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MULTIPARAMETRIC ANALYSIS OF THE MAXIMUM TOLERANCE IN LINEAR FRACTIONAL PROGRAMMING PROBLEM

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This paper proposes an alternative approach to multiparametric sensitivity analysis in linear fractional programming problems (LFPP) by using the concept of maximum volume in the tolerance region.

Key Words : Linear Fractional Programming; Sensitivity Analysis; Parametric Analysis; Tolerance Approach

1. INTRODUCTION

Sensitivity analysis is concerned with how changes in the parameters of a problem affect the optimal solution. Sensitivity analysis is important because it often makes it unnecessary to solve the problem again. A knowledge of sensitivity analysis enables the analyst to determine from the original solution, how changes in the parameters of the problem change its optimal solution. Sensitivity analysis and parametric analysis have been carried out for the linear programming problems using different approaches e.g., Gal[5], Bradley, Hax, and Magnanti[1], Wendell[6, 7, 8, 9]. Recently, Wang and Huang[11, 12] proposed the concept of maximum volume in the tolerance region to allow the parameters to be analyzed at their independent levels of sensitivity.

2. MAXIMUM VOLUME REGION WITHIN THE TOLERANCE REGION

Consider a linear programming problem as follows :

(LPP) Minimize $c^T x$

subject to $Ax = b$, $x \geq 0$,

where A is given $m \times n$ matrix with full row rank, b and c represent the estimated column vectors. Let B denote an optimal basis to LPP.

The following is the perturbed model of LPP :

(PLPP) Minimize $(c + \Delta c)^T x$

subject to $(A + \Delta A)\, x = b + \Delta b,\ x \geq 0,$

where $\Delta c = \left(\sum_{h=1}^{H} c_{jh}\, t_h \right)_{n \times 1}$ and $\Delta b = \left(\sum_{h=1}^{H} b_{ih}\, t_h \right)_{m \times 1}$ and

ΔA is an $m \times n$ perturbation matrix in the following forms when the perturbation is considered in the rth column or rth row respectively.

$$\Delta A = \begin{bmatrix} 0 & 0 \cdots & \xi_{1r}(t) \cdots 0 & 0 \\ 0 & 0 \cdots & \xi_{2r}(t) \cdots 0 & 0 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 \cdots & \xi_{mr}(t) \cdots 0 & 0 \end{bmatrix}_{m \times n} \qquad \ldots (1)$$

where $\xi_{ir}(t) = \sum_{h=1}^{H} \alpha_{ih}\, t_h,\ i = 1, 2, \ldots, m$ or

$$\Delta A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \zeta_{r1}(t) & \zeta_{r2}(t) & \cdot & \zeta_{rn}(t) \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 \end{bmatrix}_{m \times n} \qquad \ldots (2)$$

where $\zeta_{rj}(t) = \sum_{h=1}^{H} \gamma_{jh}\, t_h,\ j = 1, 2, \ldots, n,$

are the perturbations defined by the parameter $t = (t_1, \ldots, t_H)^T$.

Let

$u_h = (b_{1h}, \ldots, b_{mh})^T$: vector of the coefficients of parameter t_h in Δb.

$v_h = (c_{1h}, \ldots, c_{mh})^T$: vector of the partial coefficients of parameter t_h in Δc.

$\sigma = \{1, \ldots, m\}$: the index set of optimal basic variables to LPP.

$\bar{\sigma} = \{m + 1, \ldots, n\}$: the index set of the corresponding non basic variables.

Also, let S denotes the critical region of PLPP and for any matrix $\mathbf{K} = [k_{ij}]$, $K_{i.}$ and $K_{.j}$ denote the ith row and jth column respectively.

The main results discussed in Wang and Huang[11, 12] of simultaneous and independent perturbations are given in the following theorems.

Theorem 2.1 — *When b and c are perturbed simultaneously, the critical region S_{bc} of the PLPP is given by*

$$S_{bc} = \left\{ t = (t_1, t_2, ..., t_H)^T \mid \bar{b}_i + \sum_{h=1}^{H} (\beta_{i.}^T u_h)\, t_h \geq 0,\ i = 1, 2, ..., m \right.$$

$$\left. \text{and} \quad \bar{c}_j + \sum_{h=1}^{H} (y_{.j}^T v_h - c_{m+j,h})\, t_h \leq 0, j = 1, 2, ..., n-m \right\}.$$

A parameter t_h is focal iff $u_h \neq 0$ and $y_{.j}^T v_h \neq c_{m+j,h}$ for some $j \in \{1, ..., n - m\}$.

Column Perturbation :

Consider a perturbation on the rth column of A in the form (1) of problem LPP. Let $\alpha_{.h} = [\alpha_{1h}, \alpha_{2h}, ..., \alpha_{mh}]^T$ be the given coefficient vector of parameter t_h. Let the perturbation in the rth column corresponding to non basic variable is denoted by (NC) and corresponding to basic variable is denoted by (BC), then the following result holds true.

Theorem 2.2 — *The critical region S_{NC} of the problem PLPP, when perturbation of A is in the form of (1) and $r \in \bar{\sigma}$, is*

$$S_{NC} = \left\{ t = (t_1, t_2, ..., t_H)^T \mid \bar{c}_{r-m} + \sum_{h=1}^{H} \left[\sum_{i=1}^{m} c_i (\beta_{i.}^T \alpha_{.h})\, t_h \leq 0 \right] \right\}.$$

A parameter t_h is focal if and only if $\sum_{i=1}^{m} c_i (\beta_{i.}^T \alpha_{.h}) \neq 0.$

Theorem 2.3 — *The critical region S_{BC} of the problem PLPP, when perturbation of A is in the form of (1) and $r \in \sigma$, is*

$$S_{BC} = \left\{ t = (t_1, t_2, ..., t_H)^T \mid 1 + \sum_{h=1}^{H} (\beta_{r.}^T \alpha_{.h})\, t_h \neq 0; \right.$$

$$\left[\bar{b}_i + \sum_{h=1}^{H} (\bar{b}_i (\beta_{r.}^T \alpha_{.h}) - \bar{b}_r (\beta_{i.}^T \alpha_{.h}))\, t_h \right] \Big/$$

$$\left. \left[1 + \sum_{h=1}^{H} (\beta_{r.}^T \alpha_{.h})\, t_h \right] \geq 0,\ i = 1, 2, ..., m \text{ and} \right.$$

$$\left[\bar{c}_j + \sum_{h=1}^{H} \left(\bar{c}_j (\beta_{r\cdot}^T \alpha_{\cdot h}) - y_{rj} \sum_{i=1}^{m} c_i (\beta_{i\cdot}^T \alpha_{\cdot h}) \right) t_h \right] \Bigg/$$

$$\left[1 + \sum_{h=1}^{H} (\beta_{r\cdot}^T \alpha_{\cdot h}) t_h \right] \leq 0, \quad j = 1, 2, \ldots, n - m \Bigg\}.$$

Thus, parameter t_h *is nonfocal if and only if*

$$\beta_{r\cdot}^T \alpha_{\cdot h} = 0, \quad \bar{b}_i (\beta_{r\cdot}^T \alpha_{\cdot h}) - \bar{b}_r (\beta_{i\cdot}^T \alpha_{\cdot h}) = 0 \text{ for } i = 1, 2, \ldots, m$$

and $$\bar{c}_j (\beta_{r\cdot}^T \alpha_{\cdot h}) - y_{rj} \sum_{i=1}^{m} c_i (\beta_{i\cdot}^T \alpha_{\cdot h}) = 0 \text{ for } j = 1, 2, \ldots, n - m.$$

The above region can be decomposed in the following forms :

$$S_{d+} = \left\{ t = (t_1, t_2, \ldots, t_H)^T \mid 1 + \sum_{h=1}^{H} (\beta_{r\cdot}^T \alpha_{\cdot h}) t_h > 0 \right\}$$

$$S_{d-} = \left\{ t = (t_1, t_2, \ldots, t_H)^T \mid 1 + \sum_{h=1}^{H} (\beta_{r\cdot}^T \alpha_{\cdot h}) t_h < 0 \right\}$$

$$S_{b+} = \left\{ t \mid \bar{b}_i + \sum_{h=1}^{H} (\bar{b}_i (\beta_{r\cdot}^T \alpha_{\cdot h}) - \bar{b}_r (\beta_{i\cdot}^T \alpha_{\cdot h})) t_h \geq 0, \quad i = 1, 2, \ldots, m \right\}$$

$$S_{b-} = \left\{ t \mid \bar{b}_i + \sum_{h=1}^{H} (\bar{b}_i (\beta_{r\cdot}^T \alpha_{\cdot h}) - \bar{b}_r (\beta_{i\cdot}^T \alpha_{\cdot h})) t_h \leq 0, \quad i = 1, 2, \ldots, m \right\}$$

$$S_{c+} = \left\{ t \mid \bar{c}_j + \sum_{h=1}^{H} \left(\bar{c}_j (\beta_{r\cdot}^T \alpha_{\cdot h}) - y_{rj} \sum_{i=1}^{m} c_i (\beta_{i\cdot}^T \alpha_{\cdot h}) \right) t_h \geq 0, \quad \forall j \right\}$$

$$S_{c-} = \left\{ t \mid \bar{c}_j + \sum_{h=1}^{H} \left(\bar{c}_j (\beta_{r\cdot}^T \alpha_{\cdot h}) - y_{rj} \sum_{i=1}^{m} c_i (\beta_{i\cdot}^T \alpha_{\cdot h}) \right) t_h \leq 0, \quad \forall j \right\}.$$

Then

$$S_{BC} = \left\{ t = (t_1, \ldots, t_H)^T \mid t \in \left\{ S_{d+} \cap S_{b+} \cap S_{c-} \right\} \text{ or } t \in \left\{ S_{d-} \cap S_{b-} \cap S_{c+} \right\} \right\}.$$

Row Perturbation :

Consider a perturbation on the rth row of A in the form (2) of problem LPP. Let $\gamma_{.h} = [\gamma_{1h}, \gamma_{2h}, ..., \gamma_{mh}]^T$ be the given partial coefficient vector of parameter t_h, then the following result holds true.

Theorem 2.4 — *The critical region $S_{R'}$ of the problem PLPP when perturbation of A is in the form of (2) is*

$$S_{R'} = \left\{ t = (t_1, t_2, ..., t_H)^T \mid 1 + \sum_{h=1}^{H} (\beta_{r.}^T \gamma_{.h})\, t_h \neq 0 ; \right.$$

$$\left[\bar{b}_i + \sum_{h=1}^{H} (\bar{b}_i (\beta_{.r}^T \gamma_{.h}) - \beta_{ir} (\bar{b}^T \gamma_{.h}))\, t_h \right] \Big/ \left[1 + \sum_{h=1}^{H} (\beta_{.r}^T \gamma_{.h})\, t_h \right] \geq 0,$$

$i = 1, 2, ..., m$ *and*

$$\left[\bar{c}_j + \sum_{h=1}^{H} (\bar{c}_j (\beta_{.r}^T \gamma_{.h}) + \gamma_{m+j,h} (\beta_{.r}^T c_B) - (\beta_{.r}^T c_B) (y_{.j}^T \gamma_{.h}))\, t_h \right] \Big/$$

$$\left. \left[1 + \sum_{h=1}^{H} (\beta_{.r}^T \gamma_{.h}))\, t_h \right] \leq 0, \; j = 1, 2, ..., n - m \right\}.$$

Thus, parameter t_h is nonfocal if and only if

$$\beta_{.r}^T \gamma_{.h} = 0, \; \bar{b}_i (\beta_{.r}^T \gamma_{.h}) - \beta_{ir} (\bar{b}^T \gamma_{.h}) = 0 \text{ for } i = 1, 2, ..., m \text{ and}$$
$$\bar{c}_j (\beta_{.r}^T \gamma_{.h}) + \gamma_{m+j,h} (\beta_{.r}^T c_B) - (\beta_{.r}^T c_B) (y_{.j}^T \gamma_{.h}) = 0 \text{ for } j = 1, ..., n - m.$$

The above region can be decomposed in the following form :

$$S_{dp} = \left\{ t = (t_1, t_2, ..., t_H)^T \mid 1 + \sum_{h=1}^{H} (\beta_{r.}^T \gamma_{.h})\, t_h > 0 \right\}$$

$$S'_{dn} = \left\{ t = (t_1, t_2, ..., t_H)^T \mid 1 + \sum_{h=1}^{H} (\beta_{.r}^T \gamma_{.h})\, t_h < 0 \right\}$$

$$S_{bp} = \left\{ t \mid \bar{b}_i + \sum_{h=1}^{H} (\bar{b}_i (\beta_{.r}^T \gamma_{.h}) - \beta_{ir} (\bar{b}^T \gamma_{.h}))\, t_h \geq 0, \; i = 1, 2, ..., m \right\}$$

$$S_{bn} = \left\{ t \mid \bar{b}_i + \sum_{h=1}^{H} (\bar{b}_i (\beta_{r.}^T \gamma_{.h}) - \beta_{ir} (\bar{b}^T \gamma_{.h}))\, t_h \leq 0, \; i = 1, 2, ..., m \right\}$$

$$S_{cp} = \left\{ t \mid \bar{c}_j + \sum_{h=1}^{H} [\bar{c}_j (\beta^T_{\cdot r} \gamma_{\cdot h}) + \gamma_{m+j,h} (\beta^T_{\cdot r} c_B) - (\beta^T_{\cdot r} c_B) (y^T_{\cdot j} \gamma_{\cdot h})] t_h \geq 0, \forall j \right\}$$

$$S_{cn} = \left\{ t \mid \bar{c}_j + \sum_{h=1}^{H} [\bar{c}_j (\beta^T_{\cdot r} \gamma_{\cdot h}) + \gamma_{m+j,h} (\beta^T_{\cdot r} c_B) - (\beta^T_{\cdot r} c_B) (y^T_{\cdot j} \gamma_{\cdot h})] t_h \leq 0, \forall j \right\}$$

Then

$$S_{R'} = \left\{ t = (t_1, ..., t_H)^T \mid t \in \left\{ S_{dp} \cap S_{bp} \cap S_{cn} \right\} \text{ or } t \in \left\{ S_{dn} \cap S_{bn} \cap S_{cp} \right\} \right\}.$$

Definition 2.1 (Wang and Huang)[12] — The maximum volume region (MVR) B_S in a critical region is given by

$$B_S = \left\{ t = (t_1, t_2, ..., t_H)^T \mid\mid t_j \mid \leq k_j^*, \ j = 1, 2, ..., H \right\},$$

where $k^* = (k_1^*, k_2^*, ..., k_H^*)^T$ is the optimal solution of the following maximization problem

(MP) Maximize $V(k) = k_1 k_2 \ldots k_H$ subject to $k \in K(S)$,

where

$$K(S) = \left\{ k = (k_1, k_2, ..., k_H)^T \mid\mid t_j \mid \leq k_j, \ j = 1, 2, ..., H, \text{ implies } t = (t_1, t_2, ..., t_H)^T \in S \right\}$$

Volume of $B_S = 2^H \cdot k_1^* k_2^* \ldots k_H^*$.

Gal[5] has shown that critical region of the perturbed problem PLPP is a polyhedral set. That is, there exists $P = [p_{ij}] \in R^{I \times J}$ and $q = [q_i] \in R^I$, $I, J \in N$ represents the number of constraints and variables of S respectively such that

$$S = \left\{ t = (t_1, t_2, ..., t_J)^T \mid Pt \leq q \right\}.$$

While deleting all the nonfocal parameters of S we can assume that $p_{\cdot j} \neq 0$ for $j = 1, 2, ..., J$ and also we have $q \geq 0$.

Wang and Huang[11, 12] have obtained the following equivalent form of $K(S)$:

$$K(S) = \left\{ k = (k_1, k_2, ..., k_J)^T \mid\mid P \mid k \leq q, \ \ k \geq 0 \right\}.$$

The maximum volume region (MVR) denoted by B_S of a polyhedral set $S = \{t \mid Pt \leq q\}$ with $q > 0$ and $p_{\cdot j} \neq 0$ for all j is bounded and is defined by

$$B_S = \left\{ t = (t_1, t_2, ..., t_J)^T \mid\mid t_j \mid \leq k_j^*, j = 1, 2, ..., J \right\},$$

where $k^* = (k_1^*, k_2^*, ..., k_J^*)^T$ is the unique optimal solution of the following problem

(MP′) Maximize $V(k) = k_1 k_2 \ldots k_J$ subject to $\mid P \mid k \leq q, \ \ k \geq 0$.

Remark : When perturbation is considered in one column of the constraint matrix that corresponds to a non basic variable, the MVR can be obtained by solving (MP'). In case of perturbation in the column of a basic variable or in one row of the constraint matrix, the critical region can be neither closed nor connected, and hence there are problems in finding the MVR.

Definition 2.2 — (Wang and Huang[11]) — The symmetrically rectangular parallelepiped

$$B_{\hat{\psi}} = \left\{ t = (t_1, t_2, ..., t_H)^T \mid\mid t_j \mid \leq \bar{\lambda}_j,\ j = 1, 2, ..., H \right\}$$

of a region ψ is said to be Weak Maximal Volume Region (WMVR) of ψ if $\bar{v} = \bar{\lambda}_1 \bar{\lambda}_2 \ldots \bar{\lambda}_H$ is the least upper bound of the set

$$v = \left\{ \lambda_1 \lambda_2 \ldots \lambda_H \mid\mid t_j \mid \leq \lambda_j,\ j = 1, ..., H \text{ implies } t = (t_1, ..., t_H)^T \in \psi \right\}.$$

Also, if $B_{\hat{\psi}}$ is a subset of ψ, that is, $\bar{v}$ is also the maximum, then $B_{\hat{\psi}}$ is a Strong Maximal Volume Region (SMVR) of ψ and denote it by B_{ψ}. From Definition 2.2, it follows that MVR in case of S_b, S_c or S_{bc} is a SMVR and the volume of B_{ψ} is

$$\text{Vol } (B_{\psi}) = \text{Vol } (B_{\psi'}) = 2^H \bar{\lambda}_1 \bar{\lambda}_2 \ldots \bar{\lambda}_H$$

and it is the maximal volume.

The following results relates the WMVR and SMVR in case of perturbations corresponding to column of a basic variable or in one row of the constraint matrix :

Theorem 2.5 — $\left\{ k = (k_1, ..., k_H)^T \mid\mid t_j \mid \leq k_j,\ j = 1, ..., H \Rightarrow t = (t_1, ..., t_H)^T \in S_{BC} \right\}$

$$= \left\{ k \mid\mid t_j \mid \leq k_j, j = 1, ..., H \Rightarrow t \in S_R = S_{d+} \cap S_{b+} \cap S_{c-} \right\}$$

and both sets are empty.

Theorem 2.6 — *The WMVR of S_{BC} or S_R' is the same as the WMVR of S_R. The WMVR of S_{BC} is also a SMVR of S_{BC} if and only if WMVR of S_R is also a SMVR of S_R. Let*

$$B_{S_R} = \left\{ t = (t_1, ..., t_H)^T \mid\mid t_j \mid \leq \bar{k}_j, j = 1, ..., H \right\}$$

be a WMVR of S_R, then B_{S_R} is a SMVR of S_R if and only if

$$\left\{ t = (t_1, ..., t_H)^T \mid\mid t_j \mid = \bar{k}_j, j = 1, ..., H \right\} \subseteq S_{d+}.$$

3. MAXIMUM TOLERANCE IN LINEAR FRACTIONAL PROGRAMMING PROBLEM (LFPP)

Consider the linear fractional programming problem as follows :

(LFPP) Maximize $f(x) = \dfrac{cx + \alpha}{dx + \beta}$

subject to $x \in K$, where $K = \{x \in R^n, Ax \leq b, x \geq 0\}$ and A is an $m \times n$ matrix, b is an $m \times 1$ vector, c, d and x are $n \times 1$ vectors and α, β are scalars.

Charnes and Cooper[2] replaced the given LFPP by the following two linear programming problems using the transformation $y = sx$, where s is a scalar;

(P1) Maximize $cy + \alpha s$ subject to

$$Ay - bs \leq 0,$$

$$dy + \beta s = 1$$

$$y, s \geq 0$$

(P2) Maximize — $(cy + \alpha s)$ subject to

$$Ay - bs \leq 0$$

$$dy + \beta s = -1$$

$$y, s \geq 0.$$

They further established that if the denominator $(dx + \beta) > 0$, $\forall x \in K$ in LFPP, then problem (P2) is infeasible and if $(dx + \beta) < 0$, $\forall x \in K$ in LFPP, then problem (P1) is infeasible. Zionts[13] has shown that if the LFPP has a finite solution, then whether a feasible solution to LFPP has positive or negative denominator, it is sufficient to solve only one of the problems (P1) or (P2) depending on the sign of denominator.

Sensitivity analysis is very useful for the linear fractional programming problems (LFPP), the concept of which has been used in a number of practical situations such as agricultural programming, investment portfolios, models for minimizing wastage, economic models etc. Wolf[10] has carried out the analysis with a scalar parameter on the right-hand side of the given LFPP. Charnes and Neralic[3] discussed the sensitivity analysis in data envelopment analysis (DEA) for the CCR ratio model. Dutta, Rao, and Tiwari[4] discussed the sensitivity analysis for LFPP using the tolerance approach proposed by Wendell[8, 9].

In this paper, the analysis has been done for a multi-parametric single objective linear fractional programming problem. We adopt the approach of Wang and Huang[11, 12] to classify the parameters as focal and nonfocal. The nonfocal parameters are assumed to have unlimited variation because of their less sensitive nature, and are thus deleted from the analysis. Focal parameters are analyzed on their different levels of sensitivity and a maximum volume region within the tolerance region is characterized for them. The results are compared with Dutta *et al.*[4]

Consider the given LFPP. The following steps are used to characterize the tolerance region for the perturbation of only the focal parameters of the given LFPP.

Step 1

Determine the equivalent linear programming problem of the given LFPP as follows :

If denominator $(dx + \beta) > 0$ $\forall x \in K$, then problem (P1) is feasible. Let $(\bar{y}, \bar{s})$ be an optimal solution of (P1). The optimal solution for the LFPP is then obtained by setting $\bar{x} = (1/\bar{s})\,\bar{y}$ $(\bar{s} \neq 0)$.

Similarly, if $(\boldsymbol{dx}+\beta)<0$, $\forall \boldsymbol{x} \in K$, then the problem (P2) is an equivalent linear programming problem of LFPP.

Step 2

In case of equality constraint in the problem (P1) or (P2), an artificial vector $\boldsymbol{M}$ is obtained. Also in the optimal tableau, the artificial vector are included in order to have $\boldsymbol{B}^{-1}$ for the current basis $\boldsymbol{B}$.

Step 3

Construct the critical region by considering perturbation in objective function coefficients, right-hand side and in the constraint matrix of the equivalent linear programming problem which contains the terms of the given LFPP. Delete all the nonfocal parameters and keep only the focal parameters in the analysis.

Step 4

Solve the problem (MP′) to get the maximum volume within the tolerance region using the results given in section 2.

4. EXAMPLE

Consider the problem taken by Dutta *et al.*[4]

$$\text{(LFPP)} \quad \text{Maximize } Z=\frac{5x_1+3x_2}{5x_1+2x_2+1}$$

subject to

$$3x_1+5x_2 \le 15$$

$$5x_1+2x_2 \le 10$$

$$x_1 \ge 0, x_2 \ge 0.$$

Using the transformation $\boldsymbol{y}=s\boldsymbol{x}$ (s is scalar) and taking into account that $(\boldsymbol{dx}+\beta)>0 \ \forall \boldsymbol{x}$, in the given LFPP, the equivalent linear programming problem of the given LFPP is :

$$\text{(P1)} \quad \text{Maximize } Z_1=5y_1+3y_2+0s$$

$$\text{or Minimize } Z_1=-5y_1-3y_2-0s$$

subject to

$$3y_1+5y_2-15s \le 0$$

$$5y_1+2y_2-10s \le 0$$

$$5y_1 + 2y_2 + s = 1$$

$$y_1 \geq 0,\ y_2 \geq 0,\ s \geq 0.$$

The optimal solution of (P1) is $Z_1^* = 1.2857, y_1 = 0,\ y_2 = 0.429,\ s = 0.143$. Therefore, the optimal solution of LFPP is $Z^* = 1.2857,\ x_1 = 0, x_2 = 3.0000007$.

The initial and optimal tables are as follows :

Initial Simplex Table

Basis	C_B	c_i / b	5 / y_1	3 / y_2	0 / s	0 / y_3	0 / y_4	$-M$ / y_5
y_3	0	0	3	5	− 15	1	0	0
y_4	0	0	5	2	− 10	0	1	0
y_5	$-M$	1	5	2	1	0	0	1

Optimal Simplex Table

Basis	c_B	c_i / b	3 / $y_2(z_1)$	0 / $s(z_2)$	0 / $y_4(z_3)$	5 / $y_1(z_4)$	0 / $y_3(z_5)$	− 5000 / $y_5(z_6)$
$y_2(z_1)$	3	.4285714	1	0	0	2.2285714	.0285714	.4285714
$s(z_2)$	0	.1428571	0	1	0	.5428571	−.0571428	.1428571
$y_4(z_3)$	0	.5714285	0	0	1	5.9714286	−.6285714	.5714285
		$\bar{c}_j$	0	0	0	−1.685714	−.0857142	−5001.285

Consider the following perturbation on $\boldsymbol{b}$ and $\boldsymbol{c}$.

$$\Delta \boldsymbol{b} = [2t_1 + 5t_2 + 0t_3,\ 0t_1 + 0t_2 + t_3,\ t_1 - t_2 + t_3]^T$$

$$\Delta \boldsymbol{c} = [5t_1 - 3t_2 + 3t_3 - 6t_4,\ 10t_1 + 5t_2 - 2t_3 + 5t_4,\ 5t_1 - t_2 - t_3 - 2t_4,$$

$$46t_1 - 10.9428573t_2 - .3714286t_3 - 22.6000001t_4,$$

$$-3t_1 - 1t_2 + .4828566t_3 + .7999504t_4,$$

$$6t_1 - t_2 + .4285715t_3 - 2.9999999t_4]^T$$

$$\bar{b} = [.4285714, .1428571, .5714285]^T$$

$$u_1 = [2, 0, 1]^T,\ u_2 = [5, 0, -1]^T,\ u_3 = [0, 1, 1]^T,\ u_4 = [0, 0, 0]^T$$

$$v_1 = [5, 10, 5]^T,\ v_2 = [-3, 5, -1]^T,\ v_3 = [3, -2, -1]^T,\ v_4 = [-6, 5, -2]^T$$

$$\beta = B^{-1} = \begin{bmatrix} .0285714 & 0 & .4285714 \\ -.0571428 & 0 & .1428571 \\ -.6285714 & 1 & .5714285 \end{bmatrix}$$

$$Y = B^{-1}N = \begin{bmatrix} 2.2285714 & .0285714 & .4285714 \\ .5428571 & -.0571528 & .1428571 \\ 5.9714286 & -.6285714 & .5714285 \end{bmatrix}$$

The critical region where B remains optimal for $n = 6$ and $m = 3$ is given by

$$S_{bc} = \left\{ t = (t_1, t_2, t_3, t_4)^T \mid \bar{b}_i + \sum_{h=1}^{4} (\beta_{i.}^T u_h)\, t_h \geq 0,\ i = 1, 2, 3 \right\}$$

$$\left\{ \bar{c}_j + \sum_{h=1}^{4} (y_{.j}^T v_h - c_{3+j,h})\, t_h \leq 0,\ j = 1, 2, 3 \right\}$$

$i = 1$; $\beta_{1.}^T u_1 = .48571424$, $\beta_{1.}^T u_2 = -.2857144$, $\beta_{1.}^T u_3 = .4285714$

$i = 2$; $\beta_{2.}^T u_1 = .0285715$, $\beta_{2.}^T u_2 = -.4285711$, $\beta_{2.}^T u_3 = .1428571$

$i = 3$; $\beta_{3.}^T u_1 = -.6857143$, $\beta_{3.}^T u_2 = -3.7142855$, $\beta_{3.}^T u_3 = 1.5714285$

$j = 1$;

$y_{.1}^T v_1 - c_{41} = .428571$, $y_{.1}^T v_2 - c_{42} = 1$, $y_{.1}^T v_3 - c_{43} = 0$, $y_{.1}^T v_4 - c_{44} = 0$

$j = 2$;

$3y_{.2}^T v_1 - c_{51} = .571528$, $y_{.2}^T v_2 - c_{52} = .00005$, $y_{.2}^T v_3 - c_{53} = 0.345735$, $y_{.2}^T v_4 - c_{54} = 0$

$j = 3$;

$y_{.3}^T v_1 - c_{61} = .4285705$, $y_{.3}^T v_2 - c_{62} = .1428572$, $y_{.3}^T v_3 - c_{63} = 0$, $y_{.3}^T v_4 - c_{64} = 0$.

Since $y_{j}^T v_4 - c_{3+j,4} = 0$ for all j and $u_4 = 0$, parameter t_4 is taken as nonfocal and thus can be varied unlimitedly. The critical region defined by three focal parameters t_1, t_2, and t_3 is given by

$$S = \{ t = (t_1, t_2, t_3)^T \mid .428571t_1 + t_2 + 0t_3 \leq 1.6857143,$$

$$.571528t_1 + .00005t_2 + 0.345735t_3 \leq .0857142,$$

$$.4285705t_1 + .1428572t_2 + 0t_3 \leq 5001.285,$$

$$.48571424t_1 - .2857144t_2 + .4285714t_3 \leq .4285714,$$

$$.0285715t_1 - .4285711t_2 + .1428571t_3 \leq .1428571,$$

$$-.6857143t_1 - 3.7142855t_2 + 1.5714285t_3 \leq .5714285\}.$$

Form (MP′) to obtain the MVR of S_c when perturbation is on c only i.e. $\Delta b = 0$:

(MP′) Maximize $V(k) = k_1 \cdot k_2 \cdot k_3$ subject to

$$.428571k_1 + k_2 + 0k_3 \leq 1.6857143,$$

$$.571528k_1 + .00005k_2 + 0.345735k_3 \leq .0857142,$$

$$.4285705k_1 + .1428572k_2 + 0k_3 \leq 5001.285,$$

$$k_1, k_2, k_3 \geq 0.$$

The optimal solution is $k^* = (.0742, 1.6539, .125)^T$.

$$\text{MVR } B_S = \{t = (t_1, t_2, t_3)^T \mid |t_1| \leq .0742, |t_2| \leq 1.6539, |t_3| \leq .125\},$$

$$\text{Vol } (B_S) = 2^3 . (.0742) \cdot (1.6539) \cdot (.125) = 0.123 \text{ with } t_4 \in R.$$

Comparison 1

The tolerance region obtained by Dutta *et al.*[4] using Wendell's[8,9] approach for objective function perturbation (corresponding perturbation in the numerator of the objective function of LFPP) involving all the four parameters at equal levels of sensitivity is

$$B_T = \{t = (t_1, t_2, t_3, t_4)^T \mid |t_1| \leq .019, |t_2| \leq .019, |t_3| \leq .019, |t_4| \leq .019\}.$$

The volume of the above region in the reduced three dimensional space is $2^3 \cdot (.019) \cdot (.019) \cdot (.019) = 000055$, which is less than Vol (B_S). Also, in the analysis done by Dutta *et al.*[4] for LFPP, all the parameters were given the same tolerance region irrespective of their different levels of sensitivity while the approach adopted in this paper enables the parameters to be analyzed at their independent levels of sensitivity e.g. $t_4 \in R$.

Form (MP′) to obtain the MVR of S_b when perturbation is on b only i.e. $\Delta c = 0$:

(MP′) Maximize $V(k) = k_1 \cdot k_2 \cdot k_3$ subject to

$$.4857142k_1 + .2857144k_2 + .4285714k_3 \leq .4285714,$$

$$.0285715k_1 + .4285711k_2 + .1428571k_3 \leq .1428571,$$

$$.6857143k_1 + 3.7142855k_2 + 1.5714285k_3 \leq .5714285$$

$$k_1, k_2, k_3 \geq 0.$$

The optimal solution is $k^* = (.278, .051, .121)^T$. Therefore,

$$\text{MVR } B_S = \{t = (t_1, t_2, t_3)^T \mid |t_1| \le .278, |t_2| \le .051, |t_3| \le .121\},$$

$$\text{Vol } (B_S) = 2^3 \cdot (.278) \cdot (.051) \cdot (.121) = 0.0137 \text{ with } t_4 \in R$$

Form (MP′) to obtain MVR of S_{bc} when simultaneous perturbations on b and c is considered :

(MP′) Maximize $V(k) = k_1 \cdot k_2 \cdot k_3$ subject to

$$.428571k_1 + k_2 + 0k_3 \le 1.6857143,$$

$$.571528k_1 + .00005k_2 + 0.345735k_3 \le .0857142,$$

$$.4285705k_1 + .1428572k_2 + 0k_3 \le 5001.285,$$

$$.4257142k_1 + .2857144k_2 + .4285714k_3 \le .4285714,$$

$$.0285715k_1 + .4285711k_2 + .1428571k_3 \le .1428571,$$

$$.6857143k_1 + 3.7142855k_2 + 1.5714285k_3 \le .5714285$$

$$k_1, k_2, k_3 \ge 0.$$

The optimal solution of is $k^* = (.0896, .095, .0998)^T$. Therefore,

$$\text{MVR } B_S = \{t = (t_1, t_2, t_3)^T \mid |t_1| \le .0896, |t_2| \le .095, |t_3| \le .0998\}$$

$$\text{Vol } (B_S) = 2^3 \cdot (.0896) \cdot (.095) \cdot (.0998) = 0.0068 \text{ with } t_4 \in R.$$

To consider perturbation in the denominator of the objective function of the given LFPP, apply results of row perturbation of section 2 on the equivalent linear programme (P1).

We have $\sigma = \{1, 2, 3\}$, $\bar{\sigma} = \{4, 5, 6\}$, $m = 3$, $n = 6$. Assume the following perturbation on the third row :

$$[2t_1 + 3t_2 + t_3, t_1 + 5t_2, 0, 2t_1 + 4t_2 - t_3, 0, -t_1 + 3t_2 + 5t_3].$$

Here, $\gamma_{.1} = (2, 1, 0)$, $\gamma_{.2} = (3, 5, 0)$, $\gamma_{.3} = (1, 0, 0)$, $\gamma_{41} = 2$,

$$\gamma_{42} = 4, \gamma_{43} = -1, \gamma_{51} = 0, \gamma_{52} = 0, \gamma_{53} = 0, \gamma_{61} = -1, \gamma_{62} = 3, \gamma_{63} = 5.$$

The critical region S_R with focal parameters t_1, t_2 and t_3 where the basic variables z_1, z_2, z_3 remain optimal, is

$$S = \left\{ t = (t_1, t_2, \ldots, t_H)^T \mid t \in \left\{ S_{dp} \cap S_{bp} \cap S_{cn} \right\} \text{ or } t \in \left\{ S_{dn} \cap S_{bn} \cap S_{cp} \right\} \right\}$$

$$= \{t = (t_1, t_2, t_{3)}^T \mid 1 + 1.001t_1 + .429t_2 + 2.002t_3 > 0,$$

$$.429 + .4t_1 + .23t_2 + .172t_3 \geq 0, .143 + .2t_1 + .4t_2 + .08t_3 \geq 0,$$

$$.571 + 1.2t_1 + 2.4t_2 + .51t_3 \geq 0, -1.69 - 2.18t_1 + 3.6t_2 + 3.4t_3 \leq 0,$$

$$-.086 + .0002t_1 + .0006t_2 + .0005t_3 \leq 0,$$

$$-5001.3 - 4.8t_1 + 990.2t_2 - 145.4t_3 \leq 0\}$$

or $\{t = (t_1, t_2, t_3)^T \mid 1 + 1.001t_1 + .429t_2 + 2.002t_3 < 0,$

$$.429 + .4t_1 + .23t_2 + .172t_3 \leq 0, .143 + .2t_1 + .4t_2 + .08t_3 \leq 0,$$

$$.571 + 1.2t_1 + 2.4t_2 + .51t_3 \leq 0, -1.69 - 2.18t_1 + 3.6t_2 + 3.4t_3 \geq 0,$$

$$-.086 + .0002t_1 + .0006t_2 + .0005t_3 \geq 0,$$

$$-5001.3 - 4.8t_1 + 990.2t_2 - 145.4t_3 \geq 0\}.$$

The maximum tolerance obtained by Dutta *et al.*[4] using Wendell's[9] approach is .138. Therefore, the tolerance region for the parameters t_1, t_2 and t_3 is

$$B_T = \left\{ t = (t_1, t_2, t_3)^T \mid \mid t_1 \mid \leq .138, \mid t_2 \mid \leq .138, \mid t_3 \mid \leq .138 \right\}$$

with

$$\text{Vol } (B_T) = 2^3 \cdot (.138) \cdot (.138) \cdot (.138) = .021.$$

Form (MP′) to obtain WMVR of S_R when perturbation on third row is considered :

(MP′) Maximize $V(k) = k_1 \cdot k_2 \cdot k_3$ subject to

$$1.001k_1 + .429k_2 + 2.002k_3 < 1, .4k_1 + .23k_2 + .172k_3 \leq .429,$$

$$.2k_1 + .4k_2 + .08k_3 \leq .143, 1.2k_1 + 2.4k_2 + .51k_3 \leq .571,$$

$$2.18k_1 + 3.6k_2 + 3.4k_3 \leq 1.69, .0002k_1 + .0006k_2 + .0005k_3 \leq .086,$$

$$4.8k_1 + 990.2k_2 + 145.4k_3 \leq 5001.3, k_1, k_2, k_3 \geq 0.$$

Comparison 2

The optimal solution is $k^* = (.17, .08, .29)$. Therefore, WMVR of S_R is

$$B_{S_R} = \left\{ t = (t_1, t_2, t_3)^T \mid \mid t_1 \mid \leq .17, \mid t_2 \mid \leq .08, \mid t_3 \mid \leq .29 \right\}$$

with volume $2^3 \cdot (.17) \cdot (.08) \cdot (.29) = .035 > \text{Vol}(B_T)$.

Also, all the three parameters are analyzed at their independent levels of sensitivity. Since

$$\left\{ t = (t_1, t_2, t_3)^T \mid | t_1 | = .17, | t_2 | = .08, | t_3 | = .29 \right\} \subseteq S_{dp},$$

therefore by Theorem 2.6, the WMVR of S_R is also the SMVR.

To consider perturbation in the right-hand side of the given LFPP, apply results of section 2 corresponding to the perturbation in the column of basic variable s of the constraint matrix of problem (P1).

Assume the following perturbation in column of basic variable s $[t_1 + 2t_2 + 3t_3, t_2 + t_3, t_1 + t_3]^T$.

The critical region S_{BC} with focal parameters t_1, t_2 and t_3 where the basic variables z_1, z_2, z_3 remain optimal, is

$$S = \left\{ t = (t_1, t_2, \ldots, t_H)^T \mid t \in \left\{ S_d + \cap S_b + \cap S_c - \right\} \text{ or } t \in \left\{ S_d - \cap S_b - \cap S_c + \right\} \right\}$$

$$= \{ t = (t_1, t_2, t_3)^T \mid 1 + .086t_1 - .114t_2 - .028t_3 > 0,$$

$$.429 - .028t_1 - .057t_2 - .086t_3 \geq 0,$$

$$.571 + .0573t_1 - .0282t_2 + .0292t_3 \geq 0,$$

$$-1.69 - .891t_1 + .099t_2 - .794t_3 \leq 0,$$

$$-.086 + .071t_1 + .0191t_2 + .09t_3 \leq 0,$$

$$-5001.3 - 430.31t_1 - 570.13t_2 - 139.82t_3 \leq 0\}$$

$$\text{or} \quad \{ t = (t_1, t_2, t_3)^T \mid 1 + .086t_1 - .114t_2 - .028t_3 < 0,$$

$$.429 - .028t_1 - .057t_2 - .086t_3 \leq 0,$$

$$.571 + .0573t_1 - .0282t_2 + .0292t_3 \leq 0,$$

$$-1.69 - .891t_1 + .099t_2 - .794t_3 \geq 0,$$

$$-.086 + .071t_1 + .0191t_2 + .09t_3 \geq 0,$$

$$-5001.3 - 430.31t_1 - 570.13t_2 - 139.82t_3 \geq 0\}$$

Form (MP') to obtain WMVR of S_R when perturbation on second column (column of s) is considered :

(MP′) Maximize $V(k) = k_1 \cdot k_2 \cdot k_3$ subject to

$$.086k_1 + .114k_2 + .028k_3 < 1,$$

$$.028k_1 + .057k_2 + .086k_3 \leq .429,$$

$$.0573k_1 + .0282k_2 + .0292k_3 \leq .571,$$

$$.891k_1 + .099k_2 + .794k_3 \leq 1.69,$$

$$.071k_1 + .0191k_2 + .09k_3 \leq .086,$$

$$430.31k_1 + 570.13k_2 + 139.82k_3 \leq 5001.3$$

$$k_1, k_2, k_3 \geq 0.$$

Comparison 3

The maximum tolerance obtained by Dutta *et al.*[4] using Wendell's[9] approach is .5. Therefore, the tolerance region for the parameters t_1, t_2 and t_3 is

$$B_T = \left\{ t = (t_1, t_2, t_3)^T \mid\mid t_1 \mid \leq .5, \mid t_2 \mid \leq .5, \mid t_3 \mid \leq .5 \right\}$$

with

$$\text{Vol}(B_T) = 2^3 \cdot (.5) \cdot (.5) \cdot (.5) = 1.$$

The optimal solution of (MP′) is $k^* = (.4, 1.45, .32)$.

Therefore, WMVR of S_R is

$$B_T = \left\{ t = (t_1, t_2, t_3)^T \mid\mid t_1 \mid \leq .4, \mid t_2 \mid \leq 1.45, \mid t_3 \mid \leq .32 \right\}$$

with

$$\text{Vol. } 2^3 \text{ . } (.4) \text{ . } (1.45) \text{ . } (.32) = 1.48 > \text{Vol } (B_T).$$

Also, all the three parameters are analyzed at their independent levels of sensitivity.

Since $\left\{ t = (t_1, t_2, t_3)^T \mid\mid t_1 \mid = .4, \mid t_2 \mid = 1.45, \mid t_3 \mid = .32 \right\} \subseteq S_{d+}$,

therefore by Theorem 2.6, the WMVR of S_R is also the SMVR.

Assume the following perturbation in column of non basic variable y_1. Here r =4.

$$[t_1 + 2t_2 + 3t_3, t_2 + t_3, t_1 + t_3]^T.$$

The critical region S_{NC} with focal parameters t_1, t_2 and t_3 where the basic variables z_1, z_2, z_3 remain optimal, is

$$S = \left\{ t = (t_1, t_2, \ldots, t_H)^T \mid \bar{c}_{r-m} + \sum_{h=1}^{H} \left[\sum_{i=1}^{m} c_i (\beta_{i.}^T \alpha_{.h}) t_h \leq 0 \right] \right\}$$

$$= \left\{ t = (t_1, t_2, t_3)^T \mid -1.69 + 1.274t_1 + .174t_2 + 1.548t_3 \leq 0 \right\}.$$

Form (MP′) to obtain WMVR of S_{NC} when perturbation on column of y_1 is considered :

(MP′) Maximize $V(k) = k_1 \cdot k_2 \cdot k_3$ subject to

$$1.274k_1 + .174k_2 + 1.548k_3 \leq 1.69$$

$$k_1, k_2, k_3 \geq 0.$$

Comparison 4

The maximum tolerance obtained by Dutta, Rao, and Tiwari[4] using Wendell's[9] approach is .547. Therefore, the tolerance region for the parameters t_1, t_2 and t_3 is

$$B_T = \left\{ t = (t_1, t_2, t_3)^T \mid\mid t_1 \mid \leq .547, \mid t_2 \mid \leq .547, \mid t_3 \mid \leq .013 \right\} \text{ with}$$

$$\text{Vol } (B_T) = 2^3 \cdot (.547) \cdot (.547) \cdot (.547) = .1.31.$$

The optimal solution of (MP′) is $k^* = (.44, 3.23, .36)$. Therefore, SMVR of S_{NC} is

$$B_{S_{NC}} = \left\{ t = (t_1, t_2, t_3)^T \mid \mid t_1 \mid \leq .44, \mid t_2 \mid \leq 3.23, \mid t_3 \mid \leq .36 \right\} \text{ with}$$

$$\text{Vol } 2^3 \cdot (.44) \cdot (3.23) \cdot (.36) = 4.09 > \text{Vol } (B_T).$$

Also, all the three parameters are analyzed at their independent levels of sensitivity.

CONCLUSION

Multi-parametric analysis has been performed for the perturbation of different parameters present in the numerator, denominator and in the right-hand side of the given LFPP using the models generated by Wang and Huang[11, 12] for the equivalent linear programming problem (P1) or (P2). By applying the method of transformation of variables, the denominator of the given LFPP becomes part of the constraint matrix of the equivalent linear programming problem and thus by applying the results of row perturbation of the constraint matrix, its perturbation analysis can be performed. Also, the right-hand side of the LFPP is present as one of the columns in the constraint matrix of the equivalent linear programming problem and thus by applying results corresponding to column perturbation of the constraint matrix, its analysis can be done. Perturbation analysis in the denominator of the given LFPP can also be discussed by replacing the given LFPP by its reciprocal programming problem where the denominator and numerator of the given LFPP are interchanged with appropriate restrictions on the numerator and then applying the perturbation results to the equivalent linear programming problem.

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