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A SEMI-DYNAMICAL SYSTEM INDUCED BY COMPOSITION OPERATORS

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In this paper, we use the strong continuity of a semigroup of composition operators induced by a one parameter semigroup of holomorphic mappings of the unit disc $\mathbb{D}$ into itself to introduce a semi-dynamical system having no periodic point.

Key Words : Dynamical System; Analytic Map; Composition Operator; Semigroup; Fixed Point; Periodic Point; Effective Action

1. INTRODUCTION AND PRELIMINARIES

Let $V(X)$ denote a linear space of complex-valued functions on a non-empty set X. If $\phi : X \to X$ is a mapping such that the composite function $f \circ \phi$ defined as $(f \circ \phi)(x) = f(\phi(x))$ for all $x \in X$ belongs to $V(X)$ for every $f \in V(X)$, then the mapping $C_\phi : V(X) \to V(X)$ defined as $C_\phi f = f \circ \phi$ for every $f \in V(X)$ is a linear transformation on $V(X)$. In case $V(X)$ is a topological vector space and C_ϕ is continuous, we call it a composition operator induced by ϕ. If we take $X = \mathbb{D}$, the open unit disc in the complex plane and $V(X) = H^p(\mathbb{D})$, $1 \le p < \infty$, the Hardy space of complex-valued analytic functions f on $\mathbb{D}$ for which

$$\sup_{0<r<1} \left\{ \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta \right\} < \infty,$$

then it is known from Schwartz[5] that every analytic selfmap of the unit disc induces a composition

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operator on $H^p(\mathbb{D})$. In particular, if we take $p = 2$, then $H^2(\mathbb{D})$ becomes a Hilbert space. Further, if ϕ is an analytic function mapping the upper half-plane $\pi^+ = \{w \in \mathbb{C} : \operatorname{Im} w > 0\}$ into itself and $H^2(\pi^+)$ is the Hilbert space of all those complex-valued functions f on π^+ for which

$$\sup_{y>0} \left\{ \int_{-\infty}^{\infty} |f(x+iy)|^2 \, dx \right\} < \infty,$$

then the composition transformation C_ϕ on $H^2(\pi^+)$ defined by $C_\phi f = f \circ \phi$ need not be continuous. For counter examples, we refer to Sharma and Singh[7]. In fact, if C_ϕ maps $H^2(\pi^+)$ into itself, then by an application of closed graph theorem, one can show that C_ϕ is a continuous operator on $H^2(\pi^+)$. Sharma and Singh[7] characterized those analytic selfmaps of π^+ which induces composition operators on $H^2(\pi^+)$ in terms of the behaviour of the inducing maps in the vicinity of the point at infinity.

In this paper, a one parameter semigroup of analytic functions of π^+ is defined and it is used to construct a semi-dynamical system having no non-zero periodic point.

2. SEMI-DYNAMICAL SYSTEMS

Let us start with the following definitions and examples :

Definition 2.1 — A semi-analytical dynamical system is a triple $(\mathbb{R}^+, H(X), \Pi)$, where $\mathbb{R}^+$ is the set of non-negative real numbers, $H(X)$ is a topological vector space of holomorphic functions on a non-empty subset X of $\mathbb{C}$ or $\mathbb{C}^n$ and $\Pi : \mathbb{R}^+ \times H(X) \to H(X)$ is a mapping which satisfies the following properties :

$$\left.\begin{array}{l} (i)\ \Pi \text{ is continuous; (ii) } \Pi(0, f) = f \text{ for every } f \in H(X); \\ (iii)\ \Pi(t+s, f) = \Pi(t, \Pi(s, f)) \text{ for every } f \in H(X) \text{ and } s, t \in \mathbb{R}^+. \end{array}\right\} \quad \ldots (2.1)$$

For details about the dynamical systems, we refer to Brown[2]

Definition 2.2 — Let U be an open set in the complex plane $\mathbb{C}$. A one parameter semigroup $\{\phi_t\}$ of holomorphic mappings of U into itself is a homomorphism $t \to \phi_t$ of the additive semigroup of non-negative real numbers $\mathbb{R}^+$ into the semi-group (under composition) of all analytic mappings of U into U such that ϕ_0 is the identity map of U and $\phi_t(z)$ is continuous in (t, z) in $\mathbb{R}^+ \times U$.

Example 2.3 — For each $t \geq 0$, let $\psi_t(z) = \dfrac{e^{-t} z}{(e^{-t} - 1)z + 1}, \quad z \in \mathbb{D}$.

Then the following properties are easily verified :

(*i*) for each $t \geq 0$, ψ_t maps $\mathbb{D}$ into $\mathbb{D}$;

(*ii*) $\psi_{t+s} = \psi_t \circ \psi_s$, $t, s \geq 0$;

(*iii*) ψ_0 is the identity map of $\mathbb{D}$;

(*iv*) $\psi_t(z)$ is continuous in (t, z) on $\mathbb{R}^+ \times \mathbb{D}$. ... (2.2)

Thus, the family $\{\psi_t : t \geq 0\}$ is a one parameter semigroup of analytic functions of $\mathbb{D}$.

Example 2.4 — For each $t \geq 0$, let $\phi_t(w) = w + tw_0$, for each $w \in \pi^+$, ... (2.3)

where $w_0 = x_0 + iy_0 \in \pi^+$. The following properties are easily verified :

(*i*) for each $t \geq 0$, ϕ_t maps π^+ into π^+;

(*ii*) $\phi_{t+s} = \phi_t \circ \phi_s$, $t, s \geq 0$;

(*iii*) ϕ_0 is the identity of π^+;

(*iv*) $\phi_t(w)$ is continuous in (t, w) on $\mathbb{R}^+ \times \pi^+$. ... (2.4)

Thus, the family $\{\phi_t : t \geq 0\}$ is a one parameter semigroup of analytic functions on π^+.

Further, it is known from Cowen[3] that the composition operator C_{ϕ_t} defined on $H^2(\pi^+)$ by $C_{\phi_t} f = f \circ \phi_t$ is bounded and $\| C_{\phi_t} \| = e^{t/2} \geq 1$. Thus, the family $\{C_{\phi_t} : t \geq 0\}$ consists of bounded operators on $H^2(\pi^+)$. It is evident from the properties (*ii*) and (*iii*) of (2.4) that $\{C_{\phi_t}\}$ satisfies

(*i*) $C_{\phi_t} C_{\phi_s} = C_{\phi_{t+s}}$, $t, s \geq 0$; (*ii*) C_{ϕ_0} is the identity operator on $H^2(\pi^+)$.

Hence $\{C_{\phi_t} : t \geq 0\}$ is a semigroup under the operation of composition of operators on $H^2(\pi^+)$.

Definitions 2.5 — A one parameter semigroup of operators $\{A_t : t \geq 0\}$ on a Hilbert space H is said to be a strongly continuous semigroup of operators if the correspondence $t \to A_t$ is a continuous map from $\mathbb{R}^+$ with the usual topology to $B(H)$, the Banach algebra of bounded linear operators on H, with the strong operator topology.

Berkson and Porta[1] proved that if $\{\psi_t : t \geq 0\}$ is a one parameter semigroup of holomorphic mappings of the unit disc $\mathbb{D}$ into itself, then the induced composition semigroup $\{C_{\psi_t} : t \geq 0\}$, where $C_{\psi_t} f = f \circ \psi_t$ is strongly continuous on $H^p(\mathbb{D})$, for $1 \leq p < \infty$. We shall use this result in the following theorem to construct a semi-dynamical system on $H^2(\pi^+)$:

Theorem 2.6 — *Let* $\Pi : \mathbb{R}^+ \times H^2(\pi^+) \to H^2(\pi^+)$ *be the function defined by* $\Pi(t, f) = C_{\phi_t} f$ *for* $t \in \mathbb{R}^+$ *and* $f \in H^2(\pi^+)$ *where* ϕ_t *is as defined in (2.3). Then* $(\mathbb{R}^+, H^2(\pi^+), \Pi)$ *forms a semi-dynamical system i.e.* Π *satisfies all the three properties given in (2.1).*

PROOF : The only part remains to show that Π is continuous, which is further equivalent to show that Π is separately continuous. Since continuity of Π in the second coordinate follows

from the boundedness of C_{ϕ_t} for all $t \geq 0$, we need only to show that Π is continuous in the first coordinate. To achieve this end, let $t_n \to 0$ in $\mathbb{R}$. Then we have to show that

$$\lim_{n \to \infty} \| C_{\phi_{t_n}} f - f \| = 0.$$

In other words, we have shown that $\{C_{\phi_t} : t \geq 0\}$ is strongly continuous on $H^2(\pi^+)$. For $t \geq 0$, define ψ_t on the unit disc $\mathbb{D}$ into itself by $\psi_t = L^{-1} \circ \phi_t \circ L$ where $L : \mathbb{D} \to \pi^+$ is a linear fractional transformation defined by

$L(z) = \dfrac{i(1+z)}{1-z}$ with $L^{-1} : \pi^+ \to \mathbb{D}$ defined by $L^{-1}(w) = \dfrac{w-i}{w+i}$ (cf. Duren[4]). Then we have the following :

(*i*) for each $t \geq 0$, ψ_t maps $\mathbb{D}$ into $\mathbb{D}$;

(*ii*) $\psi_{t+s} = L^{-1} \circ \phi_{t+s} \circ L = L^{-1} \circ \phi_t \circ \phi_s \circ L = \psi_t \circ \psi_s$, $t, s \geq 0$;

(*iii*) ψ_0 is the identity map of $\mathbb{D}$;

(*iv*) $\psi_t(z)$ is continuous in (t, z) on $\mathbb{R}^+ \times \mathbb{D}$.

Thus the family $\{\psi_t : t \geq 0\}$ forms a one parameter semigroup of analytic functions on $\mathbb{D}$. Therefore, by Theorem 3.4 of Berkson and Porta[1], the induced composition semigroup $\{C_{\psi_t} : t \geq 0\}$ is strongly continuous on $H^2(\mathbb{D})$. We shall now show that $\{C_{\phi_t} : t \geq 0\}$ is strongly continuous on $H^2(\pi^+)$. Now, by a result of Singh[8],

$$C_{\phi_t} = \tilde{P} Q P^{-1} M_{\beta_t} C_{\psi_t} P Q^{-1} \tilde{P}^{-1},$$

where P and $\tilde{P}$ denote the Poisson integrals in the unit disc and in the upper half-plane respectively, the transformation Q on $L^2(m)$, defined by

$$(Qf)(x) = \frac{(f \circ L^{-1})(x)}{\sqrt{x}\,(x+i)}$$

is a well established isometric isomorphism of $L^2(m)$ onto $L^2(\mu)$, m being the normalized Lebesgue measure on the unit circle $\partial\mathbb{D}$ and μ being the Lebesgue measure on the real line $\mathbb{R}$. In fact, Q, P, $\tilde{P}$ are unitary transformations and so $U = PQ^{-1}\tilde{P}^{-1}$ is a unitary transformation from $H^2(\pi^+)$ onto $H^2(\mathbb{D})$ and M_{β_t} is the multiplication operator on $H^2(\mathbb{D})$ induced by $\beta_t = \dfrac{1-\psi_t}{1-e_1}$, where $e_1(z) = z$. Here, the boundedness of M_{β_t} is implied by the following facts :

(*i*) C_{ϕ_t} is a bounded operator on $H^2(\pi^+)$;

(*ii*) $M_{\beta_t} C_{\psi_t}$ is unitarily equivalent to C_{ϕ_t}; and

(*iii*) C_{ψ_t} is invertible on $H^2(\mathbb{D})$.

We have, for $f \in H^2(\pi^+)$,

$$\| C_{\phi_{tn}} f - f \| = \| U^{-1} M_{\beta_{tn}} C_{\psi_{tn}} Uf - U^{-1} Uf \| \leq \| M_{\beta_{tn}} C_{\psi_{tn}} g - g \|,$$

where $g = Uf \in H^2(\mathbb{D})$. In view of the above inequality, it is sufficient to prove that $\{T_t : t \geq 0\}$ is a strongly continuous semigroup of operators on $H^2(\mathbb{D})$, where $T_t = M_{\beta_t} C_{\psi_t}$ for all $t \in \mathbb{R}^+$. Let $f \in H^2(\mathbb{D})$. Then

$$T_s T_t f(z) = T_s M_{\beta_t} C_{\psi_t} f(z) = T_s \beta_t(z) f(\psi_t(z)) = (M_{\beta_s} C_{\psi_s}) \beta_t(z) f(\psi_t(z)))$$

$$= \beta_s(z) \beta_t(\psi_s(z)) f(\psi_t)) (\psi_s(z)))$$

$$= \frac{1 - \psi_s(z)}{1 - e_1} \frac{1 - \psi_t(\psi_s(z))}{1 - \psi_s(z)} f(\psi_{t+s}(z))$$

$$= \frac{1 - \psi_{t+s}(z)}{1 - e_1} C_{\psi_{t+s}} f(z)$$

$$= M_{\beta_{t+s}} C_{\psi_{t+s}} f(z)$$

$$= T_{s+t} f(z),$$

which implies that $\{T_t : t \geq 0\}$ is a semigroup of operators on $H^2(\mathbb{D})$. Since $\{C_{\psi_t} : t \geq 0\}$ is a strongly continuous semigroup of operators on $H^2(\mathbb{D})$ and $\beta_t \in H^\infty(\mathbb{D})$, by Theorem 5.3.44 of Singh and Manhas[9], $\{T_t : t \geq 0\}$ is a strongly continuous semigroup of operators on $H^2(\mathbb{D})$. This completes the proof of the theorem.

Definition 2.7 — In a semi dynamical system $(\mathbb{R}^+, H(X), \Pi)$, Π is said to be an effective action if for every $0 \neq t \in \mathbb{R}^+$ there exists $f \neq 0$ in $H(X)$ such that $\Pi(t, f) \neq f$, and it is said to be free motion if for every $0 \neq t \in \mathbb{R}^+$, $\Pi(t, f) \neq f$ for every $0 \neq f$ in $H(X)$.

It is obvious that every free action is an effective action.

In the following theorem, we shall prove that Π does not have any periodic point in $H^2(\pi^+)$. It may be recalled that a function $f \in H^2(\pi^+)$ is said to be periodic if $\Pi(t, f) = f$ for some $t \in \mathbb{R}^+$. Non-existence of periodic functions also guarantees that no function in $H^2(\pi^+)$ can be a fixed point of Π. In fact, a function $f \in H^2(\pi^+)$ is a fixed point of Π if $\Pi(t, f) = f$ for each $t \in \mathbb{R}^+$. Finally, we shall see that Π is a free motion and so it is an effective one.

Theorem 2.8 — *Π does not have any periodic point in $H^2(\pi^+)$.*

PROOF : Let, if possible, a function $0 \neq f \in H^2(\pi^+)$ be periodic. Then for a subset E of $\mathbb{R}^+$, $\Pi(t,f)=f$ for every $0 \neq t \in E$, that is $C_{\phi_t} f = f$. Thus for a fixed $t \in E, f(\phi_t(w)) = f(w)$ for each $w \in \pi^+$. Now, if $y > 0$ and n is a positive integer, then

$$|f(iy)| = |f(iy + tw_0)| = |f(\phi_t(iy + tw_0)| = |f(iy + 2tw_0)|$$

$$= |f(iy + ntw_0)| = |f(ntx_0 + i(y + nty_0)|.$$

Using Lemma 2.1 of Sharma[6], we have

$$|f(iy)| \leq \frac{\|f\|}{\sqrt{2\pi(y + nty_0)}} \text{ for each } y > 0,$$

which tends to 0 as n approaches infinity. Thus f vanishes on $\{iy : y > 0\}$. The analyticity of f implies that f is identically zero. This contradicts the fact that f is non-zero. Hence the theorem.

Corollary 2.9 — Π does not have a fixed point.

Corollary 2.10 — Π is a free motion.

Proofs of the above corollaries follow directly from Theorem 2.8 above.

REFERENCES

1. E. Berkson and H. Porta, *Michigan Math. J* **25** (1978), 101-15.
2. J. R. Brown, *Ergodic Theory and Topological Dynamics*, Academic Press, New York, 1976.
3. C. C. Cowen, *Indiana University Math. J.* **33** (1984), 305-18.
4. P. L. Duren, *Theory of H^p Spaces*, Academic Press, New York, 1970.
5. H. J. Schwartz, Composition operators on H^p, *Ph.D. Thesis*, University of Toledo, Ohio. 1969.
6. S. D. Sharma, *Acta Sci. Math (Szeged)* **46** (1983), 197-202.
7. S. D. Sharma and R. K. Singh, *Bull. Austral. math. Soc.* **20** (1979), 377-84.
8. R. K. Singh, *Mathematika Sciences* **1** (1975), 1-5.
9. R. K. Singh and J. S. Manhas, *Composition Operators on Function Spaces*, North-Holland Math. Studies 179, Amsterdam, 1993.