Indian J. pure appl. Math., 30(7) : 735-744, July 1999
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GLOBAL EXISTENCE OF SOLUTIONS OF NONLINEAR FUNCTIONAL INTEGRAL EQUATIONS

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(*Received* 10 *January* 1998; *after Revision* 7 *October* 1998; *Accepted* 2 *December* 1998)

In the present paper global existence results for certain nonlinear functional integral and inregro-differential equations are established. Our analysis is based on an application of the Leray-Schauder alternative.

Key Words : Global Existence Results; Nonlinear Functional Integral; Leray-Schauder Alternative; Volterra Integral; Fixed Point Theory

1. INTRODUCTION

Let R^n be the n-dimensional Euclidean space and $|\cdot|$ denotes the Euclidean norm on R^n. Let $C = C([-r, 0], R^n)$, $r > 0$, denote the Banach space of all continuous functions $\psi : [-r, 0] \to R^n$, endowed with the sup-norm

$$\| \psi \| = \sup \{ | \psi(\theta) | : \theta \in [-r, 0] \}.$$

If x is a continuous function from $[-r, T]$, $T > 0$, to R^n and $t \in [0, T]$, then x_t denotes the element of C given by $x_t(\theta) = x(t + \theta)$ for $\theta \in [-r, 0]$. In the present paper, we shall be concerned with the nonlinear functional integral and integrodifferential equations of the forms

$$x(t) = h(t) + \int_0^t a(t, s)\, g(s, x(s), x_s)\, ds, \qquad t \in [0, T], \qquad \ldots (1.1)$$

$$x(t) = \phi(t), \quad t \in [-r, 0] \qquad \ldots (1.2)$$

$$x'(t) = f(t, x(t), x_t, \int_0^t a(t, s)\, g(s, x(s), x_s)\, ds), \qquad t \in [0, T]; \qquad \ldots (1.3)$$

and

$$x(t) = \phi(t), \ t \in [-r, 0], \qquad \ldots (1.4)$$

where $h : [0, T] \to R^n$, $a : [0, T] \times [0, T] \to R$, $g ; [0, T] \times R^n \times C \to R^n$, $f ; [0, T] \times R^n \times C \times R^n \to R^n$ are

continuous functions and $\phi \in C$ is given.

Equations of the type (1.1) - (1.2) and (1.3) - (1.4) or their special forms arise in many areas of applied mathematics. The problems of existence, uniqueness and other properties of solutions of the special forms of these equations have been studied by many authors by using different techniques, for example, see [1-3, 6, 8, 9, 11, 12] and the references listed therein. The aim of the present paper is to study the global existence of solutions of the equations (1.1) - (1.2) and (1.3) - (1.4) by using topological transversality theorem of Granas[4], known as Leray-Schauder alternative. The main feature of the method employed here is that it yields simultaneously the existence of a solution and the maximal interval of existence.

2. STATEMENT OF RESULTS

We use the following version of the topological transversality theorem given by Grans in ([4], p. 61)

Theorem *G — Let B be a convex subset of a normed linear space E and assume $O \in B$. Let $F : B \to B$ be a completely continuous operator and let*

$$U(F) = \{x \in B : x = \lambda Fx \text{ for some } 0 < \lambda < 1.\}$$

Then either $U(F)$ is unbounded or F has a fixed point.

We list the following hypotheses for convenience :

(H_1) There exists a constant $c \geq 0$ such that

$$| h(t) | \leq c$$

for every $t \in [0, T]$.

(H_2) There exists a constant $c_1 \geq 0$ such that

$$| a(t, s) | \leq c_1 \text{ for } t \geq s \geq 0.$$

(H_3) For each $\bar{t} \in [0, T]$,

$$\lim_{t \to \bar{t}} \int_0^T | a(t, s) - a(\bar{t}, s) | \, ds = 0,$$

is satisfied for $t \in [0, T]$.

(H_4) There exists a continuous function $q : [0, T] \to R_+ = [0, \infty)$ such that

$$| g(t, u, \psi) | \leq q(t) \, G(| u | + \| \psi \|)$$

for every $t \in [0, T]$, $u \in R^n$ and $\psi \in C$, where $G : R_+ \to (0, \infty)$ is a continuous nondecreasing function.

(H_5) There exists a continuous function $p : [0, T] \to R_+$ such that

$$| f(t, u, \psi, v) | \leq p(t) \, (| u | + \| \psi \| + | v |),$$

for every $t \in [0, T]$, $u, v \in R^n$ and $\psi \in C$.

Our main results are established in the following theorems :

Theorem 1 — *Suppose that the hypotheses* (H_1) — (H_4) *hold. If T satisfies the following condition*

$$\int_0^T C_1 q(s)\, ds < \int_c^\infty \frac{ds}{G(2s)} \qquad \ldots (2.1)$$

where $c = \| \phi \|$ *and* $\phi(0) = h(0)$, *then the initial value problem (1.1) - (1.2) has a solution x(t) defined on [– r, T].*

Remark 1 : We note that in [11], Pachpatte has studied the global existence of the solution of initial value problem of the type (1.1)-(1.2) without functional argument and our result given above is a further generalization of the result given in [11, Th. 1].

Theorem 2 — *Suppose that the hypotheses* (H_2), (H_4) *and* (H_5) *hold. Further suppose that T satisfies the following condition*

$$\int_0^T M(s)\, ds < \int_{2c}^\infty \frac{ds}{2s + G(s)}, \qquad \ldots (2.2)$$

where

$$c = \| \phi \| \text{ and } M(t) = \max \{p(t), C_1, q(t)\},\ t \in [0, T]. \qquad \ldots (2.3)$$

Then the initial value problem (1.3)-(1.4) has a solution x(t) defined on [– r, T].

Remark 2 : We note that in [10], the topological transversality method is used to study the global existence of solutions of the special version of eqs. (1.3)-(1.4) when the function f is of the type,

$$A(t)\, x(t) + \int_0^t a(t, s) g(s, x_s) ds,$$

where $A(t)$ is a continuous $n \times n$ matrix for $t \in [0, T]$. Our result given here is motivated by the recent work of various investigators[3, 7,12]

3. PROOF OF THEOREM 1

First, we establish the priori bounds for $(1.1)_\lambda$-(1.2), $\lambda \in (0, 1)$ where

$$x(t) = h(t) + \lambda \int_0^t a(t, s)\, g(s, x(s), x_s)\, ds, \qquad t \in [0, T]. \qquad \ldots (1.1)_\lambda$$

Let $x(t)$ be a solution of $(1.1)_\lambda$-(1.2). By using $(1.1)_\lambda$ and hypotheses (H_1), (H_2), $H_4)$ we obtain

$$|x(t)| \leq c + \int_0^t c_1 q(s) G(|x(s)| + \|x_s\|)\, ds. \qquad \ldots (3.1)$$

Now, we consider the function $z : [-r, T] \to R$ defined by

$$z(t) = \sup \{|x(s)| : -r \leq s \leq t\}, \quad -r \leq t \leq T$$

and let $t^* \in [-r, t]$ be such that $z(t) = |x(t^*)|$. If $t^* \in [0, t]$, then by (3.1), we have,

$$z(t) = |x(t^*)| \leq c + \int_0^{t^*} c_1 q(s) G(|x(s)| + \|x_s\|)\, ds$$

$$\leq c + \int_0^t c_1 q(s) G(|x(s)| + \|x_s\|) ds$$

$$= c + \int_0^t c_1 q(s) G(2z(s))\, ds. \qquad \ldots (3.2)$$

If $t^* \in [-r, 0]$ then $z(t) = |x(t^*)| \leq \|\phi\| = C$ and hence (3.2) holds.
Let

$$u(t) = c + \int_0^t c_1\, q(s)\, G(2z(s))\, ds;$$

then $z(t) \leq u(t)$, $t \in [0, T]$, $u(0) = c$ and

$$u'(t) \leq c_1\, q(t)\, G(2u(t)),$$

which imply

$$\frac{u'(t)}{G(2u(t))} \leq c_1\, q(t). \qquad \ldots (3.3)$$

Integrating (3.3) from 0 to t and by making use of the change of variable and the condition (2.1), we obtain

$$\int_c^{u(t)} \frac{ds}{G(2s)} \leq \int_0^t c_1\, q(s) ds \leq \int_0^T c_1 q(s) ds$$

$$< \int_0^\infty \frac{ds}{G(2s)}\,. \qquad \ldots (3.4)$$

Using (3.4), we see that there is a constant α independent of $\lambda \in (0, 1)$ such that $u(t) \leq \alpha$ for

$t \in [0, T]$ and hence $z(t) \leq \alpha$ for $t \in [0, T]$. Then it follows that $|x(t)| \leq \alpha$ for $t \in [-r, T]$ and consequently, $\|x\| = \text{Sup}\{|x(t)| : t \in [-r, T]\} \leq \alpha$.

Now, we rewrite the initial value problem (1.1)-(1.2) as follows. For $\phi \in C$, define $\bar{\phi} \in B$, $B = C([-r, T], R^n)$, by

$$\bar{\phi}(t) = \begin{cases} \phi(t), & -r \leq t \leq 0 \\ h(t), & 0 \leq t \leq T. \end{cases}$$

If $y \in B$ and $x(t) = y(t) + \bar{\phi}(t)$, $t \in [-r, T]$, then, clearly, $y(t)$ satisfies

$$y(t) = y_0 = 0, \; -r \leq t \leq 0,$$

and
$$y(t) = \int_0^t a(t, s)\, g(s, y(s) + \bar{\phi}(s), y_s + \bar{\phi}_s)\, ds \qquad t \in [0, T],$$

if and only if $x(t)$ satisfies (1.1)-(1.2). Define an operator $F : B_0 \to B_0$, $B_0 = \{y \in B, y_0 = 0\}$ as follows :

$$Fy(t) = \begin{cases} 0, & -r \leq t \leq 0 \\ \int_0^t a(t, s) g(s, y(s) + \bar{\phi}(s), (y_s + \bar{\phi}_s) ds & 0 \leq t \leq T \end{cases} \qquad \ldots (3.5)$$

Then, clearly F is continuous. Next, we prove that F is completely continuous.

Let $\{w_m\}$ be a bounded sequence in B_0, i.e., $\|w_m\| \leq b$ for all m, where b is a positive constant. Then $\|w_{m_t}\| \leq b$, $t \in [0, T]$, for all m. From (3.5) and making use of hypotheses (H_1) - (H_4) and letting $\beta = \sup\{q(t) : t \in [0, T]\}$, we have

$$|F_{w_m}(t)| \leq \int_0^t c_1 q(s) G(|w_m(s) + \bar{\phi}(s)| + \|w_{m_s} + \bar{\phi}_s\|)\, ds$$

$$\leq \int_0^t c_1 \beta G(2(b + c))\, ds$$

$$\leq c_1 \beta G(2(b + c)) T.$$

Hence,

$$\|Fw_m\| \leq c_1 \beta T G(2(b + c))$$

which implies $\{Fw_m\}$ is uniformly bounded.

Now, we prove that sequence $\{Fw_m\}$ is equicontinuous. We consider the following cases :

Case (*i*) — $0 \leq t_1 \leq t_2$: Then, by using (3.5), we obtain

$$| Fw_m(t_2) - Fw_m(t_1) |$$

$$\leq \int_0^{t_2} | a(t_2, s) - a(t_1, s) | \, g(s, w_m(s) + \bar{\phi}(s), w_{m_s} + \bar{\phi}_s) \, ds$$

$$+ \int_{t_1}^{t_2} | a(t_1, s) | \, g(s, w_m(s) + \bar{\phi}(s), w_{m_s} + \bar{\phi}_s) \, ds$$

$$\leq \int_0^{t_2} | a(t_2, s) - a(t_1, s) | \, q(s) \, G(| w_m(s) + \bar{\phi}(s) | + \| W_{m_s} + \bar{\phi}_s \|) \, ds$$

$$+ \int_{t_1}^{t_2} | a(t_1, s) | \, q(s) G(| w_m(s) + \bar{\phi}(s) | + \| w_{m_s} + \bar{\phi}_s \|) \, ds$$

$$\leq \int_0^{T} | a(t_2, s) - a(t_1, s) | \, \beta G(2(b + c)) ds + \int_{t_1}^{t_2} c_1 \beta G(2(b + c)) \, ds. \qquad \ldots (3.6)$$

Case (ii) — $t_1 \leq 0 \leq t_2$: Then, using (3.5), we have

$$| Fw_m(t_2) - Fw_m(t_1) |$$

$$\leq \int_0^{t_2} | a(t_2, s) | \, | g(s, w_m(s) + \bar{\phi}(s), w_{m_s} + \bar{\phi}_s) | \, ds$$

$$\leq \int_0^{t_2} | a(t_2, s) | \, | q(s) \, G(| w_m(s) + \bar{\phi}(s) | + \| w_{m_s} + \bar{\phi}_s) \|) \, ds$$

$$\leq \int_0^{t_2} c_1 \, \beta G(2(b + c)) \, ds. \qquad \ldots (3.7)$$

Case (iii) — $t_1 \leq t_2 \leq 0$. Then, by using (3.5), we get

$$| Fw_m(t_2) - Fw_m(t_1) | = 0. \qquad \ldots (3.8)$$

Since the cases (*i*), (*ii*) and (*iii*) imply

$$\| Fw_m(t_2) - Fw_m(t_1) \| \leq \gamma | t_2 - t_1 |,$$

with constant $\gamma > 0$, for every $t_1, t_2 \in [-r, T]$, $\{Fw_m\}$ is equicontinuous and therefore, by the Arzela-Ascoli Theorem, the operator F is completely continuous.

The set

$$U(F) = \{y \in B_0 : y = \lambda Fy, \lambda \in (0, 1)\}$$

is bounded, since for every solution y in $U(F)$ the function $x = y + \bar{\phi}$ is a solution of $(1.1)_\lambda$-(1.2) for which we have proved $\| x \| \leq \alpha$ and hence $\| y \| \leq \alpha + c$. Now, by virtue of Theorem G, the operator F has a fixed point in B_0. Therefore, the initial value problem (1.1)-(1.2) has a solution. Thus, the proof of the Theorem 1 is completed.

4. PROOF OF THEOREM 2

First, we establish the priori bounds for the initial value problem $(1.3)_\lambda$ - (1.4), $\lambda \in (0, 1)$, where

$$x'(t) = \lambda f(t, x(t), x_t, \int_0^t a(t, s) g(s, x(s), x_s) ds), \quad t \in [0, T]. \qquad \ldots (1.3)_\lambda$$

Let $x(t)$ be a solution of the initial value problem $(1.3)_\lambda$-(1.4). Then it satisfies the equivalent integral equation

$$x(t) = \phi(0) + \lambda \int_0^t f(s, x(s), x_s, \int_0^s a(s, \tau)\, g(\tau, x(\tau), x_\tau)\, d\tau) ds, \quad t \in [0, T]. \qquad \ldots (4.1)$$

Using (4.1), (H_2), (H_4) and (H_5) we obtain

$$| x(t) | \leq c + \int_0^t p(s) [| x(s) | + \| x_s \| + \int_0^s c_1 q(\tau)\, G(| x(\tau) | + \| x_\tau \|)\, d\tau]\, ds. \qquad \ldots (4.2)$$

Let the function $z : [-r, T] \to R$ be defined by

$$z(t) = \sup \{| x(s) | : -r \leq s \leq t\}, \quad -r \leq t \leq T,$$

and let $t^* \in [-r, t]$ be such that $z(t) = | x(t^*) |$. If $t^* \in [0, t]$, then from (4.2) and making use of (2.3), we have

$$z(t) = | x(t^*) | \leq c + \int_0^{t^*} p(s)\ [| x(s) | + \| x_s \| + \int_0^s c_1\, q(\tau)\, G(| x(\tau) | + \| x_\tau \|)\, d\tau]\, ds$$

$$\leq c + \int_0^t p(s) [| x(s) | + \| x_s \| + \int_0^s c_1\, q(\tau)\, G(| x(\tau) | + \| x_\tau \|)\, d\tau]\, ds$$

$$\leq c + \int_0^t 2M(s)\, z(s) ds + \int_0^t M(s) \int_0^s M(\tau)\, G(2z(\tau))\, d\tau\, ds \qquad \ldots (4.3)$$

If $t^* \in [-r, 0]$ then $z(t) = |x(t^*)| \leq \|\phi\| = c$ and hence (4.3) holds. Denoting by $u(t)$ the right hand side of (4.3), we have

$$z(t) \leq u(t),\ t \in [0, T],\ u(0) = c$$

and

$$u'(t) \leq M(t)\left[2u(t) + \int_0^t M(s)\, G(2u(s)) ds\right].$$

Let

$$v(t) = 2u(t) + \int_0^t M(s)\, G(2u(s))\, ds.$$

Then

$$u(t) \leq \frac{1}{2} v(t),\ t \in [0, T],\ v(0) = 2u(0) = 2c$$

and

$$v'(t) \leq 2\, M(t) v(t) + M(t)\, G(v(t)),$$

which imply

$$\frac{v'(t)}{2v(t) + G(v(t))} \leq M(t). \qquad \ldots (4.4)$$

Integrating (4.4) from 0 to t and by making use of the change of variable and condition (2.2), we obtain

$$\int_{2c}^{v(t)} \frac{ds}{2s + G(s)} \leq \int_0^t M(s) ds \leq \int_0^T M(s)\, ds < \int_{2c}^{\infty} \frac{ds}{2s + G(s)}. \qquad \ldots (4.5)$$

By using mean value theorem, there exists a constant η independent of $\lambda \in (0, 1)$ such that $v(t) \leq \eta$ for $t \in [0, T]$ and hence $u(t) \leq \eta, z(t) \leq \eta$ for $t \in [0, T]$. Since for every $t \in [0, T]$, $\|x_t\| \leq z(t)$, we have $\|x\| = \sup\{|x(t)| : t \in [-r, T]\} \leq \eta$.

We rewrite the initial value problem (1.3)-(1.4) as follows. For $\phi \in C$, define $\bar{\phi} \in B, B = C([-r, T], R^n)$ by

$$\bar{\phi}(t) = \begin{cases} \phi(t), & -r \leq t \leq 0, \\ \phi(0), & 0 \leq t \leq T. \end{cases}$$

If $y \in B$ and $x(t) = y(t) + \bar{\phi}(t), t \in [-r, T]$ then it is easy to observe that y satisfies

$$y(t) = y_0 = 0, \quad -r \le t \le 0,$$

and $y(t) = \int_0^t f(s, y(s) + \bar{\phi}(s), y_s + \bar{\phi}_s, \int_0^s a(s, \tau)\, g(\tau, y(\tau) + \bar{\phi}(\tau), y_\tau + \bar{\phi}_\tau)\, d\tau)\, ds \in [0, T]$ if and only if x satisfies the equivalent integral equation to (1.3)-(1.4) i.e.

$$x(t) = \phi(0) + \int_0^t f(s, x(s), x_s, \int_0^s a(s, \tau)\, g(\tau, x(\tau), x_\tau)\, d\tau)\, ds, t \in [0, T]$$

$$x(t) = \phi(t), \quad -r \le t \le 0.$$

Define an operator $F : B_0 \to B_0, B_0 = \{y \in B, y_0 = 0\}$ by

$$Fy(t) = \begin{cases} 0, & -r \le t \le 0 \\ \int_0^t f(s, y(s) + \bar{\phi}(s), & y_s + \bar{\phi}_s, \\ \int_0^s a(s, \tau)\, g(\tau, y(\tau) + \bar{\phi}(\tau), y_\tau + \bar{\phi}_\tau)\, d\tau)\, ds, & t \in [0, T] \end{cases} \quad \ldots (4.6)$$

Clearly, F is continuous. The rest of the proof can be completed by following the similar arguments as in the proof of Theorem 1 with suitable modifications and hence we omit the further details. This completes the proof of the Theorem 2.

In concluding this paper, we note that one can easily extend the ideas of this paper to study the global existence of solution of the following functional Volterra integral equation of more general type

$$x(t) = h(t) + \sum_{i=1}^{n} \int_0^t a_i(t, s) g_i(s, x(s), x_s) ds, \quad t \in [0, T], \quad \ldots (4.7)$$

$$x(t) = \phi(t), \quad t \in [-r, 0] \quad \ldots (4.8)$$

under some suitable conditions on the functions involved in (4.7)-(4.8). The precise formulation of this result is very close to that of the result given in our Theorem 1 with suitable modifications and hence we do not discuss the details. For the study of special version of eqs. (4.7)-(4.8) without functional arguments by using different technique, see, Grossman[5].

It is easy to observe that our study of the eqs. (1.3)-(1.4) include the study of the following control system

$$x'(t) = f(t, x(t), x_t) + p(t)u(t), \quad t \in [0, T], \quad \ldots (4.9)$$

$$x(t) = \phi(t), \quad t \in [-r, 0], \quad \ldots (4.10)$$

where $u(t)$ is a control given by

$$u'(t) = g(t, x(t), x_t),\ t \in [0, T] \qquad \ldots (4.11)$$

$$u(t) = \psi(t),\ t \in [-r, 0], \qquad \ldots (4.12)$$

in which $f,\ g : [0,\ T] \times R^n \times C \to R^n, p : [0, T] \to R, u : [-r, T] \to R^n$ are continuous functions and $\phi, \psi \in C$ are given. Eq. (4.11) is equivalent to the following integral equation

$$u(t) = \psi(0) + \int_0^t g(s, x(s), x_s)\, ds. \qquad \ldots (4.13)$$

Using (4.13) in (4.9), we obtain the following integrodifferential equation

$$x'(t) = f(t, x(t), x_t) + p(t)\left[\psi(0) + \int_0^t g(s, x(s), x_s) ds\right]. \qquad \ldots (4.14)$$

Eq. (4.14) with (4.10) is a special case of the equations (1.3)-(1.4). Thus, the study of equations (1.3)-(1.4) includes the study of control system of the form (4.9)-(4.12).

ACKNOWLEDGEMENT

The author expresses his sincere gratitude to Professor B.G. Pachpatte for many helpful discussions and suggestions during the preparation of the paper.

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