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MINIMIZATION THEOREM WITH FUZZY VERSION UNDER QUASI-COMPATIBLE CONDITION

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A nonconvex minimization theorem has been established for generating space of quasi-metric family. A noncommuting concept named as quasi-compatibility has been introduced to prove a fixed point theorem as an application of the minimization theorem. Noncommuting fuzzy version are obtained as consequence and suitable examples are given in support of our results.

Key Words : Generating Spaces of Quasi-Metric Family; Minimization Theorem; Quasi-Compatibility; Common Fixed Points; Fuzzy Metric Space

1. INTRODUCTION

It is well known that generating space of quasi-metric family is important because of its involvement with fuzzy metric space[5] and probabilistic metric space[8]. Further, a nonconvex minimization theorem Ref. [6] is found useful to obtain famous fixed point theorems.

In this paper, we prove a nonconvex minimization theorem for generating space of quasi-metric family and introduce a noncommuting concept known as quasi-compatible condition for proving a fixed point theorem as an application of said minimization theorem. Then, we obtain fuzzy version of above results. Suitable examples which satisfy our results are also given.

In fact, our results improve the conditions of the results of [1], [3], and [4] because we use noncommuting condition known as quasicompatibility in place of commutativity in our results.

2. PRELIMINARIES

Let us recall the following definitions

Definition 2.1[2] — Let X be nonempty set and $\{d_\alpha : \alpha \in (0, 1]\}$ be a family of mappings d_α of $X \times X$ into $\mathbb{R}^+$. $(X, d_\alpha : \alpha \in (0, 1])$ is called a generating space of quasi-metric family if it satisfies the following conditions :

(QM-1) $d_\alpha(x, y) = 0$ for all $\alpha \in (0, 1]$ if and only if $x = y$.

(QM-2) $d_\alpha(x, y) = d_\alpha(y, x)$ for all $x, y \in X$ and $\alpha \in (0, 1]$,

(QM-3) For all $\alpha \in (0, 1]$, there exist a number $\mu \in (0, \alpha]$ such that

$$d_\alpha(x, y) \leq d_\mu(x, z) + d_\mu(z, y), \; x, y, z \in X,$$

(QM-4) For any $x, y \in X$, $d_\alpha(x, y)$ is nonincreasing and left continuous in α.

$\{d_\alpha : \alpha \in (0, 1]\}$ will be called a family of quasi-metrics.

Throughout the following, we assume that $k : (0, 1] \to (0, \infty)$ is a nondecreasing function satisfying the condition

(0) $M = \sup_{\alpha \in (0, 1]} k(\alpha) < \infty$.

We recall that $\lim_{n \to \infty} x_n = \bar{x}$ in X.

Now we define quasi-compatible condition as follows :

Definition 2.2 — Let S and T be mappings from a generating space of quasi metric family $(X, d_\alpha : \alpha \in (0, 1])$ into itself. The mappings S and T are said to be quasi-compatible if :

$$d_\alpha(STx_n, TSx_n) \to 0 \text{ as } n \to \infty, \text{ for all } \alpha \in (0, 1],$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n = p$ for same $p \in X$.

Following lemma is needed to prove Theorem 3.2.

Lemma 2.1 — Let S and T be quasi-compatible mappings from a generating space of quasi-metric family $(X, d_\alpha : \alpha \in (0, 1])$ into itself. Suppose that $\lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n = p$ for $p \in X$. Then we have the following :

(*i*) $\lim_{n \to \infty} STx_n = Tp$ if T is continuous

(*ii*) $STp = TSp$ and $Sp = Tp$ if S and T are continuous.

PROOF :

(*i*) Suppose that $\lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n = p$ for the same $p \in X$.

Now, since T is continuous, we have

$$\lim_{n \to \infty} TSx_n = Tp.$$

By (QM-3), we have for every $\alpha \in (0, 1]$

$$d_\alpha(STx_n, Tp) \leq d_\mu(STx_n, TSx_n) + d_\mu(TSx_n, Tp), \text{ for some } \mu \in (0, \alpha].$$

Since S and T are quasi-compatible, we have

$$\lim_{n \to \infty} STx_n = Tp.$$

(*ii*) Since S is continuous, $\lim_{n \to \infty} STx_n = Sp$

(because $\lim_{n \to \infty} Tx_n = \lim_{n \to \infty} Sx_n = p$).

Hence, by the uniqueness of limit, we have $Sp = Tp$.

Now, again

$$d_\alpha(STp, TSp) = \lim_{n \to \infty} d_\alpha(STx_n, TSx_n) = 0$$

i.e.

$$STp = TSp.$$

This complete the proof.

3. MAIN RESULTS

First we prove a minimization theorem.

Theorem 3.1 — *Let* $(X, d_\alpha : \alpha \in (0, 1])$ *and* $(Y, \delta_\alpha : \alpha \in (0, 1])$ *be the generating complete spaces of quasi-metric family,* $f : X \to Y$ *be a closed mapping and* $T : X \to X$, *continuous mapping satisfying :*

$$d_\alpha(Tx, Ty) \leq 1/2\ [d_\alpha(x, Ty) + d_\alpha(Tx, y)]$$

and

$$\delta_\alpha(f(Tx), f(Ty)) \leq 1/2\ [\delta_\alpha(f(x), f(Ty)) + \delta_\alpha(f(Tx), f(y)],$$

for every $x, y \in X$ *and every* $\alpha \in (0, 1]$. *Let* $\psi : \mathbb{R} \to \mathbb{R}$ *be a non-decreasing continuous function, bounded from below and* $\phi : f(X) \to \mathbb{R}$ *a lower semicontinuous function, bounded from below. Assume that for any* $r \in X$ *with* $\inf_{x \in X} \psi(\phi(f(x)) < \psi(\phi(f(r))$, *there exists* $s \in X$ *with* $r \neq Ts$ *and*

$$\max\ \{1/2\ [d_\alpha(s, Tr) + d_\alpha(r, Ts), c/2[\delta_\alpha(f(s), f(T(r)) + \delta_\alpha(f(r), f(Ts))]\}$$

$$\leq k(\alpha)\ [\psi(\phi(f(r))) - \psi(\phi(f(s)))].$$

for all α *(0, 1], where* c *is a given constant. Then there exists an* $x_0 \in X$ *such that*

$$\inf_{x \in X} \psi(\phi(f(x)) = \psi(\phi(f(x_0))).$$

PROOF : Let us suppose $\inf_{x \in X} \psi(\phi(f(x)))\ \psi(\phi(f(y)))$ for every $y \in X$, and choose $r \in X$ with $\psi(\phi(f(r))) < \infty$. Then, we define a sequence $\{r_n\} \subset X$ with $r_1 = r$. Suppose that r_n is known and consider

$$W_n = \{w \in X : \max \{1/2\ [d_\alpha(w, Tr_n) + d_\alpha(Tw, r_n)],$$

$$c/2\ [\delta_\alpha f(w), f(T(r_n))) + \delta_\alpha(f(Tw), f(r_n))]\} \qquad \ldots (1)$$

$$\leq k(\alpha)\ [\psi(\phi(f(r_n))) - \psi(\phi(f(w)))]\} \text{ for any } \alpha \in (0, 1].$$

W_n is nonvoid set and there exists $ww \in W_n$ such that $r_n \neq Tw$.

We can choose $r_{n+1} \in W_n$ such that $r_n \neq Tr_{n+1}$ and

$$\psi(\phi(f(r_{n+1}))) \leq \inf_{x \in X} \psi(\phi(f(x))) + 1/2\ [\psi(\phi(f(r_n))) - \inf_{x \in X} \psi(\phi(f(x))].] \qquad \ldots (2)$$

We remark that $\{\psi(\phi(f(r_{n+1})))\}$ is a non-increasing lower bounded sequence, hence is a convergent sequence.

Now we prove $\{r_n\}$ and $\{f(r_n)\}$ are Cauchy sequence, namely

$$\max\ \{d_\alpha(Tr_n, Tr_{n+1}), c\ \delta_\alpha(f(Tr_n), f(Tr_{n+1}))$$

$$\leq \max\{1/2\ (d_\alpha(r_n, Tr_{n+1}) + d_\alpha(r_{n+1}, Tr_n)),$$

$$c/2\ (\delta_\alpha(f(r_n), f(Tr_{n+1})) + \delta_\alpha(f(r_{n+1}), f(Tr_n)))\}$$

$$\leq K(\alpha)\ [\psi(\phi(f(r_n))) - \psi(\phi(f(r_{n+1})))].$$

$\forall^n_m \in \mathbb{N}\ n < m \Rightarrow$ there exists $\mu \leq \alpha$, $\mu = \mu\ (n, m)$ such that

$$\max\ \{d_\alpha(Tr_n, Tr_m), c\delta_\alpha(f(Tr_n), f(Tr_m))\}$$

$$\leq \sum_{j=n}^{m-1} \max\ \{1/2[d_\mu(r_j, Tr_{j+1})\ d_\mu(Tr_j, r_{j+1})],$$

$$c/2\ [\delta_\mu(f(r_j), f(Tr_{j+1}) + \delta_\mu(f(Tr_j), f(r_{j+1}))]\}.$$

Hence, $\forall^n_m \in \mathbb{N}\ n < m$:

$$\max\ \{d_\alpha(Tr_n, Tr_m), c\delta_\alpha(f(Tr_n), f(Tr_m))\}$$

$$\leq k(\mu) \sum_{j=n}^{m-1} [\psi(\phi(f(r_j))) - \psi(\phi(f(r_{j+1})))]$$

$$\leq k(\alpha)\ [\psi(\phi(f(r_n))) - \psi(\phi(f(r_m)))]$$

for some α_j with $0 < \alpha_{j+1} \leq \alpha_k \leq \alpha$, $j = n, \ldots, m-1$ and

$$d_\alpha(r_n, r_{n+1}) \le d_\mu(r_n, Tr_{n+1}) + d_\mu(Tr_{n+1}, Tr_n) + d_\alpha(Tr_n, r_{n+1})$$

$$\le 3\,[1/2\,(d_\mu(r_n, Tr_{n+1}) + d_\mu(r_{n+1}, Tr_n)$$

$$\le 3k(\alpha)\,[\psi(\phi(f(r_n))) - \psi(\phi(f(r_{n+1})))].$$

Then also

$$d_\alpha(r_n, r_m) \le 3\,k(\alpha)\,[\psi(\phi(f(r_n))) - \psi(\phi(f(r_m)))] \qquad \text{where } n < m.$$

In the similar manner we obtain

$$\delta_\alpha(r_n, r_m) \le 3\,k(\alpha)\,[\psi(\phi(f(r_n))) - \psi(\phi(f(r_m)))] \qquad \text{where } n < m.$$

Hence $\{r_n\}$ and $\{f(r_n)\}$ are Cauchy sequences.

Assume that $\lim_{n\to\infty} r_n = \bar{r}$ and $\lim_{n\to\infty} f(r_n) = \bar{s}$.

Since f is closed therefore $f(\bar{r}) = \bar{s} \in f(X)$.
By the continuity of ψ and lower semi-continuity of ϕ we have

$$\psi(\phi(\bar{s})) \le \lim_{n\to\infty} \psi(\phi(f(r_n))) = \lim_{n\to\infty} \psi(\phi(f(r_{n+1}))).$$

Let $\lambda_0 = \inf_{x\in X} \psi(\phi(f(x))) \in \mathbb{R}$.

From $\psi(\phi(f(r_{n+1}))) \le \lambda_0 + 1/2\,(\psi(\phi(f(r_n))) - \lambda_0)$,

we have

$$\lim_{n\to\infty} \psi(\phi(f(r_{n+1}))) \le \lambda_0/2 + 1/2 \lim_{n\to\infty} \psi(\phi(f(r_n)))$$

$$= \lambda_0/2 + 1/2 \lim_{n\to\infty} \psi(\phi(f(r_{n+1}))).$$

Finally,

$$\psi(\phi(f(\bar{r}))) = \psi(\phi(\bar{s})) \le \lim_{n\to\infty} (\psi(\phi(f(r_{n+1}))))$$

$$\le \lambda_0 = \inf_{x\in X} \psi(\phi(f(x))) \le \psi(\phi(f(\bar{r}))),$$

which is a contradiction. Therefore, there exists an $x_0 \in X$ such that

$$\psi(\phi(f(x_0))) = \inf_{x\in X} \psi(\phi(f(x)))).$$

Now we give a fixed point theorem as an application of the Theorem 3.1 under the noncommuting condition known as quasi-compatible condition.

Theorem 3.2 — *Ley* $\{X, d_\alpha : \alpha \in (0, 1]\}$ *and* $\{Y, \delta_\alpha : \alpha \in (0, 1]\}$ *be the generating complete spaces of quasi metric family. Let* $f : X \to Y$ *be a closed mapping,* $T : X \to X$ *continuous mapping satisfying*

$$d_\alpha(Tx, Ty) \leq 1/2\ [d_\alpha(x, Ty) + d_\alpha(Tx, y)]$$

and $\delta_\alpha(f(Tx), f(Ty)) \leq 1/2\ [\delta_\alpha(f(x), f(Ty)) + \delta_\alpha(f(Tx, f(y))]$ *for every* $x, y \in X$ *and every* $\alpha \in (0, 1]$, ψ: $\mathbb{R} \to \mathbb{R}$ *a nondecreasing continuous function, bounded from below, and* $\phi : f(x) \to \mathbb{R}$ *be a lower semi-continuous function, bounded from below. Let* $S : X \to X$ *be a ocntinuous mapping. Further if* S *and* T *are quasi-compatible and*

$$\max\ \{1/2\ [d_\alpha(x, TSx) + d_\alpha(Tx, Sx), c/2\ [\delta_\alpha(f(x), f(TSx)) + \delta_\alpha(f(Tx), f(Sx))]]\}$$

$$\leq k(\alpha)\ [\psi(\phi(f(x)) - \psi(\phi(f(Sx))]$$

for every $x \in X$ *and every* $\alpha \in (0, 1]$ *where* c *is a given constant, then there exists unique common fixed point* $x_0 \in X$ *i.e.* $Tx_0 = x_0 = Sx_0$.

PROOF : *Existence of Fixed Point* — We remark that if $x_0 \in X$ such that $\psi(\phi(f(x_0))) = \inf_{x \in X} \psi(\phi(f(x)))$ then $x_0 = TSx_0$, $Sx_0 = Tx_0$ and also $x_0 = Tx_0$. By the hypothesis, if $x_0 \neq TSx_0$ or $Sx_0 \neq Tx_0$. Then for some $\alpha \in (0, 1]$:

$$0 < 1/2\ [d_\alpha(x_0, TSx_0) + d_\alpha(Sx_0, Tx_0)]$$

$$\leq k(\alpha)\ [\psi(\phi(f(x_0))) - \psi(\phi(f(Sx_0)))] \leq 0,$$

which is a contradiction. Moreover for every $\alpha \in (0, 1]$ there exists $\mu \in (0, \alpha]$ such that

$$d_\alpha(x_0, Tx_0) \leq d_\mu(x_0, TSx_0) + d_\mu(TSx_0, Tx_0)$$

$$= d_\mu(TSx_0, Tx_0) \leq 1/2\ [d_\mu(Sx_0, Tx_0) + d_\mu(x_0, TSx_0)]$$

$$= 0.$$

Then $x_0 = Tx_0\ (= Sx_0)$.

Uniqueness : Now, let us suppose that there exists another fixed point $y_0 \in X$ such that $y_0 = Sy_0 = Ty_0$. By using Theorem 3.1, we have

$$\inf_{x \in X} \psi(\phi(f(x))) = \psi(\phi(f(x_0))) \leq \psi(\phi(f(y_0))) \qquad \ldots (3)$$

Let $x_0 \neq y_0$. Then by we have

$$0 < \max\ \{1/2\ [d_\alpha(x_0, Ty_0) + d_\alpha(y_0, Tx_0)], c/2\ [\delta_\alpha(f(x_0), f(Ty_0)) + \delta_\alpha(f(y_0, f(Tx_0))]]\}$$

$$\leq k(\alpha)\ [\psi(\phi(x_0))) - \psi(\phi(f(y_0)))]$$

$$\leq 0,$$

which is contradiction. Thus $x_0 = y_0$.

This completes the proof.

As a direct consequence of Theorem 3.2, we obtain the following result, which is a improved form of Downing-Kirk's fixed point theorem[3] in generating space of quasi-metric family under quasi-compatible condition.

Corollary 3.1 — Let $(X, d_\alpha : \alpha \in (0, 1]$ and $(Y, \delta_\alpha : \alpha \in (0, 1])$ be generating complete spaces of quasi-metric family, $f : X \to Y$ be a closed mapping and $\phi f(X) \to \mathbb{R}$ a lower semi-continuous function, bounded from below. Let $S : X \to X$ be mapping such that

$$\max \{d_\alpha(Sx, x),\ c\delta_\alpha(f(Sx), f(x))\} \leq k(\alpha)\ \{\phi(x) - \phi(Sx)\}$$

for any $x \in X$ and $\alpha \in (0, 1]$, where x is a given constant. Then there exists $x_0 \in X$ with $Sx_0 = x_0$.

PROOF : Considering $T = I$ and $\psi = I$, we prove the corollary.

Remark 3.1 : (1) Downing-Krik's fixed point theorem[3] can be obtained from Corollary 3.1 considering X, Y complete metric spaces.

(2) Caristi's fixed point theorem[1] can be obtained from Corollary 3.1 considering $X = Y$ a metric space, $f = I$, $c = 1$ and $k(\alpha) = 1$.

Now, we given an Example which satisfies Theorem 3.1.

Example 1 — Let $X = [0, 1]$ and $Y = [0, \infty)$, $d_\alpha = \delta_\alpha = d$ defined by

$d(x,. y) = \dfrac{|x - y|}{1 + |x - y|}$: Then mappings defined as follows

$T : [0, 1] \to [0, 1]$ as $Tx = x^2$

$f : [0, 1] \to [0, 1]$ as $fx = x$

$\phi : f(X) \to R$ as $\phi(x) = \dfrac{1}{1 - x}$

$\psi : R \to R$ as $\psi(x) = \dfrac{x^2}{2}$. And $K(\alpha) = k = 2$.

Satisfy the all conditions of Theorem 3.1.

Now, we given an example which satisfies Theorem 3.2.

Example 2 — Let $X = [0, 1]$ and $Y = [0, \infty)$. Let mappings T, f, Φ, ψ, k be defined as in Example 1. The mappings $S : [0, 1] \to [0, 1]$ is defined as $Sx = \dfrac{x^3}{2}$. Then (T, S) is quasi-compatible and satisfying the condition of Theorem 3.2, and thus 0 is a unique fixed point.

4. VERSIONS IN FUZZY METRIC SPACES

A fuzzy number is a mapping $x : \mathbb{R} \to [0, 1]$. The set $[x]_\alpha = \{p \in \mathbb{R} : x(p) \geq \alpha\}$ is called a α-level set of fuzzy number x, for all $\alpha \in [0, 1]$.

A fuzzy number x is called to be convex iff

(*a*) X is nondecreasing or

(*b*) X is nonincreasing or

(*c*) there exist t^* such that $\begin{cases} X \text{ is nondecreasing for } t < t^* \\ X \text{ is nonincreasingfor } t > t^*. \end{cases}$

Moreover if (*c*) is satisfied for s^* and t^* then X is constant in the interval of end points s^* and t^*.

If there exist a point $p \in \mathbb{R}$ such that $x(p) = 1$, then fuzzy number is said normal.

If a fuzzy number x is upper semicontinuous, convex and normal, then the α-level set of x is a closed interval $[a^\alpha, b^\alpha]$, i.e., $[x]_\alpha = [a^\alpha, b^\alpha]$, $\alpha \in (0, 1]$, where $\alpha^\alpha = -\infty$ are admissible.

A fuzzy number is called nonnegative of $x(p) = 0$ for all $p < 0$. The fuzzy number θ is defined by $\theta(p) = 1$ for $p = 0$ and $\theta(p) = 0$ for $p \neq 0$. Let G be the set of all nonnegative upper semicontinuous normal convex fuzzy numbers and L, R : $[0, 1] \times [0, 1]$ be two functions such that they are nondecreasing in both arguments, symmetric and $L(0, 0) = 0$ $R(1, 1) = 1$. Let X be a nonempty set and $d : X \times X \to G$ be a mapping and denote

$$[d(x, y)]_\alpha = [\lambda_\alpha(x, y), \rho_\alpha(x, y)], \quad x, y \in X, \quad \alpha \in (0, 1], \qquad \ldots (4)$$

where $[d(x, y)]_\alpha$ is the α-level set of fuzzy number $d(x, y) \in G$ and $\lambda_\alpha(x, y)$, $\rho_\alpha(x, y)$ are the left and right end points of the closed interval $[d(x, y)]_\alpha$, respectively.

Definition 4.1 — [5] The quadruple (X, d, L, R) is called a fuzzy metric space if the mapping $d : X \times X \to G$ satisfies the following conditions :

(FM-1) $d(x, y) = \theta$ if and only $x = y$

(FM-2) $d(x, y) = d(y, x)$ for all $x, y \in X$,

(FM-3) For any $x, y, z \in X$

(*i*) $d(x, y)(s + t) \geq L(d(x, z)(s), d(z, y)(t))$,

whenever $s \leq \lambda_1(x, z)$, $t \leq \lambda_1(z, y)$ and $s + t \leq \lambda_1(x, y)$

(*ii*) $d(x, y)(s + t) \leq R(d(x, z)(s), d(z, y)(t))$,

whenever $s \geq \lambda_1(x, z)$, $t \geq \lambda_1(z, y)$ and $s + t \geq \lambda_1(x, y)$.

Remark 4.1 : By Theorem 3.2 in [5], we know that if (X, d, L, R) is a fuzzy metric space with $\lim_{a \to 0^+} R(a, a) = 0$, then there exists a topology $\mathcal{J}_d$ on such that $(X, \mathcal{J}_d)$ is a Hausdorff topological space and

$$\mathcal{U}(x) = \{U_x(\varepsilon, \alpha) : \varepsilon > 0, \quad \alpha \in (0, 1]\} \quad x \in X$$

is a basis of neighbourhoods of the point x for the topology $\mathcal{J}_d$, where

$$U_x(\varepsilon, \alpha) = \{y \in X / \rho_\alpha(x, y) < \varepsilon\},$$

and $\rho_\alpha(x, y)$ is the right end point of $[d(x, y)]_\alpha$ defined by (4).

Proposition 4.1[2] — Let (X, d, L, R) be a fuzzy metric space with $\lim_{a \to 0^+} R(a, a) = 0$, $\lim_{t \to \infty} d(x, y)(t) = 0, x, y \in X$. Letting $[d(x, y)]_\alpha = [\lambda_\alpha(x, y), \rho_\alpha(x, y)]$, then $(X, \rho_\alpha, \alpha \in (0, 1])$ is a generating space of quasi-metric family and the topology $\mathcal{J}\{\rho_\alpha\}$ induced by the family $\{\rho_\alpha\}$ of quasi-metrics coincides with the fuzzy topology on (X, d, L, R).

PROOF : Since $\lim_{t \to \infty} d(x, y)(t) = 0$, it follows that $\rho_\alpha(x, y) < \infty$ for all $\alpha \in (0, 1]$. Now we prove that $\{(X, \rho_\alpha, \alpha \in (0, 1])\}$ is a generating space of quasi-metric family. It is clear that $(X, \rho_\alpha, \alpha \in (0, 1])$ satisfies the condition (QM-1), (QM-2) and (QM-4) in Definition 2.1. Now we show that it also satisfies the condition (QM-3). By the assumption that $\lim_{a \to 0^+} R(a, a) = 0$, for all $\alpha \in (0, 1]$, there exists an $\mu \in (0, \alpha]$ such that $R(\mu, \mu) < \alpha$. For any given $x, y, z \in X$, let $\rho_\mu(x, z) = s$, $\rho_\mu(z, y) = t$. By the definition of ρ_μ, it is easy to show that

$$s \leq \lambda_1(x, z) \text{ and } t \geq \lambda_1(z, y).$$

(*i*) If $s + t \leq \lambda_1(x, y)$, then for any $\varepsilon > 0$ it follows from (FM-3) that

$$d(x, y)(s + t + 2\varepsilon) \leq R(d(x, z)(s + \varepsilon), d(z, y)(t + \varepsilon))$$

$$\leq R(\mu, \mu) < \alpha.$$

Hence we have, $\rho_\alpha(x, y) < 2\varepsilon + s + t$. By the arbitrariness of ε, we obtain

$$\rho_\alpha(x, y) \leq s + t = \rho_\mu(x, z) + \rho_\mu(z, y).$$

(*ii*) If $s + t < \lambda_1(x, y)$ and $\mu = \lambda_1(x, y) - (s + t)$

then we have

$$1 = d(x, y)(\lambda_1(x, y)) = d(x, y)(u + s + t)$$

$$\leq R\left(d(x, z)\left(s + \frac{1}{2}u\right), d(z, y)\left(t + \frac{1}{2}u\right)\right)$$

$$\leq R(\mu, \mu) < \alpha,$$

which is contradiction. Therefore, the case (*ii*) cannot happen proves that $(X, \rho_\alpha, \alpha \in (0, 1])$ satisfies the condition (QM-3).

On the other hand by the Remark 4.1, the topology $\mathcal{J}\{\rho_\alpha\}$ on the generating space of quasi-metric family $(X, \rho_\alpha, \alpha \in (0, 1])$ coincides with the fuzzy topology $\mathcal{J}_d$ on fuzzy metric space (X, d, L, R).

Now, we give fuzzy version of our minimization theorem.

Theorem 4.1 — *Let (X_i, d_i, L, R), $i = 1, 2, \ldots$ be two complete fuzzy metric spaces with* $\lim_{t \to \infty} d_i(x, y)(t) = 0$, *for all* $x, y \in X_i$, $i = 1, 2$, *and* $\lim_{a \to 0^+} R(a, a) = 0$. *Let* $f: X_1 \to X_2$ *be a mapping,* $T: X_1 \to X_1$ *be a continuous mapping satisfying* $\rho_{1\alpha}(Tx, Ty) \leq \lambda/2\,[\rho_{1\alpha}(x, Ty) + \rho_{1\alpha}(Tx, y)]$ *and* $\rho_{1\alpha}(f(Tx), f(Ty)) \leq 1/2\,[\rho_{1\alpha}(fx), f(Ty) + \rho_{2\alpha}(f(Tx), f(y))]$, *for every* $x, y \in X_i$ *and* $\alpha \in (0, 1]$, ψ: $I\!R \to I\!R$ *be a lower semi continuous function, bounded from below. Assume that for any* $r \in X$, *with* $\inf_{x \in X_1} \psi(\phi(f(x)) < \psi(\phi(f(r))$, *there exist* $s \in X_1$ *with* $r \neq Ts$ *and max* $\{1/2\,[\rho_{1\alpha}(s, Tr) + \rho_{1\alpha}(Ts, r)],\ c/2\,[\rho_{2\alpha}(f(s), f(Tr)) + \rho_{2\alpha}(f(Ts), f(r)]\} \leq k(\alpha)\,[\psi(\phi(f(r)) - \psi(\phi(f(s))]$ *for any* $\alpha \in (0, 1]$, *where* c *is a given constant and* $\{\rho_{i\alpha} : \alpha(0, 1]\}$ *is the family of quasi-metrics on* X_i, *defined by (12). Then there exists an* $x_0 \in X$, *such that*

$$inf_{x \in X_1}\, \psi(\phi(f(x)) = \psi(\phi(f(x_0)).$$

As a direct consequence, we have the following :

Corollary 4.2 — Let (X_i, d_i, L, R), $i = 1, 2, \ldots$ be two complete fuzzy metric spaces with $\lim_{t \to \infty} d_i(x, y)(t) = 0$, $x, y \in X_i$, $i = 1, 2$, and $\lim_{a \to 0^+} R(a, a) = 0$. Let $f : X_1 \to X_2$ be a mapping $T : X_1 \to X_1$ a continuous mapping satisfying

$$\rho_{1\alpha}(Tx, Ty) \leq 1/2\,[\rho_{1\alpha}(Tx, y) + \rho_{1\alpha}(x, Ty)]$$

and

$$\rho_{2\alpha}(f(Tx)), f(Ty)) \leq 1/2\,[\rho_{2\alpha}(f(Tx), f(y)) + \rho_{2\alpha}(f(x), f(Ty)]$$

for every $x, y \in X$, and $\alpha \in (0, 1]$ and $\phi : (f(X_1)) \to I\!R$ a lower semi-continuous function, bounded from below. Assume that for any $r \in X$ with $\inf_{x \in X_1} \phi(f(x)) < \phi(f(r))$, there exists $s \in X$, with $r \neq Ts$ and max $\{1/2\,[\rho_{1\alpha}(Tr, s) + \rho_{1\alpha}(r, Ts)],\ c/2\,[\rho_{2\alpha}(f(r), f(Ts) + \rho_{2\alpha}(f(Tr), f(s))\} \leq k(\alpha)\,[\phi(f(r)) - \phi(f(s))]$, for all $\alpha \in (0, 1)$, where c is a given constant and $\{\rho_{i\alpha} : \alpha \in (0, 1]\}$ is the family of quasi metric on X_i satisfy (4). Then there exists an $x_0 \in X_1$ such that $inf_{x \in X_1} \phi(f(x)) = \phi(f(x_0))$.

PROOF : The result follows from Theorem (4.1) with $\psi = I$.

Now, we given fuzzy version of our fixed point theorem under quasi compatible condition.

Theorem 4.3 — *Let (X_i, d_i, L, R), $i = 1, 2, \ldots$ be two complete fuzzy metric spaces with* $\lim_{t \to \infty} d_i(x, y)(t) = 0$, $x, y \in X_i$, $i = 1, 2$, *and* $\lim_{a \to 0^+} R(a, a) = 0$. *Let* $f: X_1 \to X_2$ *be a mapping,* $T: X_1 \to X_1$ *a continuous mapping satisfying*

$$\rho_{1\alpha}(Tx, Ty) \leq 1/2\,[\rho_{1\alpha}(Tx, y) + \rho_{1\alpha}(x, Ty)]$$

for every $x, y \in X_1$ *and every* $\alpha \in (0, 1]$ *and*

$$\rho_{1\alpha}(f(Tx), f(Ty)) \leq 1/2\ [\rho_{2\alpha}(fx), f(Ty) + \rho_{2\alpha}(f(Tx), f(y))],$$

$\psi : \mathbb{R} \to \mathbb{R}$ *a nondecreasing continuous function, bounded from below, and* $\phi : f(X_1) \to \mathbb{R}$ *a lower semi continuous function, bounded from below. Let* $S : X_1 \to X_1$ *be mapping. Further if* S *and* T *are quasi-compatible and*

$$\max\ \{1/2[\rho_{1\alpha}(Tx, Sx) + \rho_{1\alpha}(x, TSx)],\ c/2\,[\rho_{1\alpha}(f(Tx), f(Sx)) + \rho_{2\alpha}(f(x), f(TSx))]\}$$

$$\leq k(\alpha)\,[\psi(\phi(f(x)) - \psi(\phi(f(Sx)))],$$

for every $x \in X$, *and* $\alpha \in (0, 1]$, *where* c *is a given constant and* $\{\rho_{1\alpha} : \alpha \in (0, 1]\}$ *is the family of quasi-metric on* X_i *defined by (4). Then there exists a common fixed point* $x_0 \in X_1$ *s.t.* $Tx_0 = x_0 = Sx_0$.

As a direct consequence of Corollary 4.2 and Theorem 4.2, we have following :

Corollary 4.4 — Let (X_i, d_i, L, R), $i = 1, 2, \ldots$ be two complete fuzzy metric space with $\lim_{t \to \infty} d_i(x, y)(t) = 0$, $x, y \in X_i$, $i = 1, 2$, and $\lim_{a \to 0^+} R(a, a) = 0$. Let $f : X_1 \to X_2$ be a mapping, $T : X_1 \to X_1$ a continuous mapping satisfying

$$\rho_{1\alpha}(Tx, Ty) \leq 1/2\,[\rho_{1\alpha}(Tx, y) + \rho_{1\alpha}(x, Ty)]$$

and

$$\rho_{1\alpha}(f(Tx), f(Ty)) \leq 1/2\ [\rho_{2\alpha}(fx), f(Ty) + \rho_{2\alpha}(f(Tx), f(y))]$$

for every $x, y \in X_1$ and $\alpha \in (0, 1]$, and $\phi : f(X_1) \to \mathbb{R}$ a lower semi continuous function, bounded from below. Let $S : X_1 \to X_1$ be mapping. Further if S and T are quasi-compatible and

$$\max\ \{1/2[\rho_{1\alpha}(Tx, Sx) + \rho_{1\alpha}(x, TSx)],\ c/2\,[\rho_{1\alpha}(f(Tx), f(Sx) + \rho_{2\alpha}(f(X), f(TSx))]$$

$$\leq k(\alpha)\,[\psi(\phi(f(x)) - \psi(\phi(f(Sx))]$$

for every $x \in X$, and $\alpha \in (0, 1]$, where c is given constant and $\{\rho_{1\alpha} : \alpha \in (0, 1]\}$ is the family of quasi-metric on X_i, defined Corollary 4.1. Then there exists a fixed point $x_0 \in X_1$ s.t. $Tx_0 = x_0 = Sx_0$.

As a direct consequence, we have following :

Corollary 4.5 — Let (X_i, d_i, L, R), $i = 1, 2$, be two complete fuzzy metric spaces with $\lim_{t \to \infty} d_i(x, y)(t) = 0$, $x, y \in X_i$, $i = 1, 2$, and $\lim_{a \to 0^+} R(a, a) = 0$. Let $f : X_1 \to X_2$ be a mapping and

$\phi : f(X_1) \to \mathbb{R}$ be a lower semicontinuous function, bounded from below. Let $S : X_1 \to X_1$ be mapping such that

$$\max \{\rho_{1\alpha}(Sx, x), c\rho_{2\alpha}(f(Sx)), f(x))\} \leq k(\alpha)[\phi(f(x)) - \phi(f(Sx))]$$

for every $x \in X_1$ and every $\alpha \in (0, 1]$ where c is given constant and $\{\rho_{i\alpha} : \alpha \in (0, 1]\}$ is the family of quasi-metric on X_i, defined by (4). Then there exists an $x_0 \in X_1$ with $Sx_0 = x_0$.

PROOF : Considering $T = I$ and $\psi = I$ in corollary 4.4, we obtain the corollary.

Remark 4.2 : (1) Theorem 2 of [4] is a special case of theorem 4.1 with

$$R = \text{Max.}, \quad T = I, \quad \psi = I, \quad c = 1 \text{ and } k(\alpha) = 1.$$

(2) Theorem 3 of [4] can be obtained from Corollary 4.5 with R = Max., $c = 1$, and $k(\alpha) = 1$.

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