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ON MEASURES OF *M*-ENTROPY

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The *M*-Entropy for a probability distribution $(p_1, p_2, \ldots, p_n)$, is that function of $(p_1, p_2, \ldots, p_n)$, by maximizing which, subject to some specified linear constraints on $(p_1, p_2, \ldots, p_n)$, we get probabilities which are automatically either monotonically increasing or monotonically decreasing. This monotonicity is the reason why this measure is called as *M*-Entropy measure. We also want that this function is concave and is such that the maximizing probabilities are all non negative. In the present paper, a measure of *M*-Entropy satisfying these conditions is obtained, and Maximum *M*-Entropy probability distributions are obtained, for the case when the probabilities are increasing and one of the moments (A.M. or G.M. or H.M. or μ_2') is prescribed.

Key Words : Entropy; *M*-Entropy; Maximization •

1. INTRODUCTION

Measures of Entropy have been given by Shannon[9], Renyi[8], Havarda and Charvat[1], Kapur[2,3,5,6,7] and others. All these measures of entropy have the property that they do not change when the outcomes are rearranged among themselves i.e., they are symmetric functions of $(p_1, p_2, \ldots, p_n)$.

A situation may arise in which the outcomes cannot be rearranged among themselves e.g. if we have the prior information about the probability distribution that the probabilities are in some order say in increasing order

i.e., $$0 \le p_1 \le p_2, \ldots\ldots, \le p_n \le 1 \qquad \ldots (1)$$

and $$\sum_{i=1}^{n} p_i = 1. \qquad \ldots (2)$$

In this case, when we have the information that the probabilities are increasing, then there is a need to develop a measure of entropy for this type of situation. There are many examples of this kind in our daily life where probabilities are in increasing order e.g.,

(*a*) If we consider the case of incomes of the persons we find that the proportions of persons lying in the higher income groups is less than the proportions of persons in the lower income groups.

(*b*) The proportion of families with two or less children increases with increase in the family income.

(*c*) The proportion of enrolment of children goes on decreasing, with the increase in the class level.

The inspiration for developing a measure of entropy for these kind of problems came from Kapur[4] where many Maximum Entropy Probability Distributions (MEPD) were found for the ordered random variates, and also from the desire to relax the condition of symmetry for Shannon's measure of entropy.

In the present paper, an attempt has been made to develop such a measure of entropy, which we have called as *M*-Entropy measure, by maximizing which subject to some specified linear constraints, we get probabilities which are monotonically increasing. In fact, the prefix '*M*' is used here to indicate the monotonic character of the probabilities.

2. A MEASURE OF *M*-ENTROPY

Let the probabilities be in increasing order so that

$$0 \le p_1 \le p_2, \ldots, \le p_n \le 1 \qquad \ldots (1)$$

also

$$\sum_{i=1}^{n} p_i = 1 \qquad \ldots (2)$$

and Shannon's measure of entropy is

$$-\sum_{i=1}^{n} p_i \ln p_i. \qquad \ldots (3)$$

The inequalities (1) can be written as

$$p_1 \ge 0$$

$$p_2 - p_1 \ge 0$$

$$\vdots$$

$$\vdots$$

$$p_n - p_{n-1} \ge 0$$

$$1 - p_n \ge 0. \qquad \ldots (4)$$

Now we propose *M*-Entropy measures as

$$S = -p_1 \ln p_1 - (p_2 - p_1) \ln (p_2 - p_1) - \cdots$$

$$- (p_n - p_{n-1}) \ln (p_n - p_{n-1}) - (1 - p_n) \ln (1 - p_n). \quad \ldots (5)$$

3. PROPERTIES OF THIS M-ENTROPY MEASURE

This measure is defined only when the inequalities (4) are satisfied. We shall now show that this measure has all the desirable properties we require for it.

(*i*) We first show that the measure (5) is a concave function of $(p_1, p_2, \ldots, p_n)$ so that when it is maximized subject to linear constraints its local maximum is the global maximum.

The Hessian Matrix of the M-Entropy function (5) is

$$\boldsymbol{H} = \begin{bmatrix} -1/p_1 - 1/(p_2 - p_1) & 1/(p_2 - p_1) & 0 & .. 0 \\ 1/(p_2 - p_1) & -1/(p_2 - p_1) - 1/(p_3 - p_2) & 1/(p_3 - p_2) & ..... 0 \\ 0 & & & \\ 0 & & & \\ \vdots & & & \\ \vdots & & & \\ 0 \ldots\ldots 1/(p_n - p_{n-1}) & (-1/(p_n - p_{n-1}) & -1/(1 - p_n) & \end{bmatrix}.$$

The corresponding quadratic form can be written as

$$[-1/p_1 - 1/(p_2 - p_1)]\, x_1^2 - [1/(p_2 - p_1) + 1/(p_3 - p_2)]\, x_2^2 \cdots$$

$$- [1/(p_n - p_{n-1}) + 1/(1 - p_n)]\, x_n^2 + [2/(p_2 - p_1)]\, x_1 x_2 \cdots$$

$$= -[1/p_1]\, x_1^2 - [1/(p_2 - p_1)]\, (x_1^2 + x_2^2 - 2x_1 x_2)$$

$$- [1/(p_3 - p_2)]\, (x_2^2 + x_3^2 - 2x_2 x_3) \cdots$$

$$- (1/(1 - p_n))\, x_n^2. \quad \ldots (6)$$

From expression (6) we see that the quadratic form is negative definite, so that Hessian matrix is negative definite. Hence we can say that S is a non-symmetric concave function of $(p_1, p_2, \ldots, p_n)$.

(*ii*) The minimum value of S is zero when the probability (p_n) is 1 and all others are zero.

(*iii*) The maximum value of S subject to $\sum_{i=1}^{n} p_i = 1$ will occur when :

$$(p_2 - p_1)/p_1 = (p_3 - p_2)/(p_2 - p_1) = \ldots\ldots (p_n - p_{n-1})/(p_{n-1} - p_{n-2})$$

$$= (1 - p_n)/(p_n - p_{n-1}) \quad \ldots (7)$$

After solving eq. (7) we get

$$p_1 = \frac{1}{1 + A + A^2 + \ldots\ldots + A^n}$$

$$p_2 = \frac{1 + A}{1 + A + A^2 + \ldots\ldots + A^n}$$

$$\vdots$$

$$\vdots$$

$$p_n = \frac{1 + A + A^2 + \ldots\ldots + A^{n-1}}{1 + A + A^2 + \ldots\ldots + A^n}.$$

The value of the constant A can be calculated from the equation

$$\sum_{i=1}^{n} p_i = 1$$

or
$$f(A) \equiv (n-1) + (n-2)A + \ldots\ldots + A^{n-2} - A^n = 0. \quad \ldots (8)$$

Since there is only one change of sign on the L.H.S. of eq. (8), this equation has one positive root and since $f(1) > 0$, $f(\infty) < 0$, this root is greater than unity.

After finding the value of A we can find the value of all the probabilities, and then we can calculate the value of $S_{\max}$. Let us calculate the value of $S_{\max}$ for different values of n.

TABLE I

n	4	6	8	10	12	16	20
$S_{\max}$	1.34400	1.37753	1.38426	1.38546	1.38617	1.38628	1.38629

TABLE II

n	p_1	p_2	p_3	p_4		
= 4	0.04722	0.13178	0.27984	0.54064		
n	p_1	p_2	p_3	p_4	p_5	p_6
= 6	0.00930	0.02729	0.06211	0.12947	0.25981	0.52101

In both the results above for $n = 6$ and for $n = 4$ we see that the probabilities are increasing monotonically and also $S_{\max}$ increases when n increases. It does not show that $S_{\max}$ always increases with n. This conjecture has yet to be investigated.

(*iv*) For the uniform distribution U

$$p_1 = p_2 = \ldots = p_n = 1/n \quad \ldots (9)$$

which is permissible

$$S_u = -1/n \ln (1/n) - (1 - 1/n) \ln (1 - 1/n)$$

$$= \ln (n) - [(n-1)/n] \ln (n-1). \quad \ldots (10)$$

The *M*-Entropy for the uniform distribution is less than Shannon's entropy for the same distribution except when $n = 2$.

(*v*) For $n = 4$, $S_u = \ln(4) - (3/4) \ln (3) = 0.56234$, $S_{\max} = 1.34400$,
$n = 6$, $S_u = 0.45056$, $S_{\max} = 1.37753$

so that S_u is always less than $S_{\max}$. This is expected since the maximum *M*-Entropy distribution is different from the uniform distribution.

However, we find that S_u decreases as n increases. This is in general true when $n > 2$.

Since $(d/dn)\, S_u = -(1/n^2) \ln (n-1)$,

this can alternatively be seen that the *M*-Entropy for the uniform distribution is same as the Shannon's entropy for the probability distribution $[1/n, (n-1)/n]$. The Shannon's entropy for this distribution will decrease as n increases. So that *M*-Entropy for the uniform distribution will decrease as n increases.

(*vi*) The constraints on a probability distribution may be given in the form

$$\sum_{i=1}^{n} p_i = 1, \; E[g_r(x)] = \sum_{i=1}^{n} p_i g_r(x_i) = a_r$$

$$r = 1, 2, \ldots\ldots\ldots, m. \quad \ldots (11)$$

Now maximizing the *M*-Entropy subject to above constraints (11) we get

$$\frac{p_2 - p_1}{p_1} = \exp\, [-y_0 - y_1 g_1(x_1) - y_2 g_2(x_1) \ldots\ldots\ldots - y_m g_m(x_1)]$$

$$\frac{p_3 - p_2}{p_2 - p_1} = \exp\, [-y_0 - y_1 g_1(x_2) - y_2 g_2(x_2) \ldots\ldots - y_m g_m(x_2)]$$

$$\frac{p_n - p_{n-1}}{p_{n-1} - p_{n-2}} = \exp[-y_0\; y_1 g_1(x_{n-1}) - y_2 g_2(x_{n-1}) \ldots\ldots\ldots - y_m g_m(x_{n-1})] \quad \ldots (12)$$

$$\frac{1 - p_n}{p_n - p_{n-1}} = \exp\, [-y_0 - y_1 g_1(x_n) - y_2 g_2(x_n) \ldots\ldots - y_m g_m(x_n)]$$

All the terms in (12) on the R.H.S. are positive. After multiplying all the terms, we get

$$\frac{1 - p_n}{p_1} = \text{multiplicant of all the positive factors} \quad \ldots (13)$$

Now there are two possibilities of getting positive terms in eq. (13) on the R.H.S. i.e.,

a) $1-p_n$ is positive and p_1 is positive

b) $1-p_n$ is negative and p_1 is negative

Now $1-p_n$ cannot be negative as $p_n < 1$.

Therefore, p_1 will be positive. It means possibility (*b*) is not possible in this case and also $p_1 \neq 0$ otherwise R.H.S. of eq. (13) will be ∞ which is not possible.

Therefore,

$$p_1 > 0, p_2 > 0 \ldots\ldots, p_n > 0.$$

Here we can say that the probabilities, after maximizing *M*-Entropy measure subject to some constraints will always be positive.

(*vii*) The mean m of a maximum *M*-Entropy probability distribution will always lie between

$$\frac{(n+1)}{2} \text{ and } n.$$

To see this we note that,

$$m = \sum_{i=1}^{n} ip_i,\ (n+1)/2 = \sum_{i=1}^{n} i(1/n),\ m-(n+1)/2 = \sum_{i=1}^{n} (p_i - 1/n)i$$

$$= \Sigma_1 (p_i - 1/n)i + \Sigma_2 (p_i - 1/n)i, \qquad \ldots (14)$$

where Σ_1 and Σ_2 denote the sum for those probabilities which are $< 1/n$ and $> 1/n$ respectively.

If each probability is equal to $1/n$, $m = (n+1)/2$,
otherwise some probabilities will be $< 1/n$ some others will be $> 1/n$ and some will be $= 1/n$

Those $< 1/n$ will make a negative contribution to Σ_1 and those $> 1/n$ will make a positive contribution to Σ_2. Those equal to $1/n$ will make no contributions

$$\text{Since} \quad \sum_{i=1}^{n} (p_i - 1/n) = 0 \qquad \ldots (15)$$

and multipliers i in Σ_1 are less than the multipliers i in Σ_2.

$$\Sigma_1 (p_i - 1/n)\, i$$

will be negative and $\leq$ in magnitude to

$$\Sigma_2 (p_i - 1/n)\, i \text{ i.e.,}$$

$$m - (n+1)/2 \geq 0$$

(*viii*) Geometric mean will lie between $(n!)^{1/n}$ and n.

For this

$$\sum_{i=1}^{n} p_i \ln (i) = \ln (g) \qquad \ldots (16)$$

$$\sum_{i=1}^{n} 1/n \ln (i) = \ln (n\,!)^{1/n} \qquad \ldots (17)$$

$$\sum_{i=1}^{n} (p_i - 1/n) \ln (i) = \ln ((g)/(n\,!)^{1/n}). \qquad \ldots (18)$$

By a similar argument as above, we can say

$$g \geq (n\,!)^{1/n}.$$

(*ix*) The Harmonic mean will lie between

$$n \text{ and } \frac{n}{1 + 1/2 + ... + 1/n}$$

For this we note that

$$\sum_{i=1}^{n} \frac{p_i}{i} = 1/H \qquad \ldots (19)$$

$$\sum_{i=1}^{n} \frac{1}{n \times i} = \frac{1 + 1/2 + 1/3 \ldots + 1/n}{n} \qquad \ldots (20)$$

$$\sum_{i=1}^{n} (p_i - 1/n)\, 1/i = H - \frac{1 + 1/2 + ... + 1/n}{n} \qquad \ldots (21)$$

by an argument similar to the one used above in (*vii*), so that

$$1/H \leq \frac{1 + 1/2 + ... + 1/n}{n} \qquad \ldots (22)$$

$$H \geq \frac{n}{1 + 1/2 + ... + 1/n}$$

(*x*) Similarly the $\mu_2' > (n + 1)(2n + 1)/6$

For this

$$\sum_{i=1}^{n} p_i i^2 = \mu_2' \qquad \ldots (23)$$

$$\sum_{i=1}^{n} 1/n\, i^2 = (n+1)(2n+1)/6 \qquad \ldots (24)$$

$$\sum_{i=1}^{n} (p_i - 1/n)\, i^2 = \mu_2' - (n+1)(2n+1)/6 > 0 \qquad \ldots (25)$$

$$\mu_2' > (n+1)(2n+1)/6.$$

Here, we can say that

$$AM \geq (n+1)/2,\ GM > (n!)^{1/n},\ H \geq n/(1 + 1/2 + \ldots + 1/n) \text{ and}$$

$$\mu_2' > (n+1)(2n+1)/6.$$

In general, we can say that any moment for M-entropy probability distribution > the corresponding moment for the corresponding unform distribution. Thus

$$\mu_r' \geq 1/n \sum_{r=1}^{n} i^r,\ E(fx_i) \geq 1/n \sum_{i=1}^{n} f(x_i).$$

The equality sign in each case will arise when the M-entropy distribution is the uniform distribution.

4. MAXIMUM M-ENTROPY PROBABILITY DISTRIBUTION WHEN SOME MEAN IS PRESCRIBED

Now we will find the Maximum M-Entropy Probability Distributions (MM-EPD) when some of its moments are prescribed.

(a) When Arithmetic Mean is prescribed

$$S = -p_1 \ln p_1 - (p_2 - p_1) \ln (p_2 - p_1) \ldots\ldots$$

$$- (p_n - p_{n-1}) \ln (p_n - p_{n-1}) - (1 - p_n) \ln (1 - p_n).$$

Let the prescribed mean m $(1 < m < n)$ then we have to maximize S subject to

$$\sum_{i=1}^{n} p_i = 1$$

and $$\sum_{i=1}^{n} i p_i = m.$$

The Lagrangian is

$$L = -p_1 \ln p_1 - (p_2 - p_1) \ln (p_2 - p_1) \ldots\ldots$$

$$- (p_n - p_{n-1}) \ln (p_n - p_{n-1}) - (1 - p_n) - \ln (1 - p_n)$$

$$- y\left(\sum_{i=1}^{n} p_i - 1 \right) - u\left(\sum_{i=1}^{n} i p_i - m \right). \qquad \ldots (26)$$

Maximizing this we get

$$\frac{p_2 - p_1}{p_1} = AB$$

$$\frac{p_3 - p_2}{p_2 - p_1} = AB^2$$

$$\vdots$$

$$\vdots$$

$$\frac{p_n - p_{n-1}}{p_{n-1} - p_{n-2}} = AB^{n-1}$$

$$\frac{1 - p_n}{p_n - p_{n-1}} = AB^n.$$

By solving these we get

$$p_1 = \frac{1}{1 + AB + A^2B^3 + A^3B^6 \ldots + A^nB^{n(n+1)/2}}$$

$$p_2 = \frac{1 + AB}{1 + AB + A^2B^3 + A^3B^6 \ldots + A^nB^{n(n+1)/2}}$$

$$\vdots$$

$$\vdots$$

$$p_n = \frac{1 + AB + A^2B^3 + A^3B^6 \ldots\ldots + A^{n-1}B^{n(n-1)/2}}{1 + AB + A^2B^3 + A^3B^6 \ldots + A^nB^{n(n+1)/2}}.$$

The constants A and B can be calculated from equations

$$\sum_{i=1}^{n} p_i = 1$$

and

$$\sum_{i=1}^{n} ip_i = m.$$

Now let us discuss the case for $n = 6$.

The probability distribution at different means is given in the Table (III) and the graph between S_{max} and mean is given at Fig. 1.

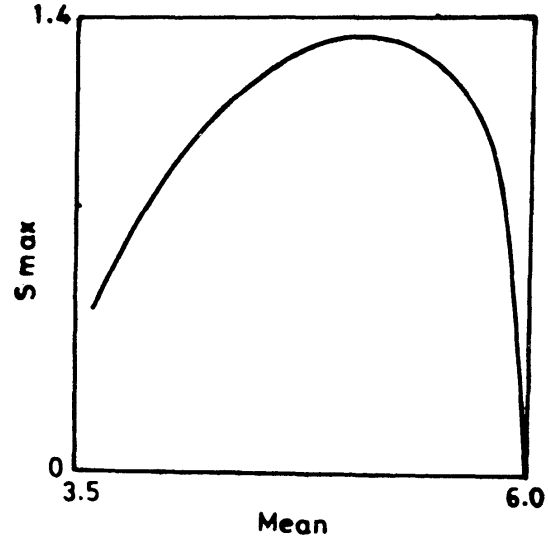

FIG. 1

TABLE III

Mean	6	5.75074	5.70329	5.56932	5.40143	5.13925
p_1	0	0.00000	0.00000	0.00000	0.00096	0.00930
p_2	0	0.00000	0.00000	0.00055	0.00639	0.02729
p_3	0	0.00014	0.00070	0.00838	0.02935	0.0621
p_4	0	0.01348	0.02530	0.06519	0.10223	0.12947
p_5	0	0.22187	0.24400	0.27303	0.27570	0.25981
p_6	1	0.76450	0.73000	0.65288	0.58537	0.51202
S_{max} =	0	1.05794	1.13285	1.26652	1.34443	1.37753

Mean	4.78775	4.44909	3.99962	3.67238	3.61354	3.55763	3.52680	3.5
p_1	0.03183	0.06092	0.10691	0.14484	0.15209	0.15915	0.16313	0.16667
p_2	0.06126	0.09360	0.13291	0.15712	0.16086	0.16402	0.16555	0.16667
p_3	0.09529	0.11989	0.14556	0.16032	0.16262	0.16569	0.16577	0.16667
p_4	0.14446	0.15161	0.15786	0.16274	0.16387	0.16510	0.16588	0.16667
p_5	0.23329	0.20903	0.18180	0.16830	0.16695	0.16626	0.16626	0.16667
p_6	0.43387	0.36495	0.27496	0.20663	0.19360	0.18077	0.17341	0.1667
$S_{max}=$	1.33605	1.22957	0.98688	0.70436	0.63542	0.55916	0.50895	0.45056

It appears from the Fig. (1) that S_{max} is a concave function of mean. The theoretical proof for the concavity of S_{max} with respect to the mean is still not developed. For the present it is just a conjecture.

(b) When Geometric Mean is Prescribed

Now we will maximize the *M*-Entropy expression (5) subject to condition

$$\sum_{i=1}^{n} p_i = 1$$

and $$\sum_{i=1}^{n} p_i \ln (i) = \ln (g)$$

where g is the geometric mean ($1 < g < n$).

The Lagrangian is

$$L = -p_1 \ln p_1 - (p_2 - p_1) \ln (p_2 - p_1) .....$$

$$- (p_n - p_{n-1}) \ln (p_n - p_{n-1}) - (1 - p_n) \ln (1 - p_n)$$

$$- y \left(\sum_{i=1}^{n} p_i - 1 \right) - u \left(\sum_{i=1}^{n} (p_i \ln (i) - \ln (g) \right). \qquad \ldots (27)$$

After maximizing we get

$$\frac{p_2 - p_1}{p_1} = A(1)^B$$

$$\frac{p_3 - p_2}{p_2 - p_1} = A(2)^B$$

:

:

$$\frac{p_n - p_{n-1}}{p_{n-1} - p_{n-2}} = A(n-1)^B$$

and

$$\frac{1 - p_n}{p_n - p_{n-1}} = A(n)^B.$$

By solving these we get

$$p_1 = \frac{1}{1 + A(1\,!)^B + A^2(2\,!)^B \ldots\ldots + A^n(n\,!)^B}$$

$$p_2 = \frac{1 + A(1\,!)^B}{1 + A(1\,!)^B + A^2(2\,!)^B \ldots\ldots + A^n(n\,!)^B} \cdot$$

:

:

$$p_n = \frac{1 + A(1\,!)^B + A^2(2\,!)^B \ldots\ldots + A^{n-1}(n-1\,!)^B}{1 + A(1\,!)^B + A^2(2\,!)^B \ldots\ldots + A^n(n\,!)^B}$$

The values of probabilities and $S_{\max}$ at different geometric means for $n = 6$ is given in Table (IV) and the graph between $S_{\max}$ and GM is given at Fig. 2.

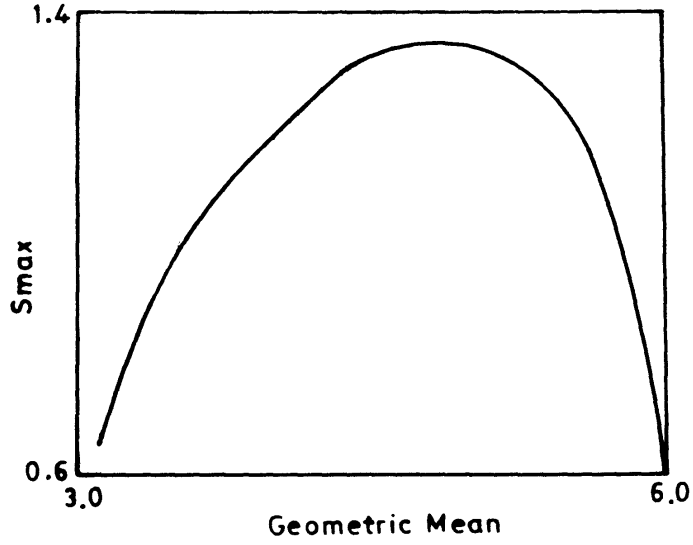

FIG. 2

TABLE IV

GM	6	5.28563	5.16377	4.96392	4.46529	4.26246	3.58907	3.24636	3.15896	3.10289
p_1	0	0.00069	0.00274	0.00930	0.02494	0.05074	0.10743	0.14140	0.15040	0.15614
p_2	0	0.00672	0.01385	0.02729	0.04896	0.07622	0.12378	0.14838	0.15466	0.15869
p_3	0	0.03307	0.04571	0.06211	0.08166	0.10183	0.13374	0.15113	0.15603	0.15936
p_4	0	0.10991	0.12030	0.12947	0.13617	0.14043	0.14738	0.15480	0.15784	0.16024
p_5	0	0.27789	0.27155	0.25981	0.24115	0.21793	0.18061	0.16639	0.16443	0.16393
p_6	1	0.57172	0.54586	0.51202	0.46712	0.41280	0.30706	0.23789	0.21664	0.20164
S_{max}	0	1.35153	1.36855	1.37753	1.36021	1.29381	1.04020	0.79539	0.70718	0.64033

(c) When Harmonic Mean is Prescribed

Maximizing the expression (5) subject to condition

$$\sum_{i=1}^{n} p_i = 1$$

and

$$\sum_{i=1}^{n} p_i / i = 1/H.$$

The Lagrangian is

$$L = -p_1 \ln p_1 - (p_2 - p_1) \ln (p_2 - p_1) - \ldots.$$

$$- (p_n - p_{n-1}) \ln (p_n - p_{n-1}) - (1 - p_n) \ln (1 - p_n)$$

$$- y \left(\sum_{i=1}^{n} p_i - 1 \right) - u \left(\sum_{i=1}^{n} (p_i / i - 1/H) \right) \qquad \ldots (28)$$

maximizing this we get

$$\frac{p_2 - p_1}{p_1} = AB$$

$$\frac{p_3 - p_2}{p_2 - p_1} = AB^{1/2}$$

$$\vdots$$

$$\frac{p_n - p_{n-1}}{p_{n-1} - p_{n-2}} = AB^{1/(n-1)}$$

$$\frac{1 - p_n}{p_n - p_{n-1}} = AB^{1/n}.$$

By solving these we get

$$p_1 = \frac{1}{1 + AB + A^2B^{1/2+1} + \ldots\ldots + A^nB^{1/n+1/(n-1)+\ldots+1/2+1}}$$

$$p_2 = \frac{1 + AB}{1 + AB + A^2B^{1/2+1} + \ldots. + A^nB^{1/n+1/(n-1)+\ldots+1/2+1}}$$

$$\vdots$$

$$\vdots$$

$$p_n = \frac{1 + AB + A^2B^{1/2+1} + \ldots\ldots + A^{n-1}B^{1/(n-1)+\ldots+1/2+1}}{1 + AB + A^2B^{1/2+1} + \ldots. + A^nB^{1/n+1/(n-1)+\ldots+1/2+1}}$$

For $n = 6$ the probability distribution and $S_{\max}$ at different Harmonic means is given in Table V and the graph between $S_{\max}$ and HM is given at Fig. 3.

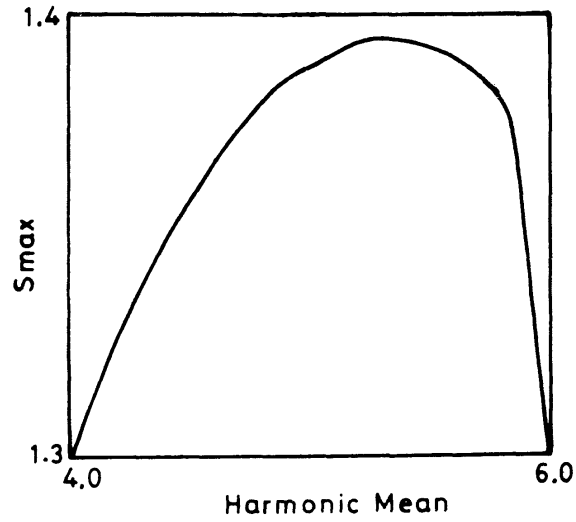

FIG. 3

TABLE V

HM	6	4.96719	4.92853	4.85512	4.68791	4.46004	4.18005	4.00378
p_1	0	0.00227	0.00309	0.00471	0.00930	0.01710	0.02880	0.03734
p_2	0	0.01479	0.01676	0.02003	0.02729	0.03706	0.04939	0.05755
p_3	0	0.04927	0.05164	0.05523	0.06211	0.07000	0.07884	0.08434
p_4	0	0.12461	0.12576	0.12728	0.12947	0.13101	0.13190	0.13222
p_5	0	0.27129	0.26948	0.26648	0.25981	0.25076	0.23922	0.23156
p_6	1	0.53777	0.53326	0.52626	0.51202	0.49407	0.47186	0.45698
$S_{\max}$	0	1.37024	1.37244	1.37516	1.37753	1.37352	1.35767	1.34090

(d) When μ_2' is Prescribed

Here we will maximize subject to condition

$$\sum_{i=1}^{n} p_i = 1$$

and

$$\sum_{i=1}^{n} i^2 p_i = \sigma^2.$$

The Lagrangian is

$$L = -p_1 \ln p_1 - (p_2 - p_1) \ln (p_2 - p_1) - \ldots\ldots\ldots$$

$$- (p_n - p_{n-1}) \ln (p_n - p_{n-1}) - (1 - p_n) \ln (1 - p_n)$$

$$- y\left(\sum_{i=1}^{n} p_i - 1 \right) - u\left(\sum_{i=1}^{n} i^2 p_i - \sigma^2 \right). \qquad \ldots (29)$$

Maximizing this, we get

$$p_1 = \frac{1}{1 + AB + A^2B^5 + \ldots + A^nB^{n(n+1)(2n+1)/6}}$$

$$p_2 = \frac{1 + AB}{1 + AB + A^2B^5 + \ldots + A^nB^{n(n+1)(2n+1)/6}}$$

$$p_n = \frac{1 + AB + A^2B^5 + ... + A^{n-1}\, B^{n(n-1)(2n-1)/6}}{1 + AB + A^2B^5 + ... + A^nB^{n(n+1)(2n+1)/6}}$$

The constants A and B can be calculated from equations

$$\sum_{i=1}^{n} p_i = 1$$

and

$$\sum_{i=1}^{n} i^2 p_i = \sigma^2.$$

Now let us see for $n = 6$.

The probabilities and S_{max} is given in Table VI and the graph between S_{max} and σ is given at Fig. 4.

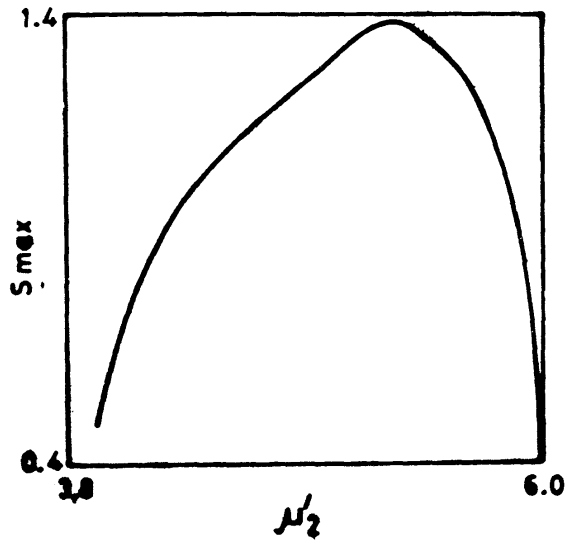

FIG. 4

TABLE VI

σ	6	5.81756	5.68110	5.26088	4.32231	4.08271	3.97739	3.89970
p_1	0	0.00000	0.00000	0.00930	0.09942	0.13219	0.14905	0.16500
p_2	0	0.00000	0.00018	0.02729	0.13543	0.15680	0.16400	0.16684
p_3	0	0.00000	0.00354	0.06211	0.15279	0.16376	0.16659	0.16689
p_4	0	0.00676	0.04202	0.12947	0.16627	0.16773	0.16771	0.16690
p_5	0	0.18358	0.25302	0.25981	0.18666	0.17374	0.16944	0.16692
p_6	1	0.80961	0.70123	0.51202	0.25942	0.20578	0.18322	0.16749
S_{max}	0	0.94913	1.19496	1.37753	0.97008	0.73915	0.60494	0.46662

5. SUMMARY

The *M*-Entropy measure for the case when probabilities are increasing is

$$S = -p_1 \ln p_1 - (p_2 - p_1) \ln (p_2 - p_1) - \ldots$$

$$- (p_n - p_{n-1}) \ln (p_n - p_{n-1}) - (1 - p_n) \ln (1 - p_n)$$

This *M*-Entropy measure is non symmetric concave function of $(p_1, p_2, \ldots, p_n)$ and we get non-negative probabilities after maximizing this measure subject to some linear constraints. The A.M., G.M., H.M. and μ_2' for this measure will satisfy the following inequalities.

$$AM \geq (n+1)/2,\ GM \geq (n!)^{1/n},\ H \geq n/(1 + 1/2 + \ldots + 1/n)$$

and $$\mu_2' \geq (n+1)(2n+1)/6.$$

The minimum value of this measure is zero and maximum value will occur when

$$(p_2 - p_1)/p_1 = (p_3 - p_2)/(p_2 - p_1) = \ldots\ldots = (p_n - p_{n-1})/(p_{n-1} - p_{n-2})$$

$$= (1 - p_n)/(p_n - p_{n-1}).$$

We have found out the MM-EPD when one of the moment (AM or GM or HM or μ_2') is prescribed.

REFERENCES

1. J. Havarda and F. Charvat, *Quantification Methods of Classificatory Processes; Concept struct.$^{\alpha}$ Entropy*, **3** (1967) 30-35
2. J. N. Kapur, *Math. Sem.* **4** (1972) 78-94
3. J. N. Kapur, *Non Additive Measures of Entropy*. Univ. Manitoba Res.Rep. (1980)
4. J. N. Kapur, (1981), *Entropy Maximization and Ordered Statistics*. Univ. of Waterloo Research Report.
5. J. N. Kapur, *J. Inf. Opt. Sci.*, **3** (1983 *a*) 207-32.
6. J. N. Kapur, *Indian J. pure appl Math.* **14**(2) (1983 *b*) 1372-84.
7. J. N. Kapur, *Indian J. Pure appl. Math*, **17** (1986) 429-49.
8. A. Renyi, *Proc. 4th Berkely Symp. Math. Stat. Prob.* Univ. California Press (1961) 547-61
9. C. E. Shannon, *Bell. Syst. Tech. J.* **27** (1948) 379-423