Indian J. pure appl. Math., 30(6) : 641-647, June 1999
© Printed in India.

SOME FIXED POINT THEOREMS BY ALTERING DISTANCES BETWEEN THE POINTS

K. P. R. SASTRY AND G. V. R. BABU

Department of Mathematics, Andhra University, Visakhapatnam 530 003

(*Received* 12 *August* 1997; *after Revision* 30 *March* 1998; *Accepted* 1 *February* 1999)

The main purpose of this paper is to obtain a fixed point theorem under a more general condition than that of Khan, Swaleh and Sessa[1]. Further generalization for a pair of selfmaps is given. Open problems are given at the end of this paper.

Key Words : Fixed Points; Metric Spaces; Selfmaps

1. INTRODUCTION

Obtaining fixed point theorems for selfmaps of a metric space by altering the distances between the points with the use of a certain continuous control function is an interesting aspect. Khan, Swaleh and Sessa[1] proved some fixed point theorems for selfmaps in metric spaces by altering distances. A comprehensive literature on fixed points of contractive type maps on complete metric space can be found in Rhoades[3].

Throughout this paper, (X, d) denotes a metric space, R^+ the set of all non-negative real numbers, N the set of all natural numbers and Φ the set of all continuous selfmaps ϕ of R^+ satisfying $\phi(t) = 0$ if and only if $t = 0$. For any subset A of X, $\bar{A}$ stands for the closure of A.

The following theorem was proved by Khan, Swaleh and Sessa[1].

Theorem 1.1 — *Let T be a continuous selfmap of a metric space (X, d) such that for some x_0 in X, the sequence $\{T^n x_0\}$ has a cluster point, z, in X; and there exist $\phi \in \Phi$ such that*

... (1.1.1)

$$\phi(d(Tx, Ty)) < c\phi(d(x, y)) + \frac{(1-c)}{2}[\phi(d(x, Tx) + \phi(d(y, Ty))] \quad \text{... (1.1.2)}$$

for all distinct x, y in X, where $0 \le c \le 1$.

Then z is the unique fixed point of T.

The main purpose of this paper is to obtain a fixed point theorem (Theorem 2.1) under a more general condition than (1.1.2) and obtain a further generalization for a pair of selfmaps (Theorem 3.1).

2. FIXED POINT THEOREMS FOR SELFMAPS

Theorem 2.1 — *Let T be a selfmap on a metric space (X, d). Suppose there exists a point x_0 in X such that the orbit $O(x_0) = \{T^n x_0 : n = 0, 1, 2, \ldots\}$ has a cluster point z in X.*

If T is orbitally continuous at z and Tz, and there exists a $\phi \in \Phi$ such that

$$\phi(d(Tx, Ty)) < \phi(d(x, y)) \qquad \ldots (2.1.1)$$

for each x, $y = Tx \in \overline{O(x_0)}$, $x \neq y$, then z is a fixed point of T.

PROOF : Define the sequence $\{x_n\}$ by $x_n = T^n x_0$ for $n = 1, 2, \ldots$ If $x_n = x_{n+1}$ for some n, then $x_m = x_n$ for $m \geq n$ so that $z = \lim_{m \to \infty} x_m$ and $Tz = z$ by the orbital continuity of T.

We now assume $x_n \neq x_{n+1}$ for every n in $\mathbb{N}$.

If $\alpha_n = \phi(d(x_n, x_{n+1}))$, then by (2.1.1) it follows that

$$\alpha_{n+1} = \phi(d(x_{n+1}, x_{n+2})) < \phi(d(x_n, x_{n+1})) = \alpha_n$$

so that $\{\alpha_n\}$ is a strictly decreasing sequence of positive numbers and hence converges, say, to $\alpha \geq 0$.

Let $k(n)$ be a sequence of positive integers such that $\{x_{k(n)}\}$ converges to z. Then $\{\alpha_{k(n)}\}$ converges to α. By the continuity of ϕ and orbital continuity of T at z,

$$\alpha = \lim_{n \to \infty} \phi(d(x_{k(n)}, x_{k(n)+1})) = \phi(d(z, Tz)).$$

It is enough to show that $\alpha = 0$.

We observe that z and Tz belong to $\overline{O(x_0)}$.

Suppose $\alpha \neq 0$. Then by (2.1.1) and the orbital continuity of T at z and Tz,

$$0 \neq \alpha = \lim_{n \to \infty} \alpha_{k(n)+1} = \lim_{n \to \infty} \phi(d(x_{k(n)+1}, x_{k(n)+2}))$$

$$= \phi(d(Tz, T(Tz))) < \phi(d(z, Tz)) = \alpha$$

a contradiction, hence $\alpha = 0$. Since ϕ vanishes at zero it follows that $Tz = z$.

The following example shows that, if we consider the inequality (2.1.1) for the points x, $y = Tx \in O(x_0)$, $x \neq y$, then z may not be a fixed point of T.

Example 2.1 — Let $X = \{1, -1, 1 + (2n)^{-1}, -1 - (2n+1)^{-1} : n = 1, 2, \ldots\}$ with the usual metric.

Define T on X by $T(1) = -1$; $T(-1) = 1$; $T(1 + (2n)^{-1}) = -1 - (2n+1)^{-1}$ and

$T(-1 - (2n+1)^{-1}) = 1 + (2(n+1))^{-1}$. For $x_0 = 1 + \frac{1}{2}$,

$$O(x_0) = \{1 + (2n)^{-1}, -1 - (2n+1)^{-1} : n = 1, 2, \ldots\}.$$

Condition (2.1.1) is satisfied on $O(x_0)$ with $\phi(t) = t$, $t \geq 0$. Take $z = 1$ in $\overline{O(x_0)}$, then $Tz = -1$ and (2.1.1) does not hold on $\overline{O(x_0)}$. Further T is continuous at z and Tz. But z is not a fixed point of T.

Theorem 2.2 — *Suppose T satisfies (1.1.1) and there exists a $\phi \in \Phi$ such that*

$$\phi(d(Tx, Ty)) < \max \{\phi(d(x, y)), \phi(d(x, Tx)), \phi(d(y, Ty))\} \quad \ldots (2.2.1)$$

for all distinct x, y, in X.

Then z (of (1.1.1)) is the unique fixed point of T.

This theorem is a corollary to Theorem 2.1 in the sense that condition (2.2.1) implies condition (2.1.1).

Remark 2.3 : We can replace condition (2.2.1) by the following inequality

$$\phi(d(Tx, Ty)) < \max \left\{ \phi(d(x, y)), \frac{\phi(d(x, Tx)) + \phi(d(y, Ty))}{2} \right\} \quad \ldots (2.3.1)$$

for all distinct points x, y in X, and conclude the same as that of Theorem 2.2. In fact, condition (2.3.1) implies condition (2.2.1).

Remark 2.4 : Since condition (1.1.2) implies condition (2.2.1), Theorem 1.1 follows as a corollary to Theorem 2.2. Further, Example 2.2 shows that Theorem 2.2 is a proper generalization of Theorem 1.1.

Example 2.2 — Let $X = [0, 2]$ with the usual metric, define $T = X \to X$ by

$$Tx = \begin{cases} 2 - x & \text{if } x \in [0, 2] \\ \dfrac{x+1}{2} & \text{if } x \in [1, 2] \end{cases}$$

with $\phi(t) = t, \quad t \geq 0.$

Evidently 1 is the unique fixed point of T. It can be easily verified that T satisfies condition (2.2.1) for all distinct x, y in X. However, T fails to satisfy condition (1.1.2) at $y = 1$, $x < 1$ for any c in [0, 1], for, in this case

$$\phi(d(Tx, Ty)) = 1 - x \text{ for any } c \text{ in } [0, 1].$$

$$c\phi(d(x, y)) + \frac{(1-c)}{2}[\phi(d(x, Tx)) + \phi(d(y, Ty))] = c(1-x) + \frac{(1-c)}{2}(2-x) = 1 - x.$$

Thus, Theorem 2.2 is stronger than Theorem 1.1.

In the following Example 2.3, $\phi(t) = t$ does not meet the requirement (satisfying (2.2.1)) and hence we take a different function to alter the distances.

Example 2.3 — Let $X = \{1, 2, 3, \ldots\}$ with the usual metric. Define $T : X \to X$ by $Tx = x^2$. Then, the condition (2.2.1) fails to hold for $\phi(t) = t$; for, in this case, when $x \neq 1$,

$d(Tx, T1) \not< \max \{d(x, 1), d(x, Tx), d(1, T1)\}$. But, however (2.2.1) holds by defining $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ by

$$\phi(t) = \begin{cases} t^2 & \text{if } 0 \leq t \leq 1 \\ t^{-1} & \text{if } t \geq 1. \end{cases}$$

In this case, for all distinct x and y in X,

$$\phi(d(Tx, Ty)) = \phi(|x^2 - y^2|) = \frac{1}{|x^2 - y^2|}$$

and

$$\phi(d(x, y)) = \phi(|x - y|) = \frac{1}{|x - y|}.$$

Since $\frac{1}{|x^2 - y^2|} < \frac{1}{|x - y|}$ for all distinct x, y in X, we have

$$\phi(d(Tx, Ty)) < \phi(d(x, y)) \leq \max \{\phi(d(x, y)), \phi(d(x, Tx)), \phi(d(y, Ty))\}$$

so that the inequality (2.2.1) holds.

By taking $\phi(t) = t$ for all $t \geq 0$, we have the following corollary (Theorem 5 of Sehgal[4]) to Theorem 2.2.

Corollary 2.4 — Suppose T is as in (1.1.1), with

$$d(Tx, Ty) < \max \{d(x, y), d(x, Tx), d(y, Ty)\}.$$

Then z is the unique fixed point of T.

Remark 2.5 : Pathak and Sharma[2] (Theorem 4) proved a fixed point theorem for T satisfying a condition similar to (1.1.2), but their result fails in the sense that T may not have a unique fixed point, but has a fixed point. If we take the statement of Theorem 4 of Pathak and Sharma[2] in this way, this becomes a corollary of Theorem 2.1.

3. FIXED POINT THEOREMS FOR A PAIR OF SELFMAPS

Theorem 3.1 — *Let S and T be selfmaps of a metric space (X, d). For $x_0 \in X$, define the sequence $\{x_n\}$ by $x_{2n+1} = Sx_{2n}$, $x_{2n+2} = Tx_{2n+1}$. Suppose either*

(A) $\{x_{2n}\}$ has a cluster point z in X; and S, ST are continuous at z or

(B) $\{x_{2n+1}\}$ has a cluster point z in X; and T, ST are continuous at z.

Assume that there exists a $\phi \in \Phi$ such that

$$\phi(d(Sx, Ty)) < \phi(d(x, y)) \qquad \ldots (3.1.1)$$

for each distinct x, $y \in \{\bar{x}_n\}$ satisfying either $x = Ty$ or $y = Sx$.

Then (1) either S or T has a fixed point in $\{x_n\}$

or (2) z is a common fixed point of S and T.

PROOF : Suppose $x_{2n} = x_{2n+1}$ for some n, then $x_{2n} = x_{2n+1} = Sx_{2n}$ so that x_{2n} is a fixed point of S.

If $x_{2n+1} = x_{2n+2}$ for some n, then $x_{2n+1} = Tx_{2n+1}$ so that x_{2n+1} is a fixed point of T.

Suppose $x_n \neq x_{n+1}$ for all n. We write $\alpha_n = \phi(d(x_n, x_{n+1}))$. Then

$$\alpha_{2n} = \phi(d(x_{2n}, x_{2n+1})) = \phi(d(Tx_{2n-1}, Sx_{2n})) < \phi(d(x_{2n+1}, x_{2n})) = \alpha_{2n-1}.$$

Similarly, $\alpha_{2n+1} < \alpha_{2n}$ holds.

Hence, $\{\alpha_n\}$ is a strictly decreasing sequence of positive terms, converges to a limit, say, α.

Suppose (*A*) holds. Then there exists a sequence $\{k(n)\}$ of positive integers such that $\{x_{2k(n)}\}$ converges to z. Then $\{\alpha_{2k(n)}\}$ converges to α. Now, by the continuity of ϕ and continuity of S at z,

$$\alpha = \lim_{n\to\infty} \alpha_{2k(n)} = \lim_{n\to\infty} \phi(d(x_{2k(n)}, x_{2k(n)+1})) = \lim_{n\to\infty} \phi(d(x_{2k(n)}, Sx_{2k(n)}))$$

$$= \phi(d(z, Sz)).$$

If $\alpha = 0$ then $Sz = z$.

If $\alpha \neq 0$ then by the continuity of S and TS at z, we have

$$\alpha = \lim_{n\to\infty} \alpha_{2k(n)+1} = \lim_{n\to\infty} \phi(d(x_{2k(n)+1}, x_{2k(n)+2})) = \lim_{n\to\infty} \phi(d(Sx_{2k(n)}, (TS)x_{2k(n)}))$$

$$= \phi(d(Sz, TSz)) < \phi(d(z, Sz))$$

so that

$$\alpha = \phi(d(Sz, TSz)) < \phi(d(z, Sz)) = \alpha$$

a contradiction. Hence, α must be zero. Therefore, $Sz = z$.

Also, $\phi(d(Sz, TSz)) = 0$ so that $Tz = z$.

Thus, z is a common fixed point of S and T.

Similarly, when (*B*) holds, the result follows.

Theorem 3.2 — *Let S and T be selfmaps of a metric space (X, d) such that for some x_0 in X, the sequence $\{(TS)^n x_0\}$ has a convergent subsequence, which converges to a point z in X and S and TS be continuous at z.* ... (3.2.1)

Let there exist a $\phi \in \Phi$ such that for all distinct x, y in X

$$\phi(d(Sx, Ty)) < \max\{\phi(d(x, y)), \phi(d(x, Sx)), \phi(d(y, Ty))\}. \quad \text{... (3.2.2)}$$

Then, exactly one of the following is true:

(*i*) *S has a fixed point and T does not have a fixed point.*

(*ii*) *T has a fixed point and S does not have a fixed point.*

(*iii*) *both S and T have unique fixed points and they coincide.*

This Theorem 3.2 is a corollary of Theorem 3.1 since (3.2.2) implies (3.1.1).

Remark 3.3 : We can replace condition (3.2.2) by

$$\phi(d(Sx, Ty)) < \max\left\{\phi(d(x, y)), \frac{\phi(d(x, Sx)) + \phi(d(y, Ty))}{2}\right\} \quad \text{... (3.3.1)}$$

for all distinct points x and y in X, and conclude the same as that of Theorem 3.2. In fact, condition (3.3.1) implies condition (3.2.2).

Remark 3.4 : When (*i*) of Theorem 3.2 holds, S may have more than one fixed point and T may not have fixed points. This is evident from the following example.

Example 3.1 — Let $X = \{0, 1\}$. Define selfmaps S and T on X by

$$Sx = \begin{cases} 0 & \text{if } x = 0 \\ 1 & \text{if } x = 1 \end{cases}$$

and

$$Tx = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x = 1 \end{cases}$$

Then S and T satisfy all the conditions of Theorem 3.2 with any $\phi \in \Phi$ (for example $\phi(t) = t^2$). S has two fixed points 0 and 1 whereas T has no fixed points.

Remark 3.5 : Theorem 2.1 (Theorem 2.2) follows as a corollary to Theorem 3.1 (Theorem 3.2) with $S = T$ in Theorem 3.1 (Theorem 3.2).

If we take $\phi(t) = t$ for all $t \geq 0$ in Theorem 3.2, we have the following corollary.

Corollary 3.6 — Assume S and T satisfy (3.2.1) together with the inequality $d(Sx, Ty) < \max \{d(x, y), d(x, Sx), d(y, Ty)\}$ for all distinct x, y in X. Then the conclusion of Theorem 3.2 holds.

Remark 3.7 : If we assume $S = T$ and $\phi(t) = t$ for all $t \geq 0$ in Theorem 3.2, then Corollary 2.4 follows.

Remark 3.8 : The following are certain situations where there exists an x_0 in X such that $\{T^n x_0\}$ has a convergent subsequence.

(*i*) (X, d) is compact.

(*ii*) $T(X)$ is relatively compact.

(*iii*) Operator T is compact.

Open Problems :

1. Find an example of a metric space X, a continuous function $\phi : R^+ \to R^+$ which vanishes only at zero and a selfmap T on X with a fixed point z satisfying the condition (2.2.1) such that for any x in X, $x \neq z$ implies $\{T^n x\}$ does not have convergent subsequence.
2. In Theorem 3.1, if (2) holds can we conclude that $x_n \to z$ as $n \to \infty$.

ACKNOWLEDGEMENT

The authors are thankful to the referee for the valuable suggestions which have improved the presentation of the paper.

This work is supported by UGC Grant No. U4/4997/97-98. One of the authors (G.V.R. Babu) thanks the University Grants Commission, India for the financial support.

REFERENCES

1. M. S. Khan, M. Swaleh and S. Sessa, *Bull. Austral. Math. S* **30** (1984), 1-9.
2. H. K. Pathak and Rekha Sharma, *Math. Edn.* **28** (1994) No. 3, 151-57.
3. B. E. Rhoades, *Trasn. Amer. Math. Soc.* **226** (1977), 257-90.
4. V. M. Sehgal, *J. London Math. Soc.* **V** (1972), 571-76.