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ON *GPR*-CONTINUOUS FUNCTIONS IN TOPOLOGICAL SPACES

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The purpose of this paper is to study on *gpr*-continuous functions. The concept of *GPR*-compact and *GPR*-connected sets have been introduced and some of their properties are investigated.

Key Words : *GPR*-Continuous Functions; Topological Spaces; Generalised Preregular; Closed Sets

1. INTRODUCTION

A weak form of continuous functions called *g*-continuous functions was introduced and studied by Balachandran, Sundaram and Maki[1]. Maki, Umehara and Noiri[5] investigated the topological properties of *gp*-closed sets from a general point of view. Recently, Gnanambal[2] has introduced and studied the properties of generalized preregular closed sets. The notion of generalized pre-regular open set, pre-regular $T_{1/2}$ space, generalized pre-regular continuity and generalized pre-regular irresoluteness are also introduced in [2]. In this paper we shall continue the investigations carried out in [2] and study the properties of generalized pre-regular continuous functions (briefly *gpr*-continuous functions) which is weaker than *g*-continuous functions. Further, we introduce the concepts of *GPR*-connected sets and *GPR*-compact sets and study their behaviour under *gpr*-continuous functions.

Throughout this paper (X, τ), (Y, σ) and (Z, μ) represent nonempty topological spaces on which no separation axioms are assumed unless otherwise stated. For a subset A of a space (X, τ), cl A and int A denote the closure of A and the interior of A respectively. (X, τ) and (Y, σ) will bedenoted by X and Y respectively if there is no chance of confusion. Let us recall the following :

Definition 1.1 — A subset A of a space (X, τ) is called

(*i*) *gp*-closed[5] if *p*cl $A \subset U$ whenever $A \subset U$ and U is open in X; and

(*ii*) *gpr*-closed[2] if *p*cl $A \subset U$ whenever $A \subset U$ and U is regular open in X.

The intersection of all preclosed sets containing A is denoted by *p*cl A. The class of all preopen (resp. preclosed, *gpr*-open and *gpr*-closed) sets of X is denoted by $PO(X)$ resp. $PC(X)$, $GPRO(X)$ and $GPRC(X)$).

Definition 1.2[2] — A space (X, τ) is called pre-regular $T_{1/2}$ if every *gpr*-closed is preclosed in X.

Definition 1.3 — Let $f: (X, \tau) \to (Y, \sigma)$ be a function. 'f' is said to be :

(*i*) p-open[3] if the image of every preopen set in X is preopen in Y; and

(*ii*) regular open if the image of every regular open set in X is regular open in Y.

2. *GPR*-CLOSED SETS

Additional results on *gpr*-closed sets are given in this section. We also prove some simple lemmas which would be useful in our later sections.

Lemma 2.1 — For an $x \in X$ its complement $X - \{x\}$ is *gpr*-closed or regular open.

PROOF : Suppose $X - \{x\}$ is not regular open. Then X is the only regular open set containing $X - \{x\}$. This implies $p\text{cl}\,(X - \{x\}) \subset X$. Hence, $X - \{x\}$ is *gpr*-closed.

Lemma 2.2 (*i*) — If (X, τ) is pre-regular $T_{1/2}$, then for each singleton $\{x\}$ of X, $\{x\}$ is preopen or $X - \{x\}$ is regular open.

(*ii*) Conversely, suppose that for each singleton $\{x\}$ of X, $\{x\}$ is preopen or $X - \{x\}$ is regular open. Then for every *gpr*-closed set A, $p\text{cl}(A) \subset A$ holds.

Proposition 2.3 — If $PO(X) = PC(X)$, then $GPRC(X) = P(X)$.

PROOF : Suppose $A \subset O$, where O is regular open in X. Since O is preopen, it is preclosed by hypothesis. Hence, $p\text{cl}\ A \subset O$ and so A is *gpr*-closed. Thus $GPRC(X) = P(X)$.

Proposition 2.4 — If A is semi-open *gpr*-closed, then it is rg-closed.

PROOF : Suppose $A \subset O$ where O is regular open. Since A is *gpr*-closed $p\text{cl}\ A \subset O$. For any semi-open set V, $p\text{cl}\ V = \text{cl}V$[7]. Hence, A is rg-closed.

Proposition 2.5 — Let (X, τ) be a topological space and $A, B \subset X$. If B is *gpr*-open and if $A \supset p\text{int}\ B$, then $A \cap B$ is *gpr*-open.

PROOF : Since B is *gpr*-open and $A \supset p\text{int}\ (B)$, $p\text{int}\ (B) \subset A \cap B \subset B$. By Theorem 4.4, $A \cap B$ is *gpr*-open.

Proposition 2.6 — Let $PO(X)$ be closed under finite intersections. If A is *gpr*-open and B is *gpr*-open then $A \cap B$ is *gpr*-open.

PROOF : Let $X - (A \cap B) = (X - A) \cup (X - B) \subset F$, where F is regular open. Then $X - A \subset F$ and $X - B \subset F$. Since A and B are *gpr*-open, $p\text{cl}(X - A) \subset F$ and $p\text{cl}(X - B) \subset F$. By hypothesis, $p\text{cl}\ ((X - A) \cup (X - B)) \subset p\text{cl}(X - A) - p\text{cl}(X - B) \subset F$. That is $p\text{cl}(X - (A \cap B)) \subset F$. This shows that $A \cap B$ is *gpr*-open.

Lemma 2.7[4] — Let A be open in X and $B \subset A$. Then $\text{int}\ B = \text{int}_A B$ where int_A denotes the interior operator in the subspace A.

Lemma 2.8[4] — Let A be closed in X and $B \subset A$. Then $\text{cl}\ B = \text{cl}_A B$ where cl_A denotes the closure operator in the subspace A.

Lemma 2.9 — Let $A \subset Y \subset X$ and Y be open in X. Then

(*i*) $A \in PO(X)$ if and only if $A \in PO(Y)$,

(*ii*) $A \in PC(Y)$ if $A \in PC(X)$ and

(*iii*) $A \in PC(Y)$ implies $A \in PC(X)$ if Y is also closed in X.

PROOF : (*i*) By Lemma 2.7 and Lemma 2.8, int (cl A) = $\text{int}_Y(cl_Y A)$. Hence, (*i*) holds.

(*ii*) Let $A \in PC(X)$. Then cl(int A) $\subset A$. Since Y is open $\text{cl}(\text{int}_Y A) \subset A$ by Lemma 2.7.

Then $\text{cl}(\text{int}_Y A) \cap Y \subset A \cap Y = A$. That is, $\text{cl}_Y(\text{int}_Y A) \subset A$. This shows $A \in PC(Y)$.

(*iii*) Suppose Y is clopen. Then $\text{cl}_Y(\text{int}_Y A) = \text{cl}(\text{int}(A)$ by Lemma 2.7 and 2.8. Hence, (*iii*) holds.

Lemma 2.10 — If $A \subset Y \subset X$ and Y is open in X, then $p\text{cl}_Y A = p\text{cl}_X A \cap Y$.

PROOF : $p\text{cl}_Y A = A \cup \text{cl}_Y(\text{int}_Y A) = A \cup \text{cl}_Y(\text{int} A) = A \cup (\text{cl}(\text{int}(A) \cap Y) = (A \cup \text{cl}(\text{int}\, A)) \cap (A \cup Y) = p\text{cl}_X A \cap Y$ since $p\text{cl}_X A = A \cup \text{cl}(\text{int}(A))$.

Lemma 2.11 — If Y is open preclosed, $p\text{cl}_Y A = p\text{cl}_X A$.

PROOF : By Lemma 2.10, $p\text{cl}_Y A = p\text{cl}_X A \cap Y$. Since Y is also preclosed, $p\text{cl}_X A \subset Y$. Hence, $p\text{cl}_Y A = p\text{cl}_X A$.

Lemma 2.12[6] — Let $A \subset X$. If A is open or dense, then

$$RO(A, \tau | A) = \{V \cap A : V \in RO(X, \tau)\}.$$

Proposition 2.13 — Let $A \subset Y \subset X$. Then :

(*i*) If Y is open in X, then $A \in GPRC(X)$ implies $A \in GPRC(Y)$; and

(*ii*) If Y is open preclosed in X, then $A \in GPRC(Y)$ implies $A \in GPRC(X)$.

PROOF : (*i*) Let A be *gpr*-closed in X. Let $A \subset O$ where O is regular open in Y. Then $O = O' \cap Y$ where O' is regular open in X by Lemma 2.12. That is, $A \subset O'$. As $A \in GPRC(X)$, $p\text{cl}_X A \subset O'$. This implies $p\text{cl}_X A \cap Y \subset O' \cap Y$. That is, $p\text{cl}_Y A \subset O$ by Lemma 2.10. Hence, $A \in GPRC(Y)$.

(*ii*) Let A be *gpr*-closed in Y. Then $A \subset U$ where U is regular open in X. Then $A = A \cap Y \subset U \cap Y$ where $U \cap Y$ is regular open in Y by Lemma 2.12. By hypothesis, $p\text{cl}_Y A \subset U \cap Y$. Then, $p\text{cl}_X A \subset U \cap Y \subset U$ by Lemma 2.11. That is $A \in GPRC(X)$.

The following example shows that the assumption Y is open preclosed in Proposition 2.13(*ii*) cannot be removed.

Example 2.14 — Consider $X = \{a, b, c\}$, $Y = \{a, c\}$ and $\tau = \{\phi, X, \{a\}, \{b\}, \{a, b\}\}$. Then $GPRC(X, \tau) = \{\phi, X, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}\}$ and $GPRC(Y, \tau | Y) = \{\phi, Y, \{c\}, \{a\}\}$. $\{a\} \in GPRC(Y, \tau | Y)$ but $\{a\} \notin GPRC(X, \tau)$. Here, Y is preclosed but not open in X.

Corollary 2.15 — If Y is open preclosed in X, then $A \in GPRC(X)$ if and only if $A \in GPRC(Y)$.

3. *GPR*-CLOSURE

In this section, the notion of *gpr*-closure is defined and some of its basic properties are studied. We define τ_η and prove that it forms a topology on X. For any $A \subset X$, *gpr*-interior of A is defined and it is proved that the complement of *gpr*-interior of A is the *gpr*-closure of the complement of A.

Definition 3.1 — For a subset A of (X, τ), gpr-cl$(A) = \bigcap \{F : A \subset F, F$ is *gpr*-closed in $X\}$.

Lemma 3.2 — For an $x \in X$, $x \in gpr\text{–}cl(A)$ if and only if $V \bigcap A \neq \phi$ for every *gpr*-open set V containing x.

PROOF : (*Necessity*) Suppose there exists a *gpr*-open set V containing x such that $V \bigcap A = \phi$. Since $A \subset X - V$, *gpr*-cl $(A) \subset X - V$. This implies $x \notin gpr\text{–cl}\, A$, a contradiction.

(*Sufficiency*) Suppose $x \notin gpr$–cl (A). Then there exists a *gpr*-closed subset F containing A such that $x \notin F$. Then $x \in X - F$ and $X - F$ is *gpr*-open. Also $(X - F) \bigcap A = \phi$, a contradiction. Hence the Lemma.

Lemma 3.3 — Let A and B be subsets of (X, τ). Then :

(*i*) *gpr*-cl$(\phi) = \phi$ and *gpr*-cl$(X) = X$;

(*ii*) If $A \subset B$, then *gpr*-cl$(A) \subset gpr\text{–}cl(B)$;

(*iii*) $A \subset gpr\text{–}cl(A)$;

(*iv*) *gpr*-cl(A) = *gpr*-cl(*gpr*-cl(A)); and

(*v*) *gpr*-cl$(A \bigcup B) \supset gpr\text{–cl}(A) \bigcup gpr\text{–}cl(B)$.

Remark 3.4 : If $A \subset (X, \tau)$ is *gpr*-closed, then *gpr*-cl$(A) = A$. But the converse is not true as seen from the following example :-

Example 3.5 — Let $X = \{a, b, c\}$, $\tau = \{\phi, X, \{a\}, \{b\}, \{a, b\}\}$. Here, $GPRC(X) = \{\phi, X, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}\}$. *gpr*-cl$\{a\} = \{a, b\} \bigcap \{a, c\} \bigcap X = \{a\}$ but $\{a\}$ is not *gpr*-closed.

Lemma 3.6 — Let A and B be subsets of (X, τ). Then *gpr*-cl$(A \bigcap B) \subset gpr\text{–}cl(A) \bigcap gpr\text{–}cl(B)$.

PROOF : Since $A \bigcap B \subset A, B$, by Lemma 3.3(*ii*), *gpr*-cl$(A \bigcap B) \subset gpr\text{–}cl(A)$ and *gpr*-cl $(A \bigcap B) \subset gpr\text{–}cl(B)$. Thus $gpr\text{–}cl(A \bigcap B) \subset gpr\text{–}cl(A) \bigcap gpr\text{–}cl(B)$.

In general, *gpr*-cl$(A) \bigcap gpr\text{–cl}(B) \not\subset gpr\text{–cl}(A \bigcap B)$.

Example 3.7 — Consider the either or topology defined on $[-1, 1]$. The open sets of this topology are ϕ, $[-1, 1]$, $(-1, 1]$, $[-1, 1)$, $(-1, 1)$ and all subsets that do not contain 0. Let $A = (-1, 0)$ and $B = (0, 1)$. Both A and B are not *gpr*-closed. *gpr*-cl $A = (-1, 0]$ and *gpr*-cl $B = [0, 1)$. *gpr*-cl $A \bigcap gpr\text{–}cl\, B = \{0\}$ but *gpr*-cl$(A \bigcap B) = \phi$. Hence *gpr*-cl $A \bigcap$ *gpr*-cl $B \not\subset gpr$–cl $(A \bigcap B)$.

Proposition 3.8 — If $GPRC(X)$ is closed under finite unions, then *gpr*-cl$(A \bigcup B) = gpr\text{–}cl(A) \bigcup gpr\text{–}cl(B)$ for every $A, B \in GPRC(X)$.

PROOF : Let A and B be *gpr*-closed in (X, τ). Then by hypothesis, $A \bigcup B$ is *gpr*-closed. Then *gpr*-cl$(A \bigcup B) = A \bigcup B =$ *gpr*-cl$(A) \bigcup$ *gpr*-cl(B).

Theorem 3.9 — *If* PC(X, τ) *is a closed under finite unions, then* GPRC(X, τ) *is closed under finite unions.*

PROOF : Let PC(X, τ) be closed under finite unions. Let $A, B \in GPRC(X, \tau)$ and let $A \bigcup B \subset U$, where U is regular open in X. Then $A \subset U$ and $B \subset U$. Hence, $pcl(A) \subset U$ and $pcl(B)$

$\subset U$. This implies $pcl(A) \cup pcl(B) \subset U$. By hypothesis, $pcl(A \cup B) \subset U$. That is $A \cup B \in GPRC(X, \tau)$.

Corollary 3.10 — If $PO(X, \tau)$ is closed under finite intersections, then $GPRO(X, \tau)$ is closed under finite intersections.

Definition 3.11 — For a subset A of (X, τ) :

(*i*) (Maki)[5] $c^*(A) = g$–closure of A; and

(*ii*) (Maki)[5] $pcl^*(A) = gp$-closure of A.

Definition 3.12 — Let (X, τ) be a topological space :

(*i*) (Maki)[5] $\tau^* = \{U \subset X / c^*(X - U) = X - U\}$.

(*ii*) (Maki)[5] ${}_p\tau^* = \{W \subset X / pcl^*(X - W) = X - W\}$.

(*iii*) $\tau_g^* = \{V \subset X / gpr\text{–cl}(X - V) = X - V\}$.

Proposition 3.13[5] — For a subset A of (X, τ), the following implications hold :

(*a*) $A \subset gpr\text{–cl}(A) \subset pcl^*(A) \subset pcl(A) \subset cl(A)$;

(*b*) $\tau \subset PO(X) \subset {}_p\tau^* \subset \tau_g^*$;

(*c*) $A \subset gpr\text{–cl}(A) \subset pcl^*(A) \subset c^*(A) \subset cl(A)$; and

(*d*) $\tau \subset GO(\tau) \subset \tau^* \subset {}_p\tau^* \subset \tau_g^*$.

The proofs follow from definitions.

Theorem 3.14 — *If* $GPRO(X, \tau)$ *is a topology, then* τ_g^* *is a topology.*

PROOF : Clearly, $\phi, X \in \tau_g^*$. Let $\{A_i : i \in \Lambda\} \in \tau_g^*$. Then $gpr\text{-cl}\ (X - (\cup A_i)) = gpr\text{–cl}\ (\cap (X - A_i)) \subset \cap\, gpr\text{–cl}\ (X - A_i) = \cap\, (X - A_i) = X - \cup A_i$. Hence, $\cup A_i \in \tau_g^*$. Let $A, B \in \tau_g^*$. Now $gpr\text{-cl}\ (X - (A \cap B)) = gpr\text{–cl}\ ((X - A) \cup (X - B)) = gpr\text{-cl}\ (X - A) \cup gpr\text{-cl}\ (X - B) = (X - A) \cup (X - B)$. Hence, $A \cap B \in \tau_g^*$. Thus τ_g^* is a topology.

Theorem 3.15 — (*i*) *For a space* (X, τ), *every* gpr*-closed set is preclosed (i.e.* (X, τ) *is pre-regular* $T_{1/2}$) *if and only if* $\tau_g^* = PO(\tau)$ *holds.*

(ii) For a space (X, τ) *every gpr-closed set is closed if and only if* $\tau_g^* = \tau$ *holds.*

(iii) For a space (X, τ) *if every gpr-closed set is gp-closed (i.e.* $GPRC(X) = GPC(X)$*) then* $\tau_g^* = {}_p\tau^*$.

PROOF : (*i*) (*Necessity*) Let $A \in \tau_g^*$. Then $gpr\text{-cl}(X - A) = X - A$. Since every *gpr*-closed set is preclosed, $pcl(X - A) = gpr\text{-cl}(X - A) = X - A$. Hence, $A \in PO(\tau)$.

(*Sufficiency*) Suppose $\tau_g^* = PO(\tau)$. Let A be a *gpr*-closed set. Then $gpr\text{-cl}\ A = A$. This implies $X - A \in \tau_g^* = PO(\tau)$. So A is preclosed.

Proof of (*ii*) is similar to (*i*).

(*iii*) Let every *gpr*-closed set be *gp*-closed. Let $A \in \tau_g^*$. Then *gpr*-cl$(X - A) = X - A$. Since every *gpr*-closed set is *gp*-closed, $pcl^*(X-A) = gpr\text{-cl}(X-A) = X-A$. That is, $A \in {}_p\tau^*$. By Proposition 3.13, ${}_p\tau^* \subset \tau_g^*$. Thus ${}_p\tau^* = \tau_g^*$.

For subsequent results we assume that

The class $GPRO(X, \tau)$ is closed under arbitrary unions. ... ($\mathcal{R}$)

Definition 3.16 — For a topological space (X, τ) let τ_η consists of exactly those sets A for which $A \cap B \in GPRO(X, \tau)$ for all $B \in GPRO(X, \tau)$.

Theorem 3.17 — *τ_η is a topology on X if ($\mathcal{R}$) holds.*

PROOF : Clearly $\phi, X \in \tau_\eta$. Suppose that $\{A_i : i \in \Lambda\} \in \tau_\eta$. Then $A_i \cap B \in GPRO(X, \tau)$ for all $B \in GPRO(X, \tau)$. Now $(\cup A_i) \cap B = \cup (A_i \cap B)$. By our assumption ($\mathcal{R}$), $(\cup A_i) \cap B$ is *gpr*-open for all $B \in GPRO(X, \tau)$ and hence $\cup A_i \in \tau_\eta$. Let $C, D \in \tau_\eta$. $(C \cap D) \cap B = C \cap (D \cap B) \in GPRO(X, \tau)$ for all $B \in GPRO(X, \tau)$. Hence, τ_η is closed under arbitrary unions and finite intersections. Thus τ_η forms a topology.

Theorem 3.18 — *For any topological space (X, τ) $\tau_\eta \subset GPRO(X, \tau)$.*

PROOF : Let $A \in \tau_\eta$. Then $A \cap B \in GPRO(X, \tau)$ for every $B \in GPRO(X, \tau)$. In particular $A \cap X = A \in GPRO(X, \tau)$ as $X \in GPRO(X, \tau)$. Hence the inclusion.

Proposition 3.19 — If $PO(X)$ is closed under finite intersections, then $GPRO(X)$ is a topology on X.

The proof follows from ($\mathcal{R}$) and Corollary 3.10.

The following example shows that τ_η need not be a subset of τ.

Example 3.20 — Consider $X = \{a, b, c\}$, $\tau = \{\phi, X, \{a\}, \{b\}, \{a, b\}\}$. Then $GPRO(X) = \{\phi, X, \{a\}, \{b\}, \{c\}, \{a, b\}\} = \tau_\eta \not\subset \tau$.

Remark 3.21 — $\tau \cap \tau_\eta$ is a topology contained in $GPRO(X)$.

Proposition 3.22 — If $PO(X)$ is closed under finite intersections, then $\tau \subset \tau_\eta$.

PROOF : Let $A \in \tau$ and $B \in GPRO(X, \tau)$. By Lemma 2.6, $A \cap B \in GPRO(X, \tau)$. Hence, $A \in \tau_\eta$.

Theorem 3.23 — *The family* GPRO(X, τ) *is a topology if and only if* GPRO$(X, \tau) = \tau_\eta$. *(Sufficiency is proved assuming ($\mathcal{R}$)).*

PROOF : (*Necessity*) Let $A \in GPRO(X, \tau)$. Since it is a topology $A \cap B \in GPRO(X, \tau)$ for every $B \in GPRO(X, \tau)$. Hence, $A \in \tau_\eta$. That is, $GPRO(X, \tau) \subset \tau_\eta$. $\tau_\eta \subset GPRO(X, \tau)$ by Theorem 3.18. Hence, $GPRO(X, \tau) = \tau_\eta$.

(*Sufficiency*) It is obvious when ($\mathcal{R}$) holds.

Remark 3.24 : For any topological space (X, τ), $\tau_\eta \subset \tau_g^*$. Since $\tau_\eta \subset GPRO(X, \tau)$ and $GPRO(X, \tau) \subset \tau_g^*$ the result follows.

Theorem 3.25 — *A subset A of a topological space (X, τ) is closed in (X, τ_η) if and only if $A \cup B$ is gpr-closed for every gpr-closed set B.*

PROOF : (*Necessity*) Let $A \subset X$ be closed in τ_η. Then $X - A \in \tau_\eta$. That is, $(X - A) \cap B \in GPRO(X)$ for every *gpr*-open set B. Hence, $A \cup (X - B)$ is *gpr*-closed for every *gpr*-closed set $X - B$.

(*Sufficiency*) Suppose $A \cup B$ be *gpr*-closed for every *gpr*-closed set B of X. Then $(X - A) \cap (X - B)$ is *gpr*-open for every *gpr*-open set $X - B$. This implies $X - A \in \tau_\eta$. That is, A is closed in τ_η.

Definition 3.26 — For any $A \subseteq X$, *gpr*-int A is defined as the union of all *gpr*-open sets contained in A.

Lemma 3.27 — $(X - gpr\text{-int}(A)) = gpr\text{-cl}(X - A)$.

PROOF : Let $x \in X - (gpr\text{-int}(A))$. Then $x \notin gpr\text{-int}(A)$. That is, every *gpr*-open set B containing x is such that $B \not\subset A$. This implies every *gpr*-open set B containing x intersects $X - A$. That means $x \in gpr\text{-cl}(X - A)$. Hence, $(X - gpr\text{-int}(A)) \subset gpr\text{-cl}(X - A)$. Conversely, let $x \in gpr\text{-cl}(X - A)$. Then every *gpr*-open set C containing x intersects $X - A$. That is, every *gpr*-open set C containing x is such that $C \not\subset A$. This implies $x \notin gpr\text{-int}(A)$. Thus $gpr\text{-cl}(X - A) \subset (X - gpr\text{-int}(A))$. Hence, $(X - gpr\text{-int}(A)) = gpr\text{-cl}(X - A)$.

Similarly, we have $(X - gpr\text{-cl}(A)) = gpr\text{-int}(X - A)$.

Proposition 3.28 — If $GPRO(X, \tau)$ is a topology, then $\tau_n = GPRO(X, \tau) = \tau_g^*$.

PROOF : It is enough if we prove $\tau_g^* \subset GPRO(X, \tau)$. Let $A \in \tau_g^*$. Then $gpr\text{-cl}(X - A) = (X - A)$. By Lemma 3.27 $X - gpr\text{-int}\ A = X - A$ which implies $gpr\text{-int}\ A = A$ which in turn implies $A \in GPRO(X)$. Now the Proportional follows from Theorem 3.23 and definition of τ_g^*.

4. *GPR*-CONTINUOUS FUNCTIONS

Here we have proved a necessary condition for *gpr*-continuous function and has shown that composition of two *gpr*-continuous functions need not be *gpr*-continuous. A pasting Lemma for *gpr*-continuous functions has also been proved. Further the notion of *gpr**-continuous function is introduced.

Definition 4.1[2] — A map $f: (X, \tau) \to (Y, \sigma)$ is called *gpr*-continuous if the inverse image of every closed set in Y is *gpr*-closed in X.

Continuity implies *gpr*-continuity but the converse need not be true.

Remarks 4.2 : (*i*) If $\tau_g^* = \tau$ in X, then continuity and *gpr*-continuity coincide.

(*ii*) Every *gpr*-continuous function defined on a pre-regular $T_{1/2}$ space is pre-continuous.

(*iii*) A map $f: (X, \tau) \to (Y, \sigma)$ is called *gpr*-continuous if and only if the inverse image of every open set in Y is *gpr*-open in X.

Theorem 4.3 — *If $f:(X,\tau)\to(Y,\sigma)$ is* gpr*-continuous,then f(*gpr*-cl(A)) $\subset$* cl *f(A) for every subset A of X.*

PROOF : Let $A\subset X$. Then cl $f(A)$ is closed in Y. By assumption $f^{-1}(\text{cl}\, f(A))$ is *gpr*-closed in Y. And $A\subset f^{-1}(f(A))\subset f^{-1}(\text{cl}\,(f(A))$ implies *gpr*-cl$(A)\subset f^{-1}(\text{cl}\,(f(A))$. Hence, $f($*gpr*-cl$(A)\subset$ cl $f(A)$.

However, the converse does not hold.

Example 4.4 — Let $X=\{1,2,3\}$, $Y=R$, $\tau=\{\phi, X, \{1\}, \{2\}, \{1,2\}\}$ and $\sigma=$ usual topology in R. Define $f:(X,\tau)\to(Y,\sigma)$ by $f(x)=x$ for every $x\in X$. In (X,τ), *gpr*-cl$(A)=A$ for every $A\subset X$. So $f($*gpr*-cl$(A)=f(A)\subset$ cl$f(A))$ for every $A\subset X$. But f is not *gpr*-continuous since $f^{-1}([0,1])=\{1\}$ is not *gpr*-closed in X.

Theorem 4.5 — *(a) Consider the map $f:(X,\tau)\to(Y,\sigma)$. If for each point $x\in X$ and each open set V containing $f(x)$ there exists a gpr-open set U containing x such that $f(U)\subset V$, then for every subset A of X, f (gpr-cl A) $\subset$ cl$(f(A))$ holds.*

(b) The following statements are equivalents

(i) For every subset A of X, f(gpr-cl A) $\subset$ cl$(f(A))$ holds.

(ii) Suppose τ_g^* is a topology. The map of $f:(X,\tau_g^*)\to(Y,\sigma)$ is continuous.

PROOF : *(a)* Let $y\in f$ (*gpr*-cl A). Let V be an open set containing y. Then by hypothesis, there exists an $x\in X$ such that $f(x)=y$ and a *gpr*-open set U containing x such that $f(U)\subset V$ and $x\in$ *gpr*-cl A. Therefore, by Lemma 3.2, $U\cap A\neq\phi$. Then $f(U\cap A)\neq\phi$. This implies $V\cap f(A)\neq\phi$. Hence, $y\in$ cl $f(A)$.

(b) *(i)* $\to$ *(ii)* Let A be closed in (Y,σ). By hypothesis, $f($*gpr*-cl$(f^{-1}(A))\subset$ cl $((ff^{-1}(A))\subset$ cl$(A)=A$. That is, *gpr*-cl $(f^{-1}(A)\subset f^{-1}(A)$. Also $f^{-1}(A)\subset$ *gpr*-cl $(f^{-1}(A))$. Thus $f^{-1}(A)$ is closed in (X,τ_g^*) and so f is continuous.

(ii) $\to$ *(i)* For every subset A of X, cl$(f(A))$ is closed in (Y,σ). Since $f:(X,\tau_g^*)\to(Y,\sigma)$ is continuous, $f^{-1}(\text{cl}(f(A))$ is closed in (X,τ_g^*) and hence *gpr*-cl $f^{-1}(cl\,f(A)))=f^{-1}(\text{cl}\, f(A)))$. Moreover, we have $A\subset f^{-1}(f(A))\subset f^{-1}(\text{cl}(f(A)))$ and by Lemma 3.3, *gpr*-cl$(A)\subset$ *gpr*-cl $(f^{-1}(\text{cl}\,(f(A))=(f^{-1}(\text{cl}(f(A)))$. Therefore, we obtain $f($*gpr*-cl$(A))\subset$ cl$(f(A))$.

We know that composition of two *gpr*-continuous functions need not be *gpr*-continuous. Refer [2, Example 6.10].

Proposition 4.6 — Let X, Y and Z be topological spaces and $\sigma_g^*=\sigma$. Then the composition $g\circ f:(X,\tau)\to(Y,\sigma)$ of the *gpr*-continuous functions $f:(X,\tau)\to(Y,\sigma)$ and $g:(Y,\sigma)\to(Z,\mu)$ is also *gpr*-continuous.

The proof follows from Theorem 3.15 *(ii)*

Regarding the restriction of a *g*-continuous map, we have the following :-

Proposition 4.7 — Let $f:(X,\tau)\to(Y,\sigma)$ be a *gpr*-continuous map and H be a regular open *gpr*-closed subset of X. Assume ($\mathcal{R}$). Then the restriction $f|H:(H,\tau|H)\to(Y,\sigma)$ is *gpr*-continuous.

PROOF : Let A be a closed subset of Y. By hypothesis and our assumption ($\mathcal{R}$), $f^{-1}(A) \cap H = H_1$ (say) is *gpr*-closed in X. Since $(f \mid H)^{-1}(A) = H_1$, we need to show that H_1 is *gpr*-closed in H. Let $H_1 \subset O_1$ where O_1 is a regular open set in H. By Lemma 2.12, $O_1 = O \cap H$ for some regular open set O in X. Then $H_1 \subset O$ implies $p\text{cl}_X H_1 \subset O$. That is, $p\text{cl}_X H_1 \cap H \subset O \cap H$. This implies $p\text{cl}_H H_1 \subset O_1$ by Lemma 2.10. Hence, H_1 is *gpr*-closed in H thereby implying $f \mid H$ is *gpr*-continuous.

Remark 4.8 : By [2, Theorem 3.10], every regular open *gpr*-closed set is clopen. Using Corollary 2.15 in Proposition 4.7, the above proof becomes very simple. Hence in the above Proposition, the assumption H is regular open *gpr*-closed can be replaced with H is clopen.

The assumption H is regular open *gpr*-closed or clopen is essential.

Example 4.9 — Consider $X = \{a, b, c, d\}$, $Y = \{a, c, d\} \subset X, Z = \{a, b, c\}$, $\tau = \{\phi, X, \{b\}, \{a, b\}, \{b, c\}, \{a, b, c\}\}$, $\sigma = \{\phi, Z, \{a\}, \{b\}, \{a, b\}, \{a, c\}\}$. Then $GPRC(Y, \tau \mid Y) = \{\phi, Y, \{d\}, \{a, c\}, \{a, d\}, \{c, d\}\}$. Define $f: (X, \tau) \to (Z, \sigma)$ by $f(a) = c$, $f(b) = a$, $f(c) = b$ and $f(d) = a$. Since every subset of X is *gpr*-closed f is *gpr*-continuous, $\{c\}$ is closed in (Z, σ) but $f^{-1}(\{c\}) = \{a\}$ is not *gpr*-closed in $(Y, \tau \mid Y)$. Hence, $f \mid Y : (Y, \tau \mid Y) \to (Z, \sigma)$ is not *gpr*-continuous.

Here Y is *gpr*-closed but not regular open in X. Also Y is not clopen.

Next we shall generalize the Pasting Lemma for *gpr*-continuous maps.

Theorem 4.10 — *Let $X = A \cup B$ be a topological space with topology τ and Y be a topological space with topology σ. Let GPRO (X, τ) be closed under finite intersections and let $f: (A, \tau \mid A) \to (Y, \sigma)$ and $g: (B, \tau \mid B) \to (Y, \sigma)$ be gpr-continuous maps such that $f(x) = g(x)$ for every $x \in A \cap B$. Suppose that both A and B are open and preclosed in X. Then their combination $f \nabla g: (X, \tau) \to (Y, \sigma)$ defined by $(f \nabla g)(x) = f(x)$ if $x \in A$ and $(f \nabla g)(x) = g(x)$ if $x \in B$ is gpr-continuous.*

PROOF : Let F be any closed set in Y. Then $(f \nabla g)^{-1}(F) = f^{-1}(F) \cup g^{-1}(F) = C \cup D$ where $C = f^{-1}(F)$ and $D = g^{-1}(F)$. Since C is *gpr*-closed in A and A is open preclosed in X, C is *gpr*-closed in X by Proposition 2.13 (*ii*). Similarly, D is *gpr*-closed in X. Also $C \cup D$ is *gpr*-closed in X by hypothesis. Therefore, $(f \nabla g)^{-1}(F)$ is *gpr*-closed in X. Hence, $f \nabla g$ is *gpr*-continuous.

Definition 4.11[2] — A map $f: (X, \tau) \to (Y, \sigma)$ is called *gpr*-irresolute if the inverse image of every *gpr*-closed set in Y is *gpr*-closed in X.

It follows easily from the definition that a map f is *gpr*-irresolute if and only if the inverse image of every gpr-open set in Y is gpr-open in X.

Remark 4.12 : If a map is *gpr*-irresolute then it is *gpr*-continuous but not conversely [2, Example 6.5]

Proposition 4.13 — If $f: (X, \tau) \to (Y, \sigma)$ is *gpr*-continuous and $\sigma_g^* = \sigma$ holds, then f is *gpr*-irresolute.

The proof follows from Theorem 3.15.

Definition 4.14 — A map $f: (X, \tau) \to (Y, \sigma)$ is said to be *gpr**-continuous if the inverse image of every preclosed set in Y is *gpr*-closed in X.

Example 4.15 — The function defined in Example 4.9 is *gpr**-continuous.

Remark 4.16 : The class of all *gpr**-continuous functions lie in-between the class of all *gpr*-irresolute functions and the class of all *gpr*-continuous functions, as seen from the following Proposition.

Proposition 4.17 — Let $f: (X, \tau) \to (Y, \sigma)$ be a function.

(*i*) If f is *gpr*-irresolute, then it is *gpr**-continuous.

(*ii*) If f is *gpr**-continuous, then it is *gpr*-continuous.

The proof is immediate.

Proposition 4.18 — If a bijection $f: (X, \tau) \to (Y, \sigma)$ is regular open *gpr**-continuous, then it is *gpr*-irresolute.

PROOF : Let A be *gpr*-closed in Y. Let $f^{-1}(A) \subset O$ where O is regular open in X. Since f is regular open, $f(O)$ is regular open in Y. $A \subset f(O)$ implies $p\text{cl}\ (A) \subset f(O)$. That is, $f^{-1}\ (p\text{cl}\ A) \subset O$. Since f is *gpr**-continuous, $p\text{cl}(f^{-1}\ (p\text{cl}\ A)) \subset O$ and so $p\text{cl}\ (f^{-1}\ (A)) \subset O$. This shows $f^{-1}\ (A)$ is *gpr*-closed in X. Hence, f is *gpr*-irresolute.

Proposition 4.19 — Let $f: (X, \tau) \to (Y, \sigma)$ be a map and Y be a submaximal (door) space. Then f is *gpr**-continuous if it is *gpr*-continuous.

The proof follows from the fact that every preopen set is open in a submaximal (door) space[8].

Proposition 4.20 — If a bijection $f: (X, \tau) \to (Y, \sigma)$ is regular open *gpr**-continuous and p-open and if X is pre-regular $T_{1/2}$ then Y is pre-regular $T_{1/2}$.

PROOF : Let $A \in GPRC(Y)$. By Proposition 4.18, $f^{-1}(A) \in GPRC(X)$. By hypothesis, $f^{-1}(A) \in PC(X)$. Since f is bijective p-open $A = f(f^{-1}(A)) \in PC(Y)$. That is, Y is pre-regular $T_{1/2}$.

5. *GPR*-COMPACTNESS

GPR-compactness is defined in this section and some of its characterizations are proved.

Definition 5.1 — A collection $\{A_i : i \in \Lambda\}$ of *gpr*-open sets in a topological space X is called a *gpr*-open cover of a subset S if $S \subset \bigcup \{A_i : i \in \Lambda\}$ holds.

Definition 5.2 — A topological space (X, τ) is *GPR*-compact if every *gpr*-open cover of X has a finite subcover.

Definition 5.3 — A subset S of a topological X is said to be *GPR*-compact relative to X, if for every collection $\{A_i : i \in \Lambda\}$ of *gpr*-open subsets of X such that $S \subset \bigcup \{A_i : i \in \Lambda\}$ there exists a finite subset Λ_0 of Λ such that $S \subset \bigcup \{A_i : i \in \Lambda_0\}$.

Definition 5.4 — A subset S of a topological space X is said to be *GPR*-compact if S is *GPR*-compact as a subspace of X.

Proposition 5.5 — A *gpr*-closed subset of a *GPR*-compact space X is *GPR*-compact relative to X.

PROOF : Let A be a *gpr*-closed subset of a *GPR*-compact space X. Then $X - A$ is *gpr*-open. Let Ω be a *gpr*-open cover for A. Then $\{\Omega, X - A\}$ is a *gpr*-open cover for X. Since X is *GPR*-compact, it has a finite subcover, say, $\{P_1, P_2, P_n\} = \Omega_1$. If $X - A \notin \Omega$, then Ω_1 is a finite subcover of A. If $X - A \in \Omega_1$, then $\Omega_1 - (X - A)$ is a subcover of A. Thus A is *GPR*-compact relative to X.

Proposition 5.6 — Let $f:(X,\tau)\to(Y,\sigma)$ be a surjective, *gpr*-continuous map. If X is *GPR*-compact, then Y is compact.

PROOF : Let $\{A_i : i\in\Lambda\}$ be an open cover of Y. Then $\{f^{-1}(A_i) : i\in\Lambda\}$ is a *gpr*-open cover of X. Since X is *GPR*-compact, it has a finite subcover, say, $\{f^{-1}(A_1), f^{-1}(A_2), \dots f^{-1}(A_n)\}$. Surjectiveness of f implies $\{A_1, A_2, \dots A_n\}$ is an open cover of Y and hence Y is compact.

Proposition 5.7 — If a map $f:(X,\tau)\to(Y,\sigma)$ is *gpr*-irresolute and a subset S of X is *GPR*-compact relative to X, then the image $f(S)$ is *GPR*-compact relative to Y.

PROOF : Let $\{A_i : i\in\Lambda\}$ be a collection of *gpr*-open sets in Y such that $f(S)\subset\bigcup\{A_i : i\in\Lambda\}$. Then $S\subset\bigcup\{f^{-1}(A_i) : i\in\Lambda\}$ where $f^{-1}(A_i)$ is *gpr*-open in X for each i. Since S is *GPR*-compact relative to X, there exists a finite subcollection $\{A_1 A_2, \dots A_n\}$ such that $S\subset\bigcup\{f^{-1}(A_i), i=1\text{to } n\}$. That is, $f(S)\subset\bigcup\{A_i, i=1 \text{ to } n\}$. Hence, $f(S)$ is *GPR*-compact relative to Y.

Remark 5.8 : If $U\times V$ is regular open in $X\times Y$, then U is regular open in X and V is regular open in Y. This follows from the fact that $\text{int}_{X\times Y}(\text{cl}_{X\times Y}((U\times V)) = \text{int}_X(\text{cl}_X(U))\times\text{int}_Y(\text{cl}_Y(V))$.

Lemma 5.9 — Let $\pi: X\times Y\to X$ be a projection. If A is *gpr*-closed in X, then $\pi^{-1}(A)=A\times Y$ is *gpr*-closed in $X\times Y$.

PROOF : Suppose $A\times Y\subset O$, where O is regular open in $X\times Y$. Then $O=U\times Y$ where U is regular open in X by Remark 5.8. Since U is a regular open set in X containing A and A is *gpr*-closed in X, we have $\text{pcl}_X A\subset U$. This implies $\text{pcl}_X A\times Y\subset U\times Y$. That is, $\text{pcl}_{X\times Y}(A\times Y)\subset U\times Y$ (Since $\text{pcl}_X A\times Y=(A\cup\text{cl}_X(\text{int}_X(A)))\times Y=(A\times Y)\cup(\text{cl}_X(\text{int}_X(A))\times Y)$ $=(A\times Y)\cup\text{cl}_{X\times Y}\text{int}_{X\times Y}(A\times Y)=\text{pcl}_{X\times Y}(A\times Y)$.

Theorem 5.10 — *If the product space of two non-empty spaces is* GPR-*compact, then each factor space is* GPR-*compact.*

PROOF : Let $X\times Y$ be the product space of the nonempty spaces X and Y and suppose $X\times Y$ is *GPR*-compact. Then the projection $\pi: X\times Y\to X$ is a *gpr*-irresolute map by Lemma 5.9. Using proposition 5.7, we see that $\pi(X\times Y)=X$ is *GPR*-compact. For the space Y, the proof is similar.

6. *GPR*-CONNECTEDNESS

In this section *GPR*-connectedness is defined and it is shown that a proper subset of a *GPR*-connected space cannot be both *gpr*-open and *gpr*-closed. We have also proved that under surjective *gpr*-continuous maps the image of a *GPR*-connected space is connected. Also we have shown that if the product of two nonempty topological space is *GPR*-connected then each factor space is *GPR*-connected.

Definition 6.1 — A topological space (X,τ) is said to be *GPR*-connected if X cannot be written as the disjoint union of two non-empty *gpr*-open sets.

A subset of X is *GPR*-connected if it is *GPR*-connected as a subspace.

Proposition 6.2 — For a topological space X, the following are equivalent :

(*i*) X is *GPR*-connected.

(*ii*) The only subsets of X which are both *gpr*-open and *gpr*-closed are the empty set ϕ and X.

(*iii*) Each *gpr*-continuous map of X into a discrete space Y with atleast two points is a constant map.

PROOF : (*i*) $\rightarrow$ (*ii*) Suppose $S \subset X$ is a proper subset which is both *gpr*-open and *gpr*-closed. Then its complement $X - S$ is also *gpr*-open and *gpr*-closed. Then $X = S \cup (X - S)$, a disjoint union of two non-empty *gpr*-open sets which contradicts the fact that X is *GPR*-connected. Hence, $S = \phi$ or X.

(*ii*) $\rightarrow$ (*i*) Suppose $X = A \cup B$ where $A \cap B = \phi, A \neq \phi, B \neq \phi$ and A and B are *gpr*-open. Since $A = X - B$, A is *gpr*-closed. But by assumption $A = \phi$ or X, which is a contradiction. Hence, (*i*) holds.

(*ii*) $\rightarrow$ (*iii*) Let $f: X \rightarrow Y$ be a *gpr*-continuous map, where Y is a discrete space with atleast two points. Then $f^{-1}(\{y\})$ is *gpr*-closed and *gpr*-open for each $y \in Y$. And $X = \cup \{f^{-1}(\{y\}) : y \in Y\}$. By assumption, $f^{-1}(\{y\}) = \phi$ or X. If $f^{-1}(\{y\}) = \phi$ for all $y \in Y$, f will not be a map. Also there cannot exist more than one $y \in Y$ such that $f^{-1}(\{y\}) = X$. Hence, there exists only one $y \in Y$ such that $f^{-1}(\{y\}) = X$ and $f^{-1}(\{y^1\}) = \phi$ where $y \neq y^1 \in Y$. This shows that f is a constant map.

(*iii*) $\rightarrow$ (*ii*) Let S be both *gpr*-open and *gpr*-closed in X. Suppose $S \neq \phi$. Let $f: X \rightarrow Y$ be a *gpr*-continuous map defined by $f(S) = \{a\}$ and $f(X - S) = \{b\}$ where $a \neq b$ and $a, b \in Y$. By assumption, f is constant. Therefore, $S = X$.

Remark 6.3 : Every *GPR*-connected space is connected but the converse need not be true.

Example 6.4 — Let $X = \{a, b, c\}$, $\tau = \{\phi, X, \{a, b\}, \{a\}\}$. Then (X, τ) is not *GPR*-connected because every subset of X is *gpr*-open. The only clopen sets of X are ϕ and X. Therefore, X is connected.

Proposition 6.5 — If (X, τ) is a topological space with atleast two points and if $PO(X) = PC(X)$, then X is not *GPR*-connected.

PROOF : By Proposition 2.3, there exists a proper subset of (X, τ) which is both *GPR*-open and *gpr*-closed. By Theorem 6.2, (X, τ) is not *GPR*-connected.

Proposition 6.6 — Suppose (X, τ) is a topological space with $\tau_g^* = \tau$. Then X is connected if and only if X is *GPR*-connected.

Proof follows from definitions and Theorem 3.15(*ii*).

Proposition 6.7 — (*i*) If $f: X \rightarrow Y$ is *gpr*-continuous and onto, and X is *GPR*-connected then Y is connected.

(*ii*) If $f: X \rightarrow Y$ is *gpr*-irresolute and onto and X is *GPR*-connected, then Y is *GPR*-connected.

PROOF : (*i*) Assume the contrary. Suppose Y is not connected. Then $Y = A \cup B$ where $A \cap B = \phi$, $A \neq \phi$, $B \neq \phi$, and A, B are open in Y. Since f is *gpr*-continuous and onto $X = f^{-1}(A) \cup f^{-1}(B)$ are disjoint non-empty *gpr*-open subsets of X. This contradicts the fact that X is *gpr*-connected. Hence the result.

(*ii*) Proof is similar to (*i*).

Proposition 6.8 — Let $f: X \to Y$ be a surjective gpr-continuous map. If H is regular open *gpr*-closed and *GPR*-connected and ($\mathcal{R}$) holds then $f(H)$ is *GPR*-connected.

PROOF : The restriction $f \mid H$ is *gpr*-continuous by Proposition 4.7. Using proposition 6.7 (*i*), the image of the *GPR*-connected space $(H, \tau \mid H)$ under $f \mid H : (H, \tau \mid H) \to (f(H), \sigma)$ is connected. Then $f(H)$ is a connected subset of Y.

Theorem 6.9 — *If the product of two nonempty topological space is GPR-connected, then each factor space is GPR-connected.*

PROOF : Let (X, τ) and (Y, σ) be two nonempty topological spaces. By Lemma 5.9, the projection $p : X \times Y \to X$ is *gpr*-irresolute. Using Proposition 6.7 (*ii*), the *gpr*-irresolute image $p(X \times Y)(= X)$ of the *GPR*-connected space $X \times Y$, is *GPR*-connected. Similarly, Y is *GPR*-connected.

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