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ON SOME VECTOR VALUED SEQUENCE SPACES DEFINED USING A MODULUS FUNCTION

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In this paper, we define and study the vector valued sequence space $F(E_k, f)$ using modulus function f. The results of this paper generalize the corresponding results of Maddox[1].

Key Words : Vector Valued Sequence Spaces; Modulus Function

1. INTRODUCTION

It would be fair to say that the fundamental definition of a composite space has been given by Jakimovski and Russell[8]. All that the studia math paper considers is the completeness of $E\mathcal{B}$[8] and the characterization of continuous dual of $E\mathcal{B}$, plus a number of important examples which arise in summability theory and approximate theory. In this paper, we define more generalized composite sequence space rather than the space[8]. Ruckle[2] used the idea of a modulus function (see definition [2.4] below) to construct the sequence space $\mathcal{L}(f)$[2]. He gave a negative answer to Wilansky's question : "Is there a smallest FK - space[2] in which the set $\{e_1, e_2, \ldots\}$ of unit vectors is bounded?". Maddox[1] generalized the well known strongly summable sequence spaces w_0, w, w_∞[6&7] using modulus function. In the present note, we introduce a more generalized space using modulus function which generalizes the work of Maddox[1], Connor[5], Jakimovski and Russel[8].

2. DEFINITIONS AND SOME KNOWN RESULTS

Let S denote the family of all sequences $\{x_n\}$ with $x_n \in C, n \geq 1$, where C = complex field and F be a subspace of S.

Definition 2.1 [12, p. 64] — A scalar valued paranormed sequence space (F, g_F), where g_F is a paranorm on F is called monotone paranormed space if $x = (x_k) \in F, y = (y_k) \in F$ and $|x_k| \leq |y_k|$ for all k implies $g_F(x) \leq g_F(y)$.

Definition 2.2 [12, p. 48] — F is called normal or solid if $y=(y_k)\in F$ whenever $|y_i|\le|x_i|, i\ge 1$ for some $x=(x_k)\in F$.

Definition 2.3 [12, p. 57] — A sequence space F with linear topology is called a K-space provided each of the maps $P_i: F\to C, P_i(x)=x_i$ is continuous, $i\ge 1$.

Remark : Let (F, g_F) be a paranormed space and $(a^n)\subset F$, where $a^n=(a_k^n)$. If $a_k^n\to 0$ as $n\to\infty$ for each $k \Rightarrow g_F(a^n)\to 0$ as $n\to\infty$, then we say that co-ordinatewise convergence implies convergence in g_F. e.g., c_0, l_1, l_∞.

Definition 2.4 [2, p. 973] — By modulus function f, we mean a function $f : [0,\infty)\to[0,\infty)$ *s. t* (*i*) $f(x) = 0$ if and only if $x = 0$, (*ii*) $f(x+y)\le f(x)+f(y)$ for all $x, y\ge 0$, (*iii*) f is an increasing function, (*iv*) f is continuous from the right at 0.

Remark : If f is a modulus function then the composite function $f^n=f\cdot f\cdot f\ldots f$ (n – times) is also a modulus function (where n is an positive integer and $f\cdot f$ implies the composition of f with f).

3. THE SPACE $F(E_k, f)$

The new class $F(E_k, f)$ is defined as follows :

Let $\{E_k\}$ be a sequence of complex Banach spaces with norms $\|\cdot\|_{E_k}$ ($k=1, 2, 3\ldots$) and F a normal sequence space (of complex sequences) with monotone paranorm g_F such that co-ordinatewise convergence implies convergence in paranorm (we denote this hypothesis by (3.1)). Further let f be a modulus function. We define

$$F(E_k, f) = \{x=(x_k) : x_k\in E_k \text{ for each } k \text{ and } (f(\|x_k\|_{E_k}))\in F\}.$$

The paranorm g_F and the condition on g_F are irrelevant as far as the definition of the sets $F(E_k, f)$ is concerned; they are needed to define a topology on $F(E_k, f)$.

If $E_x=X$ for all k and f is the identity map, then the sets $F(E_k, f)$ reduce to the sets $F(X)$ of certain X-valued sequences[3,4&8].

Another specialization yields the sets $w_0(f)$, and $w_\infty(f)$ studied by Maddox[1].

Remark : In the definition of Jakimovski and Russel's space[8] the spaces E_k (which they denote by B_k) are p-normed spaces rather than (as in this paper) fully normed, so they gain a little generality there. However, the sequence spae F (which they denote by E) is q-normed, where as in this paper it is paranormed, which gives more generality for F. It should also be noted that they (J & R) assume that the space F is AK, but so for as completeness of $F(E_k)$ is concerned, this condition is not needed here.

For $x=(x_k)\in F(E_k, f)$, define,

$$g(x) = g_F[(f(\|x_k\|_{E_k}))]. \qquad \ldots (3.2)$$

Now we have,

Theorem 3.1 — *$F(E_k,f)$ is a complete paranormed space under the paranorm g given by (3.2) where F is a k-space.*

PROOF : It is easy to verify that g is a paranorm of $F(E_k,f)$. Let (x^n) be a Cauchy sequence in $F(E_k,f)$ where $x^n=(x_k^n)\in F(E_k,f)$. Therefore, $g(x^n-x^m)\to 0$ as $n,m\to\infty$.

$$\Rightarrow \quad \| x_k^n - x_k^m \|_{E_k} \to 0 \text{ as } n,m\to\infty \quad \text{[since, } F(E_k,f) \text{ is a } k\text{-space]}.$$

$$\Rightarrow \quad \text{For each } k,(x_k^n), \text{ is a Cauchy sequence in } E_k.$$

$$\Rightarrow \quad \text{For each } k, \text{ there exists } x_k\in E_k \; s\cdot t \| x_k^n - x_k \|_{E_k}\to 0 \text{ as } n\to\infty.$$

Let $y_k^n=f(\| x_k^n-x_k\|_{E_k})$. Then for each k, $y_k^n\to 0$ as $n\to\infty$. We choose δ_k^n such that $y_k^n<\delta_k^n f(\| x_k^n\|_{E_k})$, where $0<\delta_k^n<1$. This implies $y^n\in F$ for each n. So by our hypothesis (3.1) $g_F(y^n)\to 0$ as $n\to\infty$ i.e. $g(x^n-x)\to 0$ as $n\to\infty$. Again,

$$f(\| x_k \|_{E_k}) \le (\delta_k^n+1) f(\| x_k^n \|_{E_k})$$

Since F is normal, so $x=(x_k)\in F(E_k,f)$. Therefore, $F(E_k,f)$ is a complete paranormed space.

Remark : Let $y_k^n=f\left(\| x_k^n-x_k\|_{E_k}\right)$ Then we have, $f\left(\| x_k\|_{E_k}\right)\le y_k^n + f\left(\| x_k^n\|_{E_k}\right)$ Since $\left\{f\left(\| x_k^n\|_{E_k}\right)\right\}\in F$ for each n, $y_k^n\to 0\,(n\to\infty)$ for each k, and F is normal, it is asserted that $\left\{f\left(\| x_k^n\|_{E_k}\right)\right\}\in F$. It is hard to see how this follows. If we already knew that $y^n\in F$, then it would follow. But it seems to one that what we have, for any fixed n, is

$$\text{for each } k, f(\| x_k\|_{E_k})\le \varepsilon + z_k, \text{ where } (z_k)\in F.$$

But this is not enough to establish that $\{f(\| x_k\|_{E_k})\}\in F$. That is why by choosing suitable δ_k^n, we have shown $y^n\in F$.

Let $S=\pi E_k$, where πE_k denotes product of E_k, k = 1, 2, 3, ... and for each k, E_k are normed algebra. We define,

$$M[F(E_k,f)]$$

$$= \{a=(a_k)\in S : (a_kx_k)\in F(E_k,f) \text{ for all } x=(x_k)\in F(E_k,f)\}.$$

$M[F(E_k,f)]$ is called space of multiplier of $F(E_k,f)$.

Theorem 3.2 — $l_\infty(E_k) \subset M[F(E_k,f)]$.

PROOF : Let $a=(a_k) \in l_\infty(E_k)$, $H=\sup_k \| a_k \|_{E_k}$ and $x=(x_k) \in F(E_k,f)$.

Since, E_k are normed algebra so $\| a_k x_k \|_{E_k} \leq \| a_k \|_{E_k} \| x_k \|_{E_k}$.

$\Rightarrow$ $f(\| a_k x_k \|_{E_k}) < (1+[H]) f(\| x_k \|_{E_k})$, where $[H]$ denotes integer part of H.

Since F is a normal space, so $\{f(\| a_k x_k \|_{E_k})\} \in F$.

$\Rightarrow$ $ax \in F(E_k,f)$.

$\Rightarrow$ $a \in M[F(E_k,f)]$.

$\Rightarrow$ $l_\infty(E_k) \subset M[F(E_k,f)]$.

Proposition 3.3 — If $I=(I_1, I_2, I_3, ...) \in F(E_k,f)$, then $M[F(E_k,f)] \subseteq F(E_k,f)$, where I_k is an identity element of E_k for each k.

Proposition 3.4 — There exists a modulus function f s.t $F(E_k,f) \subset M[F(E_k,f)]$.

PROOF : Let $x=(x_k) \in F(E_k,f)$ and $y=(y_k) \in F(E_k,f)$. We take $f(t) = \log(1+t)$, $t \geq 0$. For this modulus function we have, $f(u+v) \leq f(u)+f(v)$ (cf [1, p. 161]), where $u, v \geq 0$. Since, E_k are normed algebra for each k, so

$$\| x_k y_k \|_{E_k} \leq \| x_k \|_{E_k} \| y_k \|_{E_k}.$$

$\Rightarrow$ $f(\| x_k y_k \|_{E_k}) \leq f(\| x_k \|_{E_k}) + f(\| y_k \|_{E_k})$.

$\Rightarrow$ $xy \in F(E_k,f)$.

$\Rightarrow$ $F(E_k,f) \subset M[F(E_k,f)]$.

Proposition 3.5 — If $l_\infty \subset F$, and f is a bounded modulus function, then $\pi E_k = F(E_k,f)$.

PROOF : Obviously $F(E_k,f) \subseteq \pi E_k$. Since, f is bounded, so for any $x=(x_k) \in \pi E_k$, $\sup_k f(\| x_k \| E_k) < \infty$.

$\Rightarrow$ $\{f(\| x_k \|_{E_k})\} \in F$.

$\Rightarrow$ $x=(x_k) \in F(E_k,f)$.

$\Rightarrow \quad \pi E_k \subseteq F(E_k, f)$

Therefore, $\pi E_k = F(E_k, f)$.

4. Composite Space $F(E_k, f^r)$ Using Composite Modulus Function f^r (where $r(\geq 1)$ is an Integer)

Theorem 4.1 — *If f is a modulus function s. t $0 < m \leq f(t)/t \leq M$ for $t > 0$, where m, M are two positive constants, then $F(E_k, f^s) = F(E_k)$ [where $s \geq 1$ is a positive integer].*

PROOF : Consider, $m \leq f(t)/t$.

$\Rightarrow t < r^s f^2(t)$, where $r = 1 + [1/m]$ and $[1/m]$ denotes the integer part of $1/m$.

Let $x = (x_k) \in F(E_k, f^s)$.

Since, $t < r^s f^2(t)$, so $\| x_k \|_{E_k} < r^s f^s(\| x_k \|_{E_k})$.

Since, F is normal space, so $x = (x_k) \in F(E_k)$.

Therefore, $F(E_k, f^s) \subseteq F(E_k)$ ... (4.1)

Consider, $f(t)/t \leq M$.

$\Rightarrow f^s(t) < H^s t$, where $H = 1 + [M]$ and $[M]$ denotes the integer part of M.

Let $x = (x_k) \in F(E_k)$.

Since, $f^s(t) < H^s t$, so $f^s(\| x_k \|_{E_k}) < H^s \| x_k \|_{E_k}$.

Since, F is normal, so $x = (x_k) \in F(E_k, f^s)$.

Therefore, $F(E_k) \subseteq F(E_k, f^s)$. ... (4.2)

From (4.1) & (4.2, we get $F(E_k, f^s) = F(E_k)$

e.g. $f(t) = at$ ($a > 0$, a constant and $t \geq 0$), $f(t) = \log(1 + t)$ ($t \geq 0$).

Remark : If $\lim_{t \to \infty} f(t)/t > 0$ for some modulus function f, then Theorem 4.1 may not be true. Because we can not claim that the existence of above limit for some modulus function f implies the existence of two constants m and M such that

$0 < m \leq f(t)/t \leq M$ for all $t > 0$.

It can be shown that existence of above limit implies the existence of a constant m such that

$$0 < m \leq f(t)/t \text{ for all } t > 0.$$

Buf $f(t)/t$ need not be bounded. To see this we define the functions f, f_1 and f_2 by $f_1(t) = t, f_2(t) = t^{1/2}$ and $f = f_1 + f_2$ for $t \geq 0$. Then f_1 and f_2 are modulus functions (cf [1, p. 161]) and consequently f is a modulus function (cf [2, p. 973]). Further $\lim_{t \to \infty} f(t)/t = 1$, but $f(t)/t \to \infty$ $(t \to 0)$.

It can be shown that the existence of $\lim_{t \to \infty} f(t)/t$ implies the existence of a constant M such that

$$f(t)/t \leq M \text{ for all } t > 0.$$

So the existence of

$$\lim_{t \to \infty} f(t)/t \text{ and } \lim_{t \to 0} f(t)/t$$

for some modulus function f implies Theorem 4.1.

Theorem 4.2 — *Let f be a modulus s.t $f(t) \leq tM$, where M is a positive constant. If r, s be any two positive integer s.t $r > s$, then $F(E_k) \subset F(E_k, f^s) \subseteq F(E_k, f^r)$*

PROOF : Let $v = r - s \geq 1$ and $t \in [0, \infty)$.

Given, $f(t) \leq tM$.

$$\Rightarrow f^r(t) < H^v f^s(t) < H^r t, \text{ where } H = 1 + [M].$$

Let $x = (x_k) \in F(E_k)$. By the above inequality, we get

$$f^r(\| x_k \|_{E_k}) < H^v f^s(\| x_k \|_{E_k}) < H^r \| x_k \|_{E_k}.$$

Since, F is a normal space, so

$$x \in F(E_k) \Rightarrow x \in F(E_k, f^s) \Rightarrow x \in F(E_k, f^r).$$

$$\Rightarrow F(E_k) \subseteq F(E_k, f^s) \subseteq F(E_k, f^r).$$

Remark : Theorem 4.2 can not hold without some condition on f. Let $E_k = C, \| \cdot \|_{E_k} = | \cdot |$, the modulus on C, for all k, $F = l_1$ and $f(t) = t^p$ $(0 < p < 1)$. Then $F(E_k) = l_1 \supset F(E_k, f) = l_p$. In fact, the proof fails in that the inequality

$f(t) < tM$ for all $t > 0$

does not hold for $f(t) = t^{1/2}$.

5. IDEAL OF $F(E_k, f)$

Suppose, G is a normal subset of F, E_k are commutative normed algebras and F is a sequence algebra.

Define :

$$G(E_k, f) = \{x = (x_k) : x_k \in E_k \text{ for each } k \text{ and } \{f(\| x_k \|_{E_k}\} \in G\}.$$

Theorem 5.1 — *If f be a modulus function s.t $f(u \cdot v) \leq f(u) f(v)$, where $u, v \geq 0$ are scalars and G is an ideal of F, then $G(E_k, f)$ is an ideal of $F(E_k, f)$.*

PROOF : It is easy to prove that $F(E_k, f)$ is a commutative sequence algebra.

Let $x = (x_k) \in G(E_k, f)$ and $y = (y_k) \in F(E_k, f)$.

Since, E_k are normed algebra, so

$$\| x_y y_k \|_{E_k} \leq \| x_k \|_{E_k} \| y_k \|_{E_k}.$$

$$\Rightarrow \quad f(\| x_k y_k \|_{E_k}) \leq f(\| x_k \|_{E_k}) f(\| y_k \|_{E_k}).$$

Since G is an ideal of F, so

$$\{f(\| x_k \|_{E_k}) f(\| y_k \|_{E_k})\} \in G.$$

Since, G is normal, so $\{f(\| x_k y_k \|_{E_k}\} \in G$.

$$\Rightarrow \quad xy \in G(E_k, f).$$

$$\Rightarrow \quad G(E_k, f) \text{ is an ideal of } F(E_k, f).$$

Proposition 5.2 — Suppose, $F = l_\infty$. If G is a subspace of l_∞, then for any unbounded modulus function f, $G(E_k, f)$ is an ideal of $l_\infty (E_k, f)$.

PROOF : Let $x = (x_k) \in G(E_k, f)$ and $y = (y_k) \in l_\infty (E_k, f)$. Since f is unbounded, so $\sup_k f(\| y_k \|_{E_k}) < \infty$ is equivalent to $(\| y_k \|_{E_k}) \in l_\infty$.

Let $M = \sup_k \| y_k \|_{E_k}$. Since E_k are normed algebra so

$$\| x_k y_k \|_{E_k} \leq \| x_k \|_{E_k} \| y_k \|_{E_k} \leq M \| x_k \|_{E_k}.$$

$$\Rightarrow \quad f(\| x_k y_k \|_{E_k}) \leq (1 + [M]) f(\| x_k \|_{E_k}).$$

$$\Rightarrow \quad xy \in G(E_k, f).$$

$$\Rightarrow \quad G(E_k, f) \text{ is an ideal of } l_\infty (E_k, f).$$

Remark : Since, $f(x) = x (x \geq 0)$ is an unbounded modulus function so by Proposition 5.2 $G(E_k)$ is an ideal of $l_\infty (E_k)$.

Define :

$$M(x, \varepsilon) = \{k : \| x_k \|_{E_k} \geq \varepsilon \} \text{ and } \| x \| = \sup_k \| x_k \|_{E_k},$$

where $x = (x_k) \in l_\infty (E_k)$ and ε is a small positive number. Let $M \supset N$ = set of positive integers and $\chi_{M(x, \varepsilon)}$ denotes the characteristic fuynction of $M(x, \varepsilon)$. Here $\chi_{(M, \varepsilon)}$ is a sequence $y = (y_k)$ $(y_k \in E_k$ for each $k)$ which agrees with x on the indices in $M(x, \varepsilon)$ and is zero elsewhere.

Theorem 5.3 — *Let $G(E_k)$ be an ideal of $l_\infty (E_k)$ and let $x = (x_k) \in l_\infty (E_k)$. Then $x \in$ closer of $G(E_k)$ if and only if $\chi_{M(x, \varepsilon)} \in G(E_k)$.*

The proof of Theorem 5.3 is very similar to the corresponding result that appears in [5].

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