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PSEUDO-DIFFERENTIAL OPERATORS ASSOCIATED WITH A SINGULAR DIFFERENTIAL OPERATOR IN]0, +∞[

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Using the harmonic analysis associated with a singular differential operator Δ on]0, +∞[, we define two classes S^m and S_0^m of symbols with $S^m \subset S_0^m$ and the corresponding pseudo-differential operators.

We prove that a pseudo-differential operator associated with a symbol in S_0^m, is a continuous linear mapping from some subspace of the Schwartz space into itself. Next we consider symbols in S^m, and we give an integral representation of the pseudo-differential operators associated with these symbols.

Finally, we show that these pseudo-differential opeators satisfy a certain L^1-norm inequality.

Key Words : Differential Operators; Harmonic Analysis; Schwartz Space; L^1-norm Inequality; Sobolov Type Spaces

INTRODUCTION

We consider the singular differential operator on]0, +∞[

$$\Delta = \frac{1}{A(x)} \frac{d}{dx}\left[A(x) \frac{d}{dx} \right], \qquad \ldots (1)$$

where

$$A(x) = x^{2\alpha+1} B(x)$$

with $\alpha > -\frac{1}{2}$ and B is an even C^∞-function on $\mathbb{R}$ such that $B(0) = 1$.

We assume that A is nondecreasing, $\lim_{+\infty} A = +\infty$, $\frac{A'}{A}$ nonincreasing, $\lim_{+\infty} \frac{A'}{A} = 2\rho \geq 0$, and there exist $\delta > 0$ and $x_0 > 0$ such that for $x \geq x_0$, we have

$$\left(\frac{B'}{B} \right)(x) = \begin{cases} 2\rho - \dfrac{2\alpha+1}{x} + e^{-\delta x} D(x) & \text{if } \rho > 0 \\ e^{-\delta x} D(x) & \text{if } \rho = 0, \end{cases} \qquad \ldots (2)$$

where D is a C^∞-function on $[x_0, +\infty[$, bounded together with its derivatives.

Typical examples are the Bessel and the Jacobi operators for which respectively we have

$$A(x) = x^{2\alpha+1}, \quad \alpha > -\frac{1}{2}$$

and

$$A(x) = 2^{2(\alpha+\beta+1)} (sh\, x)^{2\alpha+1} (ch\, x)^{2\beta+1}, \quad \alpha \geq \beta \geq -\frac{1}{2}, \quad \alpha > -\frac{1}{2}.$$

With the help of the Harmonic Analysis associated with Δ, which has been sufficiently developed by Chébli *et al.*[1-4] we define and study pseudo-differential operators associated with Δ.

The theory developed in this work, contains as special cases, pseudo-differential operators associated with the Bessel operator, introduced by Pathak and Pandey[5] and pseudo-differential operator associated with the Jacobi operator studied by Ben Salem and Dachraoui[6].

We introduce two classes of symbols S^m and S_0^m with $S^m \subset S_0^m$. For $p \in S_0^m$, we define the associated pseudo-differential operators h_p. Using the generalized Fourier transform on the generalized Schwartz space $\mathcal{S}_*^r(IR)$, $0 < r \leq 2$, introduced by Xu[4], we establish that h_p is a continuous linear mapping from $\mathcal{S}_*^r(IR)$ into itself. Next for $p \in S^m$, we give an integral representation for the operator h_p.

Finally, we introduce Sobolev type spaces and we prove a certain L^1-norm inequality.

We conclude this introduction with a summary of the content of this paper.

In the first section, we recall properties of the eigenfunctions of Δ, and the Harish-Chandra function associated with Δ. We give the definition of the generalized Schwartz spaces, and the main results which concern the generalized Fourier transform and its inverse.

In the second section, we introduce the classes of symbols S^m and S_0^m, and the associated pseudo-differential operator h_p. For $p \in S_0^m$, we prove the continuity of h_p from $\mathcal{S}_*^r(IR)$, $0 < r \leq 2$, into itself.

In the third section, for $p \in S^m$ we given an integral representation of h_p and $\mathcal{F}(h_p)$ under assumption on the kernel of the product formula for the eigenfunctions $\varphi_\lambda(x)$ with respect to the dual variable λ.

In the fourth section, we define a Sobolev type spaces $G_*^{r,s}(IR)$, $0 < r \leq 2$, $s \in IR$, and for $p \in S^m$, we prove that h_p is continuous from $G_*^{r,m}(IR)$ into $G_*^{r,0}(IR)$.

1. Harmonic Analysis Associated with Δ

We consider the singular differential operator (0.1) given in the introduction. In the following, we recall properties of the eigenfunctions of the operator Δ, the generalized and extended Schwartz spaces, and the generalized Fourier transform associated with Δ and its inverse.

1.1 *Properties of the Eigenfunctions of the Operator Δ :*

The eigenfunction equation

$$\Delta u = -(\lambda^2 + \rho^2)\, u$$

has for every $\lambda \in \mathbb{C}$, a unique solution φ_λ such that

$$\varphi_\lambda(0) = 1 \text{ and } \varphi'_\lambda(0) = 0.$$

In the following we recall some properties of the eigenfunction φ_λ. For more details, one can see [1, 2, 3 & 4].

— If $\rho = 0$, we have

$$\forall x \geq 0, \quad \varphi_0(x) = 1. \qquad \ldots (1.1)$$

— If $\rho > 0$, there exists a positive constant C such that

$$\forall x \geq 0, \quad e^{-\rho x} \leq \varphi_0(x) \leq C(1+x)\, e^{-\rho x}. \qquad \ldots (1.2)$$

— There exists a positive constant C such that

$$\forall \lambda \in \mathbb{C}, \quad x \geq 0, \quad |\varphi_\lambda(x)| \leq C(1+x)\, e^{(|\operatorname{Im}\lambda| - \rho)x} \qquad \ldots (1.3)$$

— We have

$$\forall k \in IN, \quad \forall \lambda \in \mathbb{C}, \; x \geq 0, \quad \left| \frac{\partial^k}{\partial \lambda^k} \varphi_\lambda(x) \right| \leq C\, x^k (1+x)\, e^{(|\operatorname{Im}\lambda| - \rho)x}. \qquad \ldots (1.4)$$

— For any $k \in IN$, there exist $m_k \in IN$ and a positive constant C_k such that for all $\lambda \in IR$, $x \geq 0$, we have

$$\left| \frac{d^k}{dx^k} \varphi_\lambda(x) \right| \leq C_k (1+x)^2 (\lambda^2 + \rho^2)^{m_k} \varphi_0(x). \qquad \ldots (1.5)$$

For $\lambda \in \mathbb{C} - \{0\}$, there exist two linearly independent solutions $\phi_{\pm\lambda}$ satisfying[2]

$$\left.\begin{aligned} &\Delta \phi_{\pm\lambda}(x) = -(\lambda^2 + \rho^2)\, \phi_{\pm\lambda}(x) \\ &\sqrt{Ax}\; \phi_{\pm\lambda}(x) \cong e^{i(\pm\lambda)x}, \quad x \to +\infty \end{aligned}\right] \qquad \ldots (1.6)$$

From [7], we have for all $x > 0$, and for all $\lambda \in \mathbb{C}$ with $\operatorname{Im}\lambda > 0$:

$$\sqrt{Ax}\; \phi_\lambda(x) = e^{i\lambda x}(1 + \varepsilon(\lambda, x)), \qquad \ldots (1.7)$$

where

$$|\varepsilon(\lambda, x)| \leq \left[\exp\left(\frac{1}{|\lambda|} \int_x^{+\infty} G(t)\, dt \right) - 1 \right] \qquad \ldots (1.8)$$

and G is given by

$$G(x) = \frac{1}{4}\left(\frac{A'(x)}{A(x)} \right)^2 + \frac{1}{2}\left(\frac{A'(x)}{A(x)} \right)' - \rho^2.$$

The two eigenfunctions φ_λ and ϕ_λ are related by

$$\varphi_\lambda(x) = c(\lambda)\, \phi_\lambda(x) + c(-\lambda)\, \phi_{-\lambda}(x). \qquad \ldots (1.9)$$

The function $c : \mathbb{C} \to \mathbb{C}$ is called the Harish-Chandra function associated with Δ, and we have the following results about its behaviour[7].

There exist positive constants k, k_1, k_2 such that

— If $\rho \geq 0$ and $\alpha > -\frac{1}{2}$; for $\lambda \in \mathbb{C}$, $|\lambda| > k$, we have

$$k_1 |\lambda|^{2\alpha+1} \leq |c(\lambda)|^{-2} \leq k_2 |\lambda|^{2\alpha+1}. \qquad \ldots (1.10)$$

— If $\rho > 0$ and $\alpha > -\frac{1}{2}$; for $\lambda \in \mathbb{C}$, $|\lambda| \leq k$, we have

$$k_1 |\lambda|^{2} \leq |c(\lambda)|^{-2} \leq k_2 |\lambda|^{2}. \qquad \ldots (1.11)$$

— If $\rho = 0$ and $\alpha > 0$; for $\lambda \in \mathbb{C}$, $|\lambda| \leq k$, we have

$$k_1 |\lambda|^{2\alpha+1} \leq |c(\lambda)|^{-2} \leq k_2 |\lambda|^{2\alpha+1}. \qquad \ldots (1.12)$$

We shall need later more detail about the behaviour of the function ϕ_λ, so we establish the following proposition :

Proposition 1.1 — Let x_0 and δ be defined by (0.2), then if $\rho > 0$, we have for all $x > x_0$ and $\lambda \in \mathbb{C}$, Im $\lambda > 0$:

$$\phi_\lambda(x) = e^{(i\lambda - \rho)x}\left[1 + e^{-\frac{\delta}{2}x}\, \theta(\lambda, x) \right]. \qquad \ldots (1.13)$$

The function $\theta(\lambda, x)$ satisfies : For all $n \in IN$, $\eta > 0$, there exists a positive constant K_n such that :

$$\sup_{|\lambda| \geq \eta,\, x \geq x_0} \left| \frac{d^n}{dx^n}\, \theta(\lambda, x) \right| \leq K_n. \qquad \ldots (1.14)$$

PROOF : We write

$$\phi_\lambda(x) = e^{(i\lambda-\rho)x}\left[\, 1 + e^{-\frac{\delta}{2}x}\,\theta(\lambda, x)\,\right]$$

and we insert it in the differential equation (1.6). We obtain for $x \geq x_0$:

$$\Delta\phi_\lambda(x) + (\lambda^2+\rho^2)\,\phi_\lambda(x) = \phi_\lambda(x) + [2\rho + e^{-\delta x}\,D(x)]\,\phi'_\lambda(x) + (\lambda^2+\rho^2)\,\phi_\lambda(x) = 0,$$

where D is the function given by the relation (0.2).

We deduce that $\theta(\lambda, x)$ is a solution of the differential equation

$$\theta''(\lambda, x) + \theta'(\lambda, x)\,(2i\lambda - \delta) + \left(\frac{\delta^2}{4} - i\lambda\delta\right)\theta(\lambda, x) = h(\lambda, x), \qquad \ldots (1.15)$$

where

$$h(\lambda, x) = (\rho - i\lambda)\,D(x)\,e^{-\frac{\delta}{2}x} + \left(\rho + \frac{\delta}{2} - i\lambda\right)e^{-\delta x}\,D(x)\,\theta(\lambda, x) - \theta'(\lambda, x)\,e^{-\delta x}\,D(x).$$

$$\ldots (1.16)$$

Applying the method of variation of constants, we find that $\theta(\lambda, x)$ is solution of the integral equation

$$\theta(\lambda, x) = \int_x^{+\infty} \left\{ e^{-\frac{\delta}{2}(t-x)} - e^{(2i\lambda - \frac{\delta}{2})(t-x)} \right\} \frac{h(\lambda, t)}{2i\lambda\,(\delta - 2i\lambda)}\,dt. \qquad \ldots (1.17)$$

We denote by

$$k(t, x) = \frac{1}{2i\lambda(\delta - 2i\lambda)} \left\{ e^{-\frac{\delta}{2}(t-x)} - e^{(2i\lambda - \frac{\delta}{2})(t-x)} \right\},$$

$$\psi(x) = \int_x^{+\infty} e^{-\frac{\delta t}{2}}\,|\,D(t)\,|\,dt \;; \quad k_0 = \sup_{|\lambda| \geq \eta} \frac{1}{|\lambda|\,|\delta - 2|\lambda||}$$

and

$$k_1 = \sup_{|\lambda| \geq \eta} \frac{\delta + 2|\lambda|}{|\lambda|\,(|2|\,|\lambda| - \delta|)} \;; \quad k_2 = \sup_{|\lambda| \geq \eta} \frac{3|\lambda| + \rho + 2\delta}{|\lambda|\,|2|\,|\lambda| - \delta|}.$$

We verify that all conditions of lemma 10.2 of [O][3a] p.219, are satisfied for the integral eq. (1.17). So we deduce that for $|\lambda| \geq \eta$, $x \geq x_0$, we have

$$|\,\theta(\lambda, x)\,| \leq k_0\,\psi(x_0)\exp\{k_2\,\psi(x_0)\} = K_0$$

and

$$|\,\theta'(\lambda, x)\,| \leq k_1\,\psi(x_0)\exp\{k_2\,\psi(x_0)\} = K_1.$$

We obtain the relation (1.14) for $n \geq 2$, by induction and by using the eq. (1.15), and the majoration of the function $D(x)$ and its derivatives.

1.2 *Generalized and Extended Schwartz Spaces*

We introduce the following functions spaces[4].

— $S_*(\mathbb{R})$ the space of even C^∞-function on $\mathbb{R}$, rapidly decreasing together with their derivatives, equipped with the usual Schwartz topology.

— For $0 < r \leq 2$, let $S_*^r(\mathbb{R})$ be the generalized Schwartz space defined by

$$S_*^r(\mathbb{R}) = \{g \in C^\infty(\mathbb{R}),\ g \text{ is even and, } \forall k,\ l \in \mathbb{N},$$

$$N_{k,l}(g) = \sup_{x \in \mathbb{R}} (1+x)^l \left| \varphi_0^{-\frac{2}{r}}(x)\, g^{(k)}(x) \right| < +\infty\}. \qquad \ldots (1.18)$$

The space $S_*^r(\mathbb{R})$ is topologized by means of the semi-norms $N_{k,l}(g)$. The space $S_*^r(\mathbb{R})$ is invariant under Δ.

— H_*^r, $0 < r \leq 2$, is the extended Schwartz space defined by all functions h that are even and holomorphic in the interior of $D_r = \left\{ z \in \mathbb{C} \,/\, |\operatorname{Im} z| \leq \left(\frac{2}{r} - 1 \right) \rho \right\}$ and such that h together with all its derivatives extend continuously to D_r and satisfy

$$P_{n,m}(h) = \sup_{\lambda \in D_r} (1 + |\lambda|)^n \left| h^{(m)}(\lambda) \right| < +\infty. \qquad \ldots (1.19)$$

We remark that if $\rho = 0$, for all $0 < r \leq 2$, $H_*^r = S_*^r(\mathbb{R}) = S_*(\mathbb{R})$.

Notations

We denote by

$$\left.\begin{aligned} dm(t) &= A(t)\, dt \\ d\nu(\lambda) &= (2\pi)^{-1} |c(\lambda)|^{-2}\, d\lambda, \end{aligned}\right] \qquad \ldots (1.20)$$

where $c(\lambda)$ is given by (1.12),

— $L^p(m)$, $p \geq 1$, the space of measurable functions f on $[0, +\infty[$, such that

$$\|f\|_{p,m} = \left[\int_0^{+\infty} |f(t)|^p\, dm(t) \right]^p < +\infty.$$

— $L^p(d\nu)$, $p \geq 1$, the space of measurable functions h on $[0, +\infty[$ such that

$$\| h \|_{p,\nu} = \left[\int_0^{+\infty} | h(\lambda) |^p \, d\nu(\lambda) \right]^p < +\infty.$$

1.3. *Generalized Fourier Transform Associated with Δ and Its Inverse*

For $f \in L^1(m)$, we define its generalized Fourier transform by

$$\forall \lambda \in \mathbb{C}, \quad \mathcal{F}(f)(\lambda) = \int_0^{+\infty} f(x) \, \varphi_\lambda(x) \, dm(x). \qquad \ldots (1.21)$$

The inverse of the generalized Fourier transform $\mathcal{F}$ is defined for $h \in L^1(d\nu)$ by

$$\forall x \geq 0, \quad \mathcal{F}^{-1}(h)(x) = \int_{-\infty}^{+\infty} h(\lambda) \, \varphi_\lambda(x) \, d\nu(\lambda). \qquad \ldots (1.22)$$

Using the relation (1.9) we get for $h \in L^1(d\nu)$:

$$\forall x > 0, \quad \mathcal{F}^{-1}(h)(x) = \int_{-\infty}^{+\infty} h(\lambda) \, \phi_\lambda(x) \frac{d\lambda}{2\pi c(-\lambda)}. \qquad \ldots (1.23)$$

Theorem 1.1 — *For $f \in L^1(m)$ such that $\mathcal{F}(f) \in L^1(d\nu)$, we have the following inversion formula*

$$f(x) = \int_0^{+\infty} \mathcal{F}f(\lambda) \, \varphi_\lambda(x) \, d\nu(\lambda) \text{ a.e.} \qquad \ldots (1.24)$$

Theorem 1.2 —

(*i*) *For $f \in L^1(m) \cap L^2(m)$ we have the Plancherel formula*

$$\int_0^{+\infty} | f(x) |^2 \, dm(x) = \int_0^{+\infty} | \mathcal{F}(f)(\lambda) |^2 \, d\nu(\lambda). \qquad \ldots (1.25)$$

(*ii*) *The transform $\mathcal{F}$ defined on $L^1(m) \cap L^2(m)$ extends to an isometric isomorphism from $L^2(m)$ on $L^2(d\nu)$.*[1&2]

Theorem 1.3 — *The generalized Fourier transform $\mathcal{F}$ is an isomorphism between $S_*^r(\mathbb{R})$ and H_*^r, $0 < r \leq 2$. The inverse transform is given by (1.22) and (1.23)*[4].

Remark : It is clear that for all $f \in S_*^r(\mathbb{R})$, $0 < r \leq 2$, we have :

$$\forall \lambda \in \mathbb{R}, \quad \mathcal{F}(\Delta f)(\lambda) = -(\lambda^2 + \rho^2) \, \mathcal{F}(f)(\lambda). \qquad \ldots (1.26)$$

Proposition 1.2 — For $\lambda_1, \lambda_2 \in \mathbb{R}$, there exists a function $\lambda \to a(\lambda_1, \lambda_2, \lambda)$ in $L^2(d\nu)$, such that if $\lambda \to a(\lambda_1, \lambda_2, \lambda)$ belongs to $L^1(d\nu)$ we have for all $x \geq 0$:

$$\varphi_{\lambda_1}(x)\,\varphi_{\lambda_2}(x) = \int_0^{+\infty} \varphi_\lambda(x)\, a(\lambda_1, \lambda_2, \lambda)\, d\nu(\lambda). \qquad \ldots (1.27)$$

PROOF : From the relations (1.2) and (1.3), the function $x \to \varphi_{\lambda_1}(x)\,\varphi_{\lambda_2}(x)$ belongs to $L^1(m) \cap L^2(m)$. We put

$$a(\lambda_1, \lambda_2, \lambda) = \mathcal{F}(\varphi_{\lambda_1}\varphi_{\lambda_2})(\lambda) = \int_0^{+\infty} \varphi_{\lambda_1}(x)\,\varphi_{\lambda_2}(x)\,\varphi_\lambda(x)\, dm(x). \qquad \ldots (1.28)$$

From Theorem 1.2, the map $\lambda \to a(\lambda_1, \lambda_2, \lambda)$ belong to $L^2(d\nu)$.

If we suppose further that $\lambda \to a(\lambda_1, \lambda_2, \lambda)$ belongs to $L^1(d\nu)$ we deduce from Theorem 1.1 that

$$\varphi_{\lambda_1}(x)\,\varphi_{\lambda_2}(x) = \int_0^{+\infty} \varphi_\lambda(x)\, a(\lambda_1, \lambda_2, \lambda)\, d\nu(\lambda).$$

Remark : When Δ is the Jacobi operator, the function $a(\lambda_1, \lambda_2, \lambda)$ is positive and belongs to $L^1(d\nu)$.[8]

2. PSEUDO-DIFFERENTIAL OPERATORS ASSOCIATED WITH Δ

Definition 2.1 — Let m be a real. The function $p : \mathbb{R} \times \mathbb{C} \to \mathbb{C}$ is called a symbol in the class S_0^m if it satisfies

(*i*) For fixed x in $\mathbb{R}$, the function $\lambda \to p(x, \lambda)$ is even, and holomorphic in the interior of D_r, $0 < r \leq 2$, and C^∞ in D_r.

(*ii*) For fixed λ in D_r, $0 < r \leq 2$, the function $x \to p(x, \lambda)$ is an even C^∞ function on $\mathbb{R}$.

(*iii*) For all $k, n \in \mathbb{N}$, there exists $C_{k,m,n} > 0$, such that for every $x \geq 0$ and $\lambda \in D_r$, $0 < r \leq 2$:

$$\left| \frac{\partial^k}{\partial x^k} \frac{\partial^n}{\partial \lambda^n} p(x, \lambda) \right| \leq C_{k,m,n} (1 + |\lambda|)^{m-n}. \qquad \ldots (2.1)$$

Definition 2.2 — Let m be a real, the function $p : \mathbb{R} \times \mathbb{C} \to \mathbb{C}$ is called a symbol in the class S^m if it satisfies (*i*), (*ii*) of Definition 2.1 and

(*iii*) For all $k, l, n \in IN$, there exists $C_{k,l,m,n} > 0$ such that for every $x \geq 0$ and $\lambda \in D_r$, $0 < r \leq 2$:

$$\left| (1+x)^l \varphi_0^{-\frac{2}{r}}(x) \frac{\partial^k}{\partial x^k} \frac{\partial^n}{\partial \lambda^n} p(x, \lambda) \right| \leq C_{k,l,m,n} (1 + |\lambda|)^{m-n}. \quad \ldots (2.2)$$

Remark : We notice that $S^m \subset S_0^m$. It is clear that the symbol p defined by $p(x, \lambda) = (\lambda^2 + \rho^2)^m$, is an element of S_0^{2m}, $m \geq 0$, nevertheless it does not belong to S^{2m}.

Proposition 2.1 — Let $p \in S^m$, then for fixed λ in D_r, the function $x \to p(x, \lambda)$ belongs to $S_*^r(\mathbb{R})$, $0 < r \leq 2$, and for fixed $x \geq 0$, the function $\lambda \to p(x, \lambda)$ belongs to H_*^r, $0 < r \leq 2$.

Definition 2.3 — Let $p \in S_0^m$, the pseudo-differnetial operator associated with the symbol p is defined on $S_*^r(\mathbb{R})$, $0 < r \leq 2$, by

$$\forall u \in S_*^r(\mathbb{R}),\ \forall x \geq 0,\ h_p(u)(x) = \int_0^{+\infty} p(x, \lambda)\ \mathcal{F}(u)(\lambda)\ \varphi_\lambda(x)\, d\nu(\lambda). \quad \ldots (2.3)$$

Remarks :

(*i*) If $p(x, \lambda) = -(\lambda^2 + \rho^2)$, using (1.26), we recover $h_p = \Delta$.

(*ii*) More generally if $p(x, \lambda)$ possesses a finite power series expansion in $-(\lambda^2 + \rho^2)$ with variable coefficients depending only on x, that is

$$p(x, \lambda) = \sum_{k=0}^{N} p_k(x)\, (-1)^k\, (\lambda^2 + \rho^2)^k \quad \ldots (2.4)$$

in the same way, using (1.26) we have

$$h_p(u)(x) = \sum_{k=0}^{N} p_k(x)\, \Delta^k\, u(x).$$

Theorem 2.1 — *If $\rho = 0$, let $p \in S_0^m$. Then the pseudo-differential operator h_p associated with p is a linear continuous mapping from $S_*(\mathbb{R})$ into itself.*

PROOF : From Leibnitz formula and (1.5), for all $n \in IN$, there exist $k_n \in IN$, $C_n > 0$, such that for $x \geq 0$, $\lambda \in \mathbb{R}$ we have

$$\left| \frac{d^n}{dx^n} [p(x, \lambda)\, \varphi_\lambda(x)] \right| \leq C_n (1 + |\lambda|)^m (1+x)^2\, \lambda^{2k_n}.$$

So,

$$\forall x \geq 0, \quad \left| \frac{d^n}{dx^n} h_p(u)(x) \right| \leq C_n (1+x)^2 \int_0^{+\infty} (1+|\lambda|)^m (\lambda^2)^{k_n} |\mathcal{F}(u)(\lambda)| \, d\nu(\lambda).$$

There exists a constant γ such that

$$\forall \lambda \in \mathbb{R}, \quad (1+|\lambda|)^{m-2} \leq \gamma (1+|\lambda|^2)^{|m-2|}.$$

From (1.10), we deduce that there exists an integer m_1 satisfying $m_1 > |m-2|$, such that

$$\forall x \geq 0, \quad \left| \frac{d^n}{dx^n} h_p(u)(x) \right| \leq \gamma C_n (1+x)^2 \sum_{j=0}^{m_1} \binom{m_1}{j} \int_0^{+\infty} |\lambda|^{2(j+k_n)}$$

$$|\mathcal{F}(u)(\lambda)| \lambda^{2\alpha+1} (1+|\lambda|)^{-2} \, d\lambda.$$

Thus, there exists a positive constant C_n' such that

$$\forall x \geq 0, \quad \left| \frac{d^n}{dx^n} h_p(u)(x) \right| \leq \gamma C_n' (1+x)^2 \sup_{x \in \mathbb{R}} |\mathcal{F}(\Delta^s u)(\lambda)|,$$

where s is a positive integer depending on m_1, k_n and α.

Taking into account the continuity of Δ and $\mathcal{F}$ on $\mathcal{S}_*(\mathbb{R})$, we deduce that there exists two nonnegative integers j_0 and l_0 and a positive constant $A^0_{m,n}$ such that

$$\forall x \geq 0, \quad \left| \frac{d^n}{dx^n} h_p(u)(x) \right| \leq A^0_{m,n} N_{j_0, l_0}(u),$$

where $N_{j_0 l_0}$ is given by (1.18).

So we deduce that

$$\forall x \geq 0, \quad \left| (1+x)^l \frac{d^n}{dx^n} h_p(u)(x) \right| \leq A^0_{m,n} N_{j_0+l, l_0}(u).$$

Hence, we complete the proof of the theorem.

In the following, we give an analogous result corresponding to the case $\rho > 0$, however, we need some condition on the Harish-Chandra function $c(\lambda)$. Precisely, we suppose that for each $n \in \mathbb{N}$, there exists $N_n \in \mathbb{N}$ and a positive constant k_n, such that for all $\lambda \in \mathbb{C}$

$$\left| \frac{d^n}{d\lambda^n} c(\lambda)^{-1} \right| \leq k_n (1+|\lambda|)^{N_n}.$$

This inequality is satisfied when Δ is the Jacobi operator[6].

Theorem 2.2 — *Let $\rho > 0$. Then the pseudo-differential operator associated with $p \in S_0^m$, is a linear continuous mapping from $S_*^r(\mathbb{R})$, $0 < r \leq 2$, into itself.*

For the proof of Theorem 2.2, we need the following technical lemmas :

Lemma 2.1 — Let x_0 be given by (0.2). For each $l, k \in IN$, there exist a positive constant $D_{m,l,k}$ and two nonnegative integers j_1, l_1 such that

$$\forall x \in [0, x_0], \quad \left| (1+x)^l \varphi_0^{-2/r}(x) \frac{d^k}{dx^k} h_p(u)(x) \right| \leq D_{m,l,k} N_{j_1,l_1}(u). \qquad \ldots (2.6)$$

PROOF : Using formulas (1.3), (2.1) and the Leibnitz formula we observe that for each $k \in IN$, there exist $l_k \in IN$ and a positive constant $C_{k,m}$ such that

$$\forall \lambda \in \mathbb{R}, \ \forall x \in [0, x_0], \quad \left| \frac{d^k}{dx^k} (p(x,\lambda) \varphi_\lambda(x)) \right| \leq C_{k,m} (1+|\lambda|)^m (\lambda^2+\rho^2)^{l_k}.$$

But there exists a positive constant $\gamma(\rho)$ such that

$$\forall \lambda \in \mathbb{R}, \ (1+|\lambda|)^m \leq \gamma(\rho) (\lambda^2+\rho^2)^{|m|}.$$

Then

$$\forall x \in [0, x_0], \quad \left| \frac{d^k}{dx^k} h_p(u)(x) \right| \leq \gamma(\rho) C_{k,m} \sup_{\lambda \in D_r} |\mathcal{F}(\Delta^s u)(\lambda)| \int_0^{+\infty} \frac{d\lambda}{(1+|\lambda|)^2},$$

where s is a positive integer depending on m l_k and α.

From the continuity of Δ and $\mathcal{F}$ on $S_*^r(\mathbb{R})$, $0 < r \leq 2$, there exist a positive constant $A^1_{m,k}$ and two integers j_1, l_1 such that

$$\forall x \in [0, x_0], \quad \left| \frac{d^k}{dx^k} h_p(u)(x) \right| \leq A^1_{m,k} N_{j_1,l_1}(u)$$

using (1.6), we obtain the result.

Notation — Let $\eta = \left(\frac{2}{r} - 1\right)\rho$. We note for $x \geq 0$:

$$g_1(x) = \int_{-\infty}^{+\infty} p(x, \xi + i\eta) \mathcal{F}(u)(\xi + i\eta) c(-\xi - i\eta)^{-1} e^{ix\xi} d\xi$$

and for $x \geq x_0$:

$$g_2(x) = \int_{-\infty}^{+\infty} p(x, \xi + i\eta) \mathcal{F}(u)(\xi + i\eta) c(-\xi - i\eta)^{-1} \theta(\xi + i\eta, x) e^{ix\xi} d\xi,$$

where θ is given by (1.13).

Lemma 2.2 — (*i*) g_1 belongs to $\mathcal{S}_*^r(\mathbb{R})$.

(*ii*) g_2 is C^∞ for $x \geq x_0$.

PROOF : (*i*) We have

$$\forall x \geq 0, \quad \frac{d^n}{dx^n} g_1(x) = \sum_{j=0}^{n} \binom{n}{j} \int_{-\infty}^{+\infty} \frac{d^{n-j}}{dx^{n-j}} p(x, \xi + i\eta)$$

$$\mathcal{F}(u)(\xi + i\eta)\, c(-\xi - i\eta)^{-1} (i\xi)^j e^{ix\xi}\, d\xi.$$

By using formulas (1.10) and (2.1) we deduce that there exists a positive constant $C_{n,\rho}$ such that

$$\forall x \geq 0, \quad \left| \frac{d^n}{dx^n} g_1(x) \right| \leq C_{n,\rho} \int_{-\infty}^{+\infty} (1 + |\xi + i\eta|)^m$$

$$|\mathcal{F}(u)(\xi + i\eta)|\, (1 + |\xi + i\eta|)^{\alpha + \frac{1}{2}} (1 + |\xi|)^n\, d\xi.$$

But there exists a constant γ_1 such that :

$$\forall \xi, \eta \in \mathbb{R}, \quad (1 + |\xi + i\eta|)^m \leq \gamma_1 |(\xi + i\eta)^2 + \rho^2|^{|m|}$$

and

$$\forall \xi, \eta \in \mathbb{R}, \quad (1 + |\xi + i\eta|)^{\alpha + \frac{1}{2}} \leq \gamma_1 |(\xi + i\eta)^2 + \rho^2|^{\alpha + \frac{1}{2}}.$$

We have

$$\forall x \geq 0, \quad \left| \frac{d^n}{dx^n} g_1(x) \right| \leq \gamma_1 C_{n,\rho} \int_{-\infty}^{+\infty} |\mathcal{F}(\Delta^s u)(\xi + i\eta)|\, (1 + |\xi|)^{-2}\, d\xi,$$

where s is a positive integer depending on m, n and α.

Consequently

$$\forall x \geq 0, \quad \left| \frac{d^n}{dx^n} g_1(x) \right| \leq 2\gamma_1 C_{n,\rho} \sup_{\lambda \in D_r} |\mathcal{F}(\Delta^s u)(\lambda)|. \qquad \ldots (2.7)$$

On the other hand for $k \in \mathbb{N}$, we have for all $x \geq 0$:

$$x^k \frac{d^n}{dx^n} g_1(x) = (-i)^k \sum_{j=0}^{n} \binom{n}{j} \int_{-\infty}^{+\infty} \frac{\partial^{n-j}}{\partial x^{n-j}} p(x, \xi + i\eta)$$

$$\mathcal{F}(u)(\xi + i\eta)\, c(-\xi - i\eta)^{-1} (i\xi)^j \frac{\partial^k}{\partial \xi^k} (e^{ix\xi})\, d\xi.$$

Using integrations by parts, the formula (2.5), the definition of the symbol p, we can show that g_1 is rapidly decreasing.

(*ii*) Using formula (1.14), Definition 2.1, we can see that g_2 is C^∞ on $]x_0, +\infty[$.

PROOF OF THEOREM 2.2 : Let $u \in \mathcal{S}_*^r(\mathbb{R}), 0 < r \leq 2$, we have

$$\forall x \geq 0, \quad h_p(u)(x) = \int_0^{+\infty} p(x, \lambda)\ \mathcal{F}(u)(\lambda)\ \varphi_\lambda(x)\, d\nu(\lambda).$$

From the formula (1.23) we obtain

$$\forall x > 0, \quad h_p(u)(x) = (2\pi)^{-1} \int_{-\infty}^{+\infty} p(x, \lambda)\ \mathcal{F}(u)(\lambda)\ \phi_\lambda(x)\, c(-\lambda)^{-1}\, d\lambda.$$

By Cauchy's Theorem, we find for $\eta = \left(\frac{2}{r} - 1\right)\rho$ and for all $x > 0$:

$$h_p(u)(x) = (2\pi)^{-1} \int_{-\infty}^{+\infty} p(x, \xi + i\eta)\ \mathcal{F}(u)(\xi + i\eta)\ \phi_{\xi + i\eta}(x)\, c(-\xi - i\eta)^{-1}\, d\xi.$$

Using Proposition (1.1) we get for $x \geq x_0$

$$h_p(u)(x) = (2\pi)^{-1} \int_{-\infty}^{+\infty} p(x, \xi + i\eta)\ \mathcal{F}(u)(\xi + i\eta)\ e^{i\xi x - (\eta + \rho)x}$$

$$c(-\xi - i\eta)^{-1} \left(1 + e^{-\frac{\delta}{2}x}\, \theta(\xi + i\eta, x)\right) d\xi$$

$$= (2\pi)^{-1} e^{-\frac{2\rho}{r}x} \left[g_1(x) + e^{-\frac{\delta}{2}x}\, g_2(x) \right].$$

So from (1.2) there exists a positive constant C such that for all $k \in IN$, $n \in IN$, we have

$$\forall x \geq x_0, \quad \left| (1+x)^k \left(\varphi_0^{-\frac{2}{r}}(x)\right) \frac{d^n}{dx^n} h_p(u)(x) \right|$$

$$\leq C(1+x)^k \sum_{j=0}^{n} \left\{ \left| \frac{\partial^j}{\partial x^j} g_1(x) \right| + \left| \frac{\partial^j}{\partial x^j} e^{-\frac{\delta}{2}x}\, g_2(x) \right| \right\}.$$

From (2.7), taking into account the continuity of Δ and $\mathcal{F}$ on $\mathcal{S}_*^r(\mathbb{R}), 0 < r \leq 2$, we deduce that there exist two nonnegative integer q_j, l_j and a positive constant $A_{m,j}^1$ such that

$$\forall x \geq x_0, \quad \left| \frac{\partial^j}{\partial x^j} g_1(x) \right| \leq A^1_{m,j} N_{q_j, l_j}(u).$$

In the same way, we prove that there exist a positive constant $A^2_{m,j}$ and two nonnegative integers q'_j, l'_j such that

$$\forall x \geq x_0, \quad \left| \frac{\partial^j}{\partial x^j} \left(e^{-\frac{\delta}{2}x} g_2(x) \right) \right| \leq A^2_{m,j} N_{q'_j, l'_j}(u).$$

Accordingly, there exist two positive integers q, l and a positve constant $A_{m,n}$ such that for $x \geq x_0$:

$$\left| (1+x)^k \varphi_0^{-\frac{2}{r}}(x) \frac{d^n}{dx^n} h_p(u)(x) \right| \leq A_{m,n} N_{q+k,l}(u)$$

Hence, we complete the proof by using Lemma 2.1.

3. An Integral Representation

In this section, we suppose that there exist a positive constant K which does not depend on $\lambda_1, \lambda_2 \in \mathbb{R}$ such that the function $a(\lambda_1, \lambda_2, \lambda)$ given by (1.34) satisfies :

$$\int_0^{+\infty} |a(\lambda_1, \lambda_2, \lambda)| \, d\nu(\lambda) \leq K. \qquad \ldots (3.1)$$

We give an integral representation for the pseudo-differential operator h_p associated with the symbol p in S^m.

Remark : Ben Salem[8] has proved that when Δ is the Jacobi operator the inequality (3.1) is an equality with $K = 1$.

Notation — Let $p(x, \lambda)$ be a symbol in S^m, we denote by

- $U_\lambda(x) = \varphi_\lambda(x)\, p(x, \lambda)$.

- $P(\lambda_1, \lambda_2) = \mathcal{F}(U_{\lambda_1}(\cdot))(\lambda_2), \quad \lambda_1, \lambda_2 \in \mathbb{R}$.

Lemma 3.1 — For all $\lambda_1, \lambda_2 \in \mathbb{R}$, we have

$$P(\lambda_1, \lambda_2) = \int_0^{+\infty} \mathcal{F}(p(\cdot, \lambda_1))\, a(\lambda_1, \lambda_2, \lambda)\, d\nu(\lambda). \qquad \ldots (3.2)$$

PROOF : We have for $\lambda_1, \lambda_2 \in \mathbb{R}$:

$$P(\lambda_1, \lambda_2) = \int_0^{+\infty} p(x, \lambda_1)\, \varphi_{\lambda_1}(x)\, \varphi_{\lambda_2}(x)\, dm(x).$$

Using formula (1.27) we find

$$P(\lambda_1, \lambda_2) = \int_0^{+\infty} p(x, \lambda_1) \left(\int_0^{+\infty} \varphi_\lambda(x)\, a(\lambda_1, \lambda_2, \lambda)\, d\nu(\lambda) \right) dm(x).$$

Now by Fubini's theorem, we obtain

$$P(\lambda_1, \lambda_2) = \int_0^{+\infty} \left(\int_0^{+\infty} p(x, \lambda_1)\, \varphi_\lambda(x)\, dm(x) \right) a(\lambda_1, \lambda_2, \lambda)\, d\nu(\lambda)$$

$$= \int_0^{+\infty} \mathcal{F}(p(\cdot, \lambda_1))\,(\lambda)\, a(\lambda_1, \lambda_2, \lambda)\, d\nu(\lambda).$$

Lemma 3.2 — Let $\rho = 0$ and p be a symbol in S^m. Then for all $k \in IN$, there exists $C_{m,k} > 0$ such that for all $\lambda_1, \lambda_2 \in \mathbb{R}$, we have

$$| \mathcal{F}(p(\cdot, \lambda_1))\,(\lambda_2) | \leq C_{m,k} (1 + | \lambda_1 |)^m (1 + | \lambda_2 |^{2k})^{-1} \qquad \ldots (3.3)$$

$$| P(\lambda_1, \lambda_2) | \leq C_{m,k} (1 + | \lambda_1 |)^m (1 + | \lambda_2 |^{2k})^{-1}. \qquad \ldots (3.4)$$

PROOF : For all $k \in IN$, we have for $\lambda_1, \lambda_2 \in \mathbb{R}$:

$$(-1)^k (\lambda_2^2)^k\, \mathcal{F}(p(\cdot, \lambda_1))\,(\lambda_2) = \mathcal{F}(\Delta^k p(\cdot, \lambda_1))\,(\lambda_2)$$

$$= \int_0^{+\infty} (1+x)^l\, \varphi_0^{-\frac{2}{r}}(x)\, \Delta^k p(x, \lambda_1)\, (1+x)^{-l}\, \varphi_0^{\frac{2}{r}}(x)\, A(x)\, dx.$$

We recall that

$$A(x) = O(x^{2\alpha+1}), \text{ for } x \to +\infty.$$

Using the fact that $x \to p(x, \lambda_1)$ belongs to $S_*^r(\mathbb{R}), 0 < r \leq 2$, the continuity of Δ on $S_*^r(\mathbb{R})$, and the definition of the symbol p, we deduce that there exists a positive constant $C_{m,k}$ such that

$$\forall \lambda_1, \lambda_2 \in \mathbb{R},\ | \lambda_2 |^{2k} | \mathcal{F}(p(\cdot, \lambda_1))\,(\lambda_2) | \leq C_{m,k} (1 + | \lambda_1 |)^m$$

then

$$\forall \lambda_1, \lambda_2 \in \mathbb{R},\ (1+|\lambda_2|^{2k})\,|\,\mathcal{F}p(\cdot,\lambda_1))(\lambda_2)\,| \le (C_{m,k}+C_{m,0})(1+|\lambda_1|)^m$$

and we obtain the inequality (3.3).

The inequality (3.4) follows from (3.3), (3.1) and Lemma 3.1.

Lemma 3.3 — Let $\rho > 0$ and p be a symbol in S^m, then for all $k \in IN$, there exists $C_{m,k} > 0$ such that for all $\lambda_1, \lambda_2 \in \mathbb{R}$, we have

$$|\,\mathcal{F}p(\cdot,\lambda_1))(\lambda_2)\,| \le C_{m,k}(1+|\lambda_1|)^m(\lambda_2^2+\rho^2)^{-k} \qquad \ldots (3.5)$$

$$|\,P(\lambda_1,\lambda_2)\,| \le C_{m,k}(1+|\lambda_1|)^m(\lambda_2^2+\rho^2)^{-k}. \qquad \ldots (3.6)$$

PROOF : We obtain the results by using (1.23), and the same proof as of Lemma 3.2.

Theorem 3.1 — *Let p be a symbol in S^m. Then the associated pseudo-differential operator h_p can be represented as follows*

$$\forall u \in \mathcal{S}_*^r(\mathbb{R}),\ 0 < r \le 2,\ \forall x \ge 0,$$

$$h_p(u)(x) = \int_0^{+\infty}\int_0^{+\infty} P(\lambda_1,\lambda_2)\,\mathcal{F}(u)(\lambda_1)\,\varphi\lambda_2(x)\,d\nu(\lambda_1)\,d\nu(\lambda_2). \qquad \ldots (3.7)$$

Furthermore, we have for all $\lambda_2 \in \mathbb{R}$:

$$\mathcal{F}(h_p(u))(\lambda_2) = \int_0^{+\infty} P(\lambda_1,\lambda_2)\,\mathcal{F}(u)(\lambda_1)\,d\nu(\lambda_1). \qquad \ldots (3.8)$$

PROOF : We have

$$\forall u \in \mathcal{S}_*^r(\mathbb{R}),\ \forall x \ge 0,\ h_p(u)(x) = \int_0^{+\infty} p(x,\lambda_1)\,\varphi_{\lambda_1}(x)\,\mathcal{F}(u)(\lambda_1)\,d\nu(\lambda_1).$$

But we remark that the map $\lambda_2 \to P(\lambda_1,\lambda_2)$ belongs to H_*^r, $0 < r \le 2$, so from Theorem 1.3, we have

$$\forall \lambda_1 \in \mathbb{R},\ \forall x \ge 0,\ p(x,\lambda_1)\,\varphi_{\lambda_1}(x) = \int_0^{+\infty} P(\lambda_1,\lambda_2)\,\varphi_{\lambda_2}(x)\,d\nu(\lambda_2).$$

Consequently,

$$\forall x \geq 0,\ h_p(u)(x) = \int_0^{+\infty} \left(\int_0^{+\infty} P(\lambda_1, \lambda_2)\, \varphi_{\lambda_2}(x)\, d\nu(\lambda_2) \right) \mathcal{F}(u)(\lambda_1)\, d\nu(\lambda_1).$$

Now for all $k > \alpha + 1$, we deduce from Lemma 3.2 that there exists a positive constant $c_{m,k}$ such that for all $\lambda_1, \lambda_2 \in \mathbb{R}$:

$$| P(\lambda_1, \lambda_2)\, \mathcal{F}(u)(\lambda_1)\, \varphi_{\lambda_2}(x) |$$

$$\leq \begin{cases} C_{m,k} (1 + |\lambda_1|)^m (1 + |\lambda_2|^{2k})^{-1}\, |\mathcal{F}(u)(\lambda_1)|, & \text{if } \rho = 0 \\ C_{m,k} (1 + |\lambda_1|)^m (\lambda_2^2 + \rho^2)^{-k}\, |\mathcal{F}(u)(\lambda_1)|, & \text{if } \rho > 0. \end{cases}$$

Since the maps $\lambda_1 \to (1 + |\lambda_1|)^m\, \mathcal{F}(u)(\lambda_1)$, $\lambda_2 \to (1 + |\lambda_2|^{2k})^{-1}$ and $\lambda_2 \to (\lambda_2^2 + \rho^2)^{-k}$ belong to $L^1(d\nu)$ we deduce that the map

$$(\lambda_1, \lambda_2) \to P(\lambda_1, \lambda_2)\, \mathcal{F}(u)(\lambda_1)$$

is integrable on $[0, +\infty[\times [0, +\infty[$ with respect to the measure $\nu \otimes \nu$. So by Fubini's theorem we obtain the first equality (3.7). The second equality (3.8) is obtained from Theorem 1.3.

4. SOBOLEV TYPE SPACES AND AN L^1-NORM INEQUALITY

In the following we define subspaces of $\mathcal{S}_*^r(\mathbb{R})$, $0 < r \leq 2$, which are Sobolev type spaces and we prove a certain L^1-norm inequality.

Definition 4.1 — (Sobolev type spaces). The space $G_*^{r,s}(\mathbb{R})$, $0 < r \leq 2$, $s \in \mathbb{R}$ is defined to be the subspace of all elements $u \in \mathcal{S}_*^r(\mathbb{R})$, $0 < r \leq 2$, satisfying

$$\| u \|_{G_*^{r,s}} = \| (1 + |\lambda|)^s\, \mathcal{F}(u)(\lambda) \|_{1,\nu} < +\infty.$$

Theorem 4.1 — *Let p be a symbol in S^m and h_p the pseudo-differential operator associated with p. Then there exists $C > 0$ such that*

$$\| h_p(u) \|_{G_*^{r,o}} \leq C \| u \|_{G_*^{r,m}}.$$

PROOF : From Theorem 3.1 and Lemma 3.1, we have

$$\forall \lambda_2 \in \mathbb{R},\ \mathcal{F}(h_p(u))(\lambda_2) = \int_0^{+\infty} P(\lambda_1, \lambda_2)\, \mathcal{F}(u)(\lambda_1)\, d\nu(\lambda_1)$$

$$= \int_0^{+\infty} \left(\int_0^{+\infty} \mathcal{F}p(\cdot, \lambda_1))(\lambda)\, a(\lambda_1, \lambda_2, \lambda)\, d\nu(\lambda) \right) \mathcal{F}(u)(\lambda_1)\, d\nu(\lambda_1).$$

Let n_0 be an integer such that $n_0 > \alpha + 1$. From Lemma 3.2, there exist $C_{m, n_0} > 0$ such that

— if $\rho = 0$.

$$\forall \lambda_2 \in \mathbb{R},\ |\ \mathcal{F}(h_p(u))(\lambda_2)\ |$$

$$\leq C_{m, n_0} \int_0^{+\infty} \int_0^{+\infty} (1 + |\ \lambda_1\ |)^m\, (1 + |\ \lambda\ |^{2n_0})^{-1}\, |\ a(\lambda_1, \lambda_2, \lambda)\ |\ |\ \mathcal{F}(u)(\lambda_1)\ |\, d\nu(\lambda)\, d\nu(\lambda_1)$$

— if $\rho > 0$.

$$\forall \lambda_2 \in \mathbb{R},\ |\ \mathcal{F}(h_p(u))(\lambda_2)\ |$$

$$\leq C_{m, n_0} \int_0^{+\infty} \int_0^{+\infty} (1 + |\ \lambda_1\ |)^m\, (\lambda^2 + \rho^2)^{-n_0}\, |\ a(\lambda_1, \lambda_2, \lambda)\ |\ |\ \mathcal{F}(u)(\lambda_1)\ |\, d\nu(\lambda)\, d\nu(\lambda_1).$$

We put

$$\forall \lambda \in \mathbb{R},\quad \chi(\lambda) = \begin{cases} (1 + |\ \lambda\ |)^{2n_0 - 1}, & \text{if} \quad \rho = 0. \\ (\lambda^2 + \rho^2)^{-n_0}, & \text{if} \quad \rho > 0. \end{cases}$$

and

$$\forall \lambda \in \mathbb{R},\ \psi(\lambda) = (1 + |\ \lambda\ |)^m\ |\ \mathcal{F}(u)(\lambda)\ |\ .$$

It is clear that χ and ψ belong to $L^1(d\nu)$. From inequality (3.1), we deduce

$$\|\ \mathcal{F}(h_p(u))\ \|_{1, \nu} \leq K C_{m, n_0}\, \|\ \chi\ \|_{1, \nu}\, \|\ \psi\ \|_{1, \nu}$$

But

$$\|\ \psi\ \|_{1, \nu} = \|\ u\ \|_{G_*^{r, m}}.$$

Hence, we obtain

$$\|\ h_p(u)\ \|_{G_*^{r, o}} \leq C\, \|\ u\ \|_{G_*^{r, m}}$$

where

$$C = C_{m, n_0}\, K\, \|\ \chi\ \|_{1, \nu}$$

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