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# ON THE COMMUTATIVE NEUTRIX PRODUCT OF DISTRIBUTIONS

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Let $f$, $g$ be distributions in $\mathcal{D}'$ and let $f_n = f * \delta_n$, $g_n = g * \delta_n$, where $\{\delta_n\}$ is a certain sequence converging to the Dirac $\delta$-function. The neutrix product $f \square g$ is said to exist and be equal to $h$ if

$$N\text{-}\lim_{n \to \infty} \langle f_n g_n, \phi \rangle = \langle h, \phi \rangle$$

for all $\phi$ in $\mathcal{D}$. Neutrix products of the form $\ln^p x_+ \square \delta^{(s)}(x)$ and $x_+^r \ln x_+ \square x_-^{-s}$ are evaluated from which further neutrix products are obtained.

**Key Words : Distribution; Delta-function; Neutrix; Neutrix Limit; Neutrix Product**

In the following, we let $N$ be the neutrix, see van der Corput[7], having domain $N' = \{1, 2, ..., n, ...\}$ and range the real numbers, with negligible functions finite linear sums of the functions

$$n^{\lambda} \ln^{r-1} n, \quad \ln^r n : \lambda > 0, \quad r = 1, 2, \ldots$$

and all functions which converge to zero in the normal sense as $n$ tends to infinity.

We now let $\rho(x)$ be any infinitely differentiable function having the following properties :

(*i*) $\rho(x) = 0$ for $|x| \geq 1$,

(*ii*) $\rho(x) \geq 0$,

(*iii*) $\rho(x) = \rho(-x)$

and

$$(iv) \quad \int_{-1}^{1} \rho(x)dx = 1.$$

Putting $\delta_n(x) = n\rho(nx)$ for $n = 1, 2, ...$, it follows that $\{\delta_n(x)\}$ is a regular sequence of infinitely differentiable functions converging to the Dirac $\delta$-function $\delta(x)$.

Now let $\mathcal{D}$ be the space of infinitely differentiable functions with compact support and let $\mathcal{D}'$ be the space of distributions defined on $\mathcal{D}$. Then if $f$ is an arbitrary distribution in $\mathcal{D}'$, we define

$$f_n(x) = (f * \delta_n)(x) = \langle f(t), \delta_n(x - t) \rangle$$

for $n = 1, 2, \ldots$. It follows that $\{f_n(x)\}$ is a regular sequence of infinitely differentiable functions converging to the distribution $f$.

The following definition for the product of two distributions was given in [2].

*Definition* 1 — Let $f$ and $g$ be distributions in $\mathcal{D}'$ and let $f_n = f * \delta_n$ and $g_n = g * \delta_n$. We say that the neutrix product. $f \square g$ of $f$ and $g$ exists and is equal to the distribution $h$ on the interval $(a, b)$ if

$$N\text{–}\lim_{n \to \infty} \langle f_n g_n, \phi \rangle = \langle h, \phi \rangle$$

for all functions $\phi$ in $\mathcal{D}$ with support contained in the interval $(a, b)$. If

$$\lim_{n \to \infty} \langle f_n g_n, \phi \rangle = \langle h, \phi \rangle,$$

we simply say that the product $f \cdot g$ exists and equals $h$.[1]

This definition of the neutrix product is clearly commutative. A non-commutative neutrix product, denoted by $f \circ g$, was considered in [4].

The following theorem was proved in Kiliçman and Fisher[6].

***Theorem*** 1 — *Let $f$ and $g$ be distributions in $\mathcal{D}'$ and suppose that the neutrix products $f \square g^{(i)}$ exist on the interval(a, b) for $i = 0, 1, \ldots, r$. Then the neutrix products $f^{(k)} \square g$ exist on the interval $(a, b)$ and*

$$f^{(k)} \square g = \sum_{i=0}^{k} \binom{k}{i} (-1)^i [f \square g^{(i)}]^{(k-i)} \qquad \ldots (1)$$

$k = 1, 2, \ldots, r$.

The following Theorem holds in Kiliçman and Fisher[6], for $p = 1$. We now prove the following generalization of this result for $p = 1, 2, \ldots$ .

***Theorem*** 2 — *The neutrix products $\ln^p x_+ \square \delta^{(s)}(x)$, $\ln^p x_- \square \delta^{(s)}(x)$ and $\ln^p |x| \square \delta^{(s)}(x)$ exist and*

$$\ln^p x_+ \square \delta^{(s)}(x) = (-1)^s b_s^p \delta^{(s)}(x) \qquad \ldots (2)$$

$$= \ln^p x_- \square \delta^{(s)}(x) \qquad \ldots (3)$$

$$= \frac{1}{2} \ln^p |x| \square \delta^{(s)}(x), \qquad \ldots (4)$$

*where*

$$b_s^p = \frac{1}{s!} \int_{-1}^{1} v^s \rho^{(s)}(v) \int_{-1}^{v} \ln^p (v-u)\, \rho(u)\, du\, dv$$

*for* $s = 0, 1, 2, \ldots$ *and* $p = 1, 2, \ldots$ .

PROOF : We put

$$(\ln^p x_+)_n = \ln^p x_+ * \delta_n(x) = \int_{-1/n}^{x} \ln^p (x-t)\, \delta_n(t) dt$$

on the interval $(-1/n, 1/n)$. Then

$$\int_{-1/n}^{1/n} (\ln^p x_+)_n\, \delta_n^{(s)}(x)\, x^i\, dx = \int_{-1/n}^{1/n} x^i\, \delta_n^{(s)}(x) \int_{-1/n}^{x} \ln^p (x-t)\, \delta_n(t) dt\, dx$$

$$= n^{s-i} \sum_{k=0}^{p} \binom{p}{k} (-1)^k \int_{-1}^{1} v^i\, \rho^{(s)}(v)$$

$$\times \int_{-1}^{v} \ln^{p-k} (v-u) \ln^k n\rho(u)\, du\, dv$$

$$= n^{s-i} \int_{-1}^{1} v^i\, \rho^{(s)}(v) \int_{-1}^{v} \ln^p (v-u)\, \rho(u)\, du\, dv$$

$$+ n^{s-i} \sum_{k=1}^{p} \binom{p}{k} (-1)^k \int_{-1}^{1} v^i\, \rho^{(s)}(v)$$

$$\times \int_{-1}^{v} \ln^{p-k} (v-u) \ln^k n\rho(u)\, du\, dv \qquad \ldots (5)$$

on making the substitutions $nt = u$ and $nx = v$, for $i = 0, 1, 2, \ldots$ .

It follows that

$$N\text{–}\lim_{n \to \infty} \int_{-1/n}^{1/n} (\ln^p x_+)_n\, \delta_n^{(s)}(x) x^i\, dx = 0, \qquad \ldots (6)$$

for $i = 0, 1, 2, \ldots, s-1$ and

$$N\text{–}\lim_{n\to\infty}\int_{-1/n}^{1/n}(\ln^p x_+)_n\,\delta_n^{(s)}(x)x^s\,dx=\int_{-1}^{1}v^s\,\rho^{(s)}(v)\int_{-1}^{v}\ln^p(v-u)\,\rho(u)\,du\,dv$$

$$=s!\,b_s^p, \qquad \ldots (7)$$

$$\lim_{n\to\infty}\int_{-1/n}^{1/n}|(\ln^p x_+)_n\,\delta_n^{(s)}(x)\,x^{s+1}|\,dx=0. \qquad \ldots (8)$$

Now let $\phi$ be an arbitrary function in $\mathcal{D}$. Then

$$\phi(x)=\sum_{i=0}^{s}\frac{\phi^{(i)}(0)}{i!}x^i+\frac{\phi^{(s+1)}(\xi x)}{(s+1)!}x^{s+1},$$

where $0<\xi<1$. Using eqs. (6), (7) and (8), it follows that

$$N\text{–}\lim_{n\to\infty}\langle(\ln^p x_+)_n\,\delta_n^{(s)}(x),\phi(x)\rangle=(-1)^s b_s^p\,\phi^{(s)}(0)=(-1)^s\langle b_s^p\,\delta^{(s)}(x),\phi(x)\rangle,$$

proving eq. (2) for $s = 0, 1, 2, \ldots$ .

Eq. (3) follows on replacing $x$ by $-x$ in eq. (2) and eq. (4) then follows on noting that $\ln^p|x|=\ln^p x_+ + \ln^p x_-$.

Note that in the following, the distributions $x_+^{-r}$ and $x_-^{-r}$ are defined by

$$x_+^{-r}=\frac{(-1)^{r-1}}{(r-1)!}(\ln x_+)^{(r)},\ x_-^{-r}=-\frac{1}{(r-1)!}(\ln x_-)^{(r)}$$

for $r = 1, 2, \ldots$ and not as in Gelfand and Shilov[5].

The next theorem was given in Kiliçman and Fisher[6].

***Theorem*** 3 — *The neutrix products* $x_+^{-r}\,\square\,x_-^s$ *and* $x_-^{-r}\,\square\,x_+^s$ *exist and*

$$x_+^{-r}\,\square\,x_-^s=\sum_{i=s+1}^{r}\frac{(-1)^{r-s+i}\,s!}{(r-1)!}b_{i-s-1}^1\,\delta^{(r-s-1)}(x)$$

$$=(-1)^{r-s-1}\,x_-^{-r}\,\square\,x_+^s, \qquad \ldots (9)$$

*for* $r = 1, 2, \ldots$ *and* $s = 0, 1, \ldots, r-1$.

We now prove the following theorem.

***Theorem*** 4 — *The neutrix product* $x_+^{-r} \square x_-^s$ *exists and*

$$x_+^{-r} \square x_-^s = \frac{(-1)^{s-1} a_r^s \delta^{(s+r-1)}(x)}{(r-1)!\,(s-1)!\,(s+r-1)!}$$

$$= (-1)^{s+r-1} x_-^{-r} \square x_+^{-s}, \qquad \ldots (10)$$

*where*

$$a_r^s = \int_{-1}^{1} \rho^{(s)}(y) \int_{y}^{1} \rho^{(r)}(z) \int_{0}^{1} [\ln^2(z-y) + \ln(z-y) \ln v$$

$$+ \ln(z-y) \ln(1-v) + \ln v \ln(1-v)]\,(y+(z-y)v)^{s+r-1}\,(z-y)\,dv\,dz\,dy.$$

PROOF : Putting

$$(x_-^{-s})_n = -\frac{1}{(s-1)!} \int_{x}^{1/n} \ln(t-x)\,\delta_n^{(s)}(t)dt$$

and

$$(x_+^{-r})_n = \frac{(-1)^{r-1}}{(r-1)!} \int_{-1/n}^{x} \ln\,(x-t)\,\delta_n^{(r)}(t)\,dt.$$

Then

$$(-1)^r (r-1)!\,(s-1)! \int_{-1/n}^{1/n} (x_-^{-s})_n (x_+^{-r})_n x^m\,dx$$

$$= \int_{-1/n}^{1/n} \delta_n^{(s)}(t) \int_{t}^{1/n} \delta_n^{(r)}(w) \int_{t}^{w} \ln(x-t) \ln(w-x) x^m\,dx\,dw\,dt.$$

Making on the substitution $x - t = (w - t)\,v$, we have

$$\int_{t}^{w} \ln(x-t) \ln(w-x) x^m\,dx = \int_{0}^{1} \ln((w-t)v) \ln((w-t)(1-v))\,(t+(w-t)v)^m\,(w-t)\,dv.$$

Then putting $\delta_n(t) = n\rho(nt)$ and $\delta_n(w) = n\rho(nw)$ and making the substitution $nw = z$, $nt = y$ it follows that

$$(-1)^r (r-1)!\,(s-1)! \int_{-1/n}^{1/n} (x_-^{-s})_n (x_+^{-r})_n x^m\,dx$$

$$= n^{s+r-m-1} \int_{-1}^{1} \rho^{(s)}(y) \int_{y}^{1} \rho^{(r)}(z) \int_{0}^{1} [\ln^2(z/n - y/n) + \ln(z/n - y/n) \ln v$$

$$+ \ln(z/n - y/n) \ln(1 - v) + \ln v \ln(1 - v)] (y + (z - y)v)^m (z - y)\, dv\, dz\, dy.$$

Thus

$$N\text{–}\lim_{n \to \infty} \left[ (-1)^r (r-1)!\,(s-1)! \int_{-1/n}^{1/n} (x_-^{-s})_n (x_+^{-r})_n x^m\, dx \right] = 0 \qquad \ldots (11)$$

for $m = 0, 1, 2, \ldots, s + r - 2$, and

$$\lim_{n \to \infty} \int_{1/n}^{1/n} |(x_-^{-s})_n (x_+^{-r})_n x^m dx| = 0 \qquad \ldots (12)$$

for $m = s + r, s + r + 1, \ldots$ . In the particular case $m = s + r - 1$, we have that.

$$N\text{–}\lim_{n \to \infty} (-1)^r (r-1)!\,(s-1)! \int_{-1/n}^{1/n} (x_-^{-s})_n (x_+^{-r})_n x^{s+r-1}\, dx = a_r^s. \qquad \ldots (13)$$

Now let $\phi$ be an arbitrary function in $\mathcal{D}$. Then

$$\phi(x) = \sum_{k=0}^{s+r-1} \frac{x^k}{k!} \phi^{(k)}(0) + \frac{x^{s+r}}{(s+r)!} \phi^{(r+s)}(\xi x),$$

where $0 < \xi < 1$. On using equations (11), (12) and (13) it follows that

$$N\text{–}\lim_{n \to \infty} \langle (x_-^{-s})_n (x_+^{-r})_n, \phi(x) \rangle = \frac{(-1)^{s-1} a_r^s \delta^{(s+r-1)}(x)}{(r-1)!\,(s-1)!\,(s+r-1)!}$$

***Theorem*** 5 — *The neutrix product* $(x_+^r \ln x_+) \square x_-^{-s}$ *exists and*

$$(x_+^r \ln x_+) \square x_-^{-s} = (x_+^r \ln x_+) x_-^{-s} = 0, \qquad \ldots (14)$$

*for* $r = s, s + 1, s + 2, \ldots$ *and* $s = 1, 2, \ldots$ *and*

$$(x_+^r \ln x_+) \square x_-^{-s} = (-1)^r \binom{s-1}{r} r!\, a_0^1\, \delta^{(s-r-1)}(x)$$

$$+ \sum_{i=r+1}^{s-1} \binom{s-1}{i} (-1)^r r!\, a_{i-r}^1\, \delta^{(s-r-1)}(x)$$

$$- \psi(r) \sum_{i=r+1}^{s} \frac{(-1)^{r+i} r!\, b_{i-r-1}^{1}}{(s-1)!} \delta^{(s-r-1)}(x), \qquad \ldots (15)$$

*for* $r = 0, 1, 2, \ldots, s-1$, *where*

$$\psi(r) = \begin{cases} 0, & r = 0, \\ \Sigma_{i=1}^{r} i^{-1}, & r \geq 1 \end{cases}$$

PROOF : We first of all prove that

$$\ln x_+ \,\square\, x_-^{-1} = a_0^1 \delta(x). \qquad \ldots (16)$$

We put

$$(x_-^{-1})_n = -\int_x^{1/n} \ln(t-x)\, \delta_n'(t)\, dt$$

and define

$$(\ln x_+)_n = \int_{-1/n}^{x} \ln(x-t)\delta_n(t)dt$$

on the interval $(-1/n, 1/n)$, the intersection of the supports of $(\ln x_+)_n$ and $(x_-^{-1})_n$

Then

$$\langle (\ln x_+)_n, (x_-^{-1})_n \rangle = \int_{-1/n}^{1/n} (\ln x_+)_n (x_-^{-1})_n\, dx$$

$$= -\int_{1/n}^{1/n} \delta_n(t) \int_t^{1/n} \delta_n'(w) \int_t^w \ln(x-t)\ln(w-x)dx\, dw\, dt$$

on making the substitutions $x - t = (w - t)\, v$, it follows that

$$N\text{–}\lim_{n\to\infty} \langle (\ln x_+)_n, (x_-^{-1})_n \rangle = -\int_{-1}^{1} \rho(y) \int_y^1 \rho'(z) \int_0^1 [\ln^2(z-y) + \ln(z-y)\ln v$$

$$+ \ln(z-y)\ln(1-v) + \ln v \ln(1-v)](z-y)\, dv\, dz\, dy = a_0^1,$$

on putting $\delta_n(t) = n\rho(nt)$, $\delta_n(w) = n\rho(nw)$ and making substitution $nw = z$, $nt = y$.

Further, it follows as above that

$$|\langle (\ln x_+)_n, x(x_-^{-1})_n \rangle| \leq \int_{-1/n}^{1/n} |(\ln x_+)_n (x_-^{-1})_n x| \, dx$$

$$\leq \int_{-1/n}^{1/n} |\delta_n(t)| \int_t^{1/n} |\delta_n'(w)| \int_t^w |\ln(x-t) \ln(w-x) x| \, dx \, dw \, dt$$

$$= O(n^{-1} \ln n).$$

Now let $\phi$ be an arbitrary function in $\mathcal{D}$. Then

$$\phi(x) = \phi(0) + x\phi'(\xi x),$$

where $0 < \xi < 1$. Hence, it follows that

$$\langle (\ln x_+)_n (x_-^{-1})_n, \phi(x) \rangle - \phi(0) \langle (\ln x_+)_n, (x_-^{-1})_n \rangle = \langle (\ln x_+)_n, x(x_-^{-1})_n \phi'(\xi x) \rangle$$

$$= O(n^{-1} \ln n). \qquad \ldots (17)$$

Thus

$$N\text{–}\lim_{n \to \infty} \langle (\ln x_+)_n (x_-^{-1})_n, \phi(x) \rangle = N\text{–}\lim_{n \to \infty} \phi(0) \langle (\ln x_+)_n (x_-^{-1})_n \rangle$$

$$= a_0^1 \phi(0)$$

on using eqs. (17).

We now define the function $f(x_+, r)$ by

$$f(x_+, r) = \frac{x_+^r \ln x_+ - \psi(r) x_+^r}{r!}$$

and it follows easily by induction that

$$f^{(i)}(x_+, r) = f(x_+, r-i),$$

for $i = 0, 1, \ldots, r$. In particular,

$$f^{(r)}(x_+, r) = \ln x_+,$$

so that

$$f^{(i)}(x_+, r) = (-1)^{i-r-1} (i-r-1)! \, x_+^{-i+r},$$

for $i = r + 1, r + 2, \ldots$ . Now the product of the functions $x_+^i$ and $x_+^i \ln x_+$ and the distribution $x_-^{-1}$ exists by Definition 1 and it is easily seen that

$$x_+^i x_-^{-1} = (x_+^i \ln x_+) x_-^{-1} = 0, \qquad \ldots (18)$$

for $i = 1, 2, \ldots, r$. Using eq. (16) we have

$$f^{(r)}(x_+, r) \square x_-^{-1} = a_0^1 \delta(x) \qquad \ldots (19)$$

and using eq. (10) and replacing $r = 1$, and putting $s = -i + r$ we have

$$f^{(i)}(x_+, r) \square x_-^{-1} = (-1)^{i-r-1} (i-r-1)! \, a_{i-r}^1 \delta^{(i-r)}(x) \qquad \ldots (20)$$

for $i = r + 1. r + 2. \ldots$ .

Using eqs. (1) and (18) we now have

$$(s-1)! f(x_+, r) \square x_-^{-s} = \sum_{i=0}^{s-1} \binom{s-1}{i} (-1)^i [f^{(i)}(x_+, r) \square x_-^{-1}]^{(s-i-1)}$$

$$= \frac{(s-1)!}{r!} [x_+^r \ln x_+ - \psi(r) x_+^r] x_-^{-s}$$

$$= 0,$$

for $r = s, s + 1, s + 2, \ldots$ and $s = 1, 2, \ldots$ . Eq. (14) follow on using eq. (18).

When $r < s$ we have

$$(s-1)! f(x_+, r) \square x_-^{-s} = \sum_{i=0}^{s-1} \binom{s-i}{i} (-1)^i [f^{(i)}(x_+, r) \square x_-^{-1}]^{(s-i-1)}$$

$$= \binom{s-1}{r} (-1)^r a_0^1 \delta^{(s-r-1)}(x)$$

$$- \sum_{i=r+1}^{s-1} \binom{s-1}{i} (-1)^r (i-r-1)! \, a_{i-r}^1 \delta^{(s-r-1)}(x)$$

on using eqs. (1), (18), (19) and (20). It now follows that

$$(x_+^r \ln x_+) \square x_-^{-s} = r! f(x_+, r) \square x_-^{-s} + \psi(r) x_+^r \square x_-^{-s}.$$

*Corollary* 1 — The neutrix products $(x_-^r \ln x_-) \square x_+^{-s}$ exists and

$$(x_-^r \ln x_-) \square x_+^{-s} = (x_-^r \ln x_-) x_+^{-s} = 0, \qquad \ldots (21)$$

for $r = s, s+1, s+2, \ldots$ and $s = 1, 2, \ldots$ and

$$(x_-^r \ln x_-) \square x_+^{-s} = \frac{(-1)^{s+1}(s-1)!}{(s-r-1)!} a_0^1 \delta^{(s-r-1)}(x)$$

$$+ \sum_{i=r+1}^{s-1} \binom{s-1}{i} (-1)^{s+1} r!\,(i-r-1)!\, a_{i-r}^1 \delta^{(s-r-1)}(x)$$

$$+ \psi(r) \sum_{i=r+1}^{s} \frac{(-1)^{s+i} r!\, b_{i-r-1}^1}{(s-1)!} \delta^{(s-r-1)}(x), \qquad \ldots (22)$$

for $r = 0, 1, 2, \ldots, s-1$

PROOF : The results follow immediately on replacing $x$ by $-x$ in eqs. (14) and (15).

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