25)/3$. The integral analytically is given by $I(u)=(z_{m+}4z_m-0.25)\,\pi$. In the following table we give the difference between $I(u)$ and the quadrature sum for different values of z_m.

M	z_m	error
1	- 6.32231659E-04	1.497785197E-04
2	4.996344136-E-01	2.2181300592E-03
	- 5.0072985074E-01	1.3735139801E-03
3	7.0688322120E-01	3.615132064E-03
	- 6.3222316539E-04	- 1.0920828572E-04
	- 7.0777703710E-01	6.2726148317E-03
4	8.0886832392E-01	4.03950622930E-03
	3.0853593087E-01	1.0381433822E-03
	- 3.0973844137E-01	- 8.0038327724E-05
	- 8.0961115872E-01	1.3417118738E-02

	8.6592001402E-01	4.8561236617E-03
	4.9963494136E-01	2.1339047016E-03
5	- 6.322231659E-04	- 1.6516251708E-04
	- 5.0072985074E-01	1.3011170377E-03
	- 8.6655177559E-01	2.2462436376E-02

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