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A COMMON FIXED POINT THEOREM UNDER A NEW CONDITION

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The aim of the present paper is to obtain a common fixed point theorem by using a new continuity condition. Our theorem generalizes a multitude of common fixed point theorems and is, perhaps, the first theorem guaranteeing the existence of a common fixed point even when all the mappings may be discontinuous and some of the mappings may not satisfy the compatibility condition.

Key Words : Common Fixed Points; Compatible Maps; Contractive Conditions; Reciprocal Continuity

INTRODUCTION

In recent years, some interesting common fixed point theorems for contractive type mappings have been reported in the literature, e.g., Jachymski[2], Jungck *et al.*[4] and Pant[6,7]. These theorems invariably require a compatibility condition and a contractive condition besides assuming continuity of atleast one of the mappings and each theorem aims at weakening one or more of these conditions. The present paper is an attempt to obtain a common fixed point theorem by using fewer compatibility and contractive conditions and by using a new continuity condition. We also show that in the setting of common fixed point theorems for compatible maps satisfying contractive conditions, the new notion of continuity is weaker than the assumption of continuity of one of the mappings.

Two selfmaps A and S of a metric space (X, d) are called compatible if $\lim_n d(ASx_n, SAx_n) = 0$ whenever $\{x_n\}$ is a sequence such that $\lim_n Ax_n = \lim_n Sx_n = t$ for some t in X. This notion is due to Jungck[3] and implies commutativity at coincidence points.

Two selfmaps A and S of a metric space (X, d) will be called reciprocally continuous if $\lim_n ASx_n = At$ and $\lim_n SAx_n = St$ whenever $\{x_n\}$ is a sequence such that $\lim_n Ax_n = \lim_n Sx_n = t$ for some t in X.

If A and S are both continuous then they are obviously reciprocally continuous but the converse is not true. Moreover, in the setting of common fixed point theorems for compatible maps satisfying contractive conditions, continuity of one of the mappings A or S implies their reciprocal continuity but not conversely. We illustrate these points in the following pages.

MAIN RESULTS

If $\{A_i\}$, $i = 1, 2, 3, \ldots$, S and T be selfmappings of a metric space (X, d), in the sequel we shall denote

$$M_{1i}(x, y) = \max \{d(Sx, Ty), d(A_1x, Sx), d(A_i y, Ty), [d(A_1x, Ty) + d(A_i y, Sx)]/2\}.$$

Also, let $\phi : R_+ \to R_+$ denote an upper semicontinuous function such that $\phi(t) < t$ for each $t > 0$.

Theorem — *Let $\{A_i\}$, $i = 1, 2, 3, \ldots$, S and T be selfmappings of a complete metric space (X, d) such that*

(i) $A_1X \subseteq TX$ *and* $A_iX \subseteq SX$ *when* $i > 1$

(ii) $d(A_1x, A_2y) \leq \phi(M_{12}(x, y))$ *whenever* $M_{12}(x, y) > 0$

(iii) $d(A_1x, A_iy) < M_{1i}(x, y)$.

Let S be compatible with A_1 and T be compatible with A_k for some $k > 1$. If the mappings in one of the compatible pairs (A_1, S) or (A_k, T) be reciprocally continuous then all the A_i, S and T have a unique common fixed point.

PROOF : Let x_0 be any point in X. Define sequences $\{x_n\}$ and $\{y_n\}$ in X given by

$$y_{2n} = A_1x_{2n} = Tx_{2n+1}, y_{2n+1} = A_2x_{2n+1} = Sx_{2n+2}.$$

This can be done by virtue of *(i)*. Then, using *(ii)* we obtain

$$d(y_{2n}, y_{2n+1}) \leq \phi(d(y_{2n-1}, y_{2n})) < d(y_{2n-1}, y_{2n}) \quad \ldots (1)$$

and

$$d(y_{2n-1}, y_{2n}) \leq \phi(d(y_{2n-2}, y_{2n-1})) < d(y_{2n-2}, y_{2n-1}). \quad \ldots (2)$$

We thus see that $\{d(y_n, y_{n+1})\}$ is a strictly decreasing sequence of positive numbers and hence tends to a limit $r \geq 0$. Suppose $r > 0$. Then the inequality (1) on making $n \to \infty$ and in view of upper semicontinuity of ϕ yields $r \leq \phi(r) < r$, a contradiction. Hence, $r = \lim_{n \to \infty} d(y_n, y_{n+1}) = 0$. We show that $\{y_n\}$ is a Cauchy sequence. Suppose it is not. Then there exists $\varepsilon > 0$ and a subsequence $\{y_{n_i}\}$ of $\{y_n\}$ such that $d(y_{n_i}, y_{n_{i+1}}) > 2\varepsilon$. Since $\lim_{n \to \infty} d(y_n, y_{n+1}) = 0$, there exist integers m_i satisfying $n_i < m_i < n_{i+1}$ such that $d(y_{n_i}, y_{m_i}) \geq \varepsilon$. If not, then

$$d(y_{n_i}, y_{n_{i+1}}) \leq d(y_{n_i}, y_{n_{i+1}-1}) + d(y_{n_{i+1}-1}, y_{n_{i+1}})$$

$$< \varepsilon + d(y_{n_{i+1}-1}, y_{n_{i+1}}) < 2\varepsilon,$$

a contradiction. If m_i be the smallest integer such that $d(y_{n_i}, y_{m_i}) > \varepsilon$, then

$$\varepsilon < d(y_{n_i}, y_{m_i}) \le d(y_{n_i}, y_{m_i-2}) + d(y_{m_i-2}, y_{m_i-1}) + d(y_{m_i-1}, y_{m_i})$$

$$< \varepsilon + d(y_{m_i-2}, y_{m_i-1}) + d(y_{m_i-1}, y_{m_i}).$$

That is, there exist integers m_i satisfying $n_i < m_i < n_{i+1}$ such that $d(y_{n_i}, y_{m_i}) \ge \varepsilon$ and

$$\lim_{n_i \to \infty} d(y_{n_i}, y_{m_i}) = \varepsilon. \qquad \ldots (3)$$

Moreover, m_i can be chosen in such a manner that m_i is even when n_i is odd and m_i is odd when n_i is even. Suppose that n_i is odd and m_i even. Then, by virtue of (*ii*), we get

$$d(y_{n_i+1}, y_{m_i+1}) = d(A_1 x_{n_i+1}, A_2 x_{m_i+1}) \le \phi(M_{12}(x_{n_i+1}, x_{m_i+1})).$$

On letting $n_i \to \infty$ and in view of (3) and upper semicontinuity of ϕ, the above inequality yields $\varepsilon \le \phi(\varepsilon) < \varepsilon$, a contradiction. Hence, $\{y_n\}$ is a Cauchy sequence. Since X is complete, there exists a point z in X such that $y_n \to z$. Also

$$y_{2n} = A_1 x_{2n} = Tx_{2n+1} \to z \text{ and } y_{2n+1} = A_2 x_{2n+1} = Sx_{2n+2} \to z.$$

We show that $A_i x_{2n+1} \to z$ for each $i > 2$. If $\lim_n A_i x_{2n+1} \ne z$ for some $i > 1$, then there exist a subsequence $\{A_i x_{2m+1}\}$ of $\{A_i x_{2n+1}\}$, a number $r > 0$ and a positive integer M such that for each $m \ge M$ we have $d(A_1 x_{2m}, A_i x_{2m+1}) \ge r$, $d(A_i x_{2m+1}, z) \ge r$ and

$$d(A_1 x_{2m}, A_i x_{2m+1}) < M_{1i}(x_{2m}, x_{2m+1}) = d(A_1 x_{2m}, A_i x_{2m+1}),$$

a contradiction. Hence, $A_i x_{2m+1} \to z$ for each $i > 1$.

Suppose that T is a compatible with A_k for some $k > 1$ and T and A_k are reciprocally continuous. Then the reciprocal continuity of A_k and T implies that $\lim_n A_k Tx_{2n+1} = A_k z$ and $\lim_n TA_k x_{2n+1} = Tz$. Compatibility of A_k and T yields $\lim_n d(A_k Tx_{2n+1}, TA_k x_{2n+1}) = 0$, that is, $d(A_k z, Tz) = 0$. Hence, $A_k z = Tz$. Since $A_k X \subseteq SX$, there exists a point u in X such that $A_k z = Su$. We show that $Su = A_1 u$. If not, the inequality

$$d(A_1 u, A_k z) < M_{1k}(u, z) = d(A_1 u, A_k z)$$

yields a contradiction. Hence, $Tz = A_k z = Su = A_1 u$. Compatibility of A_1 and S implies that $A_1 Su = SA_1 u$ and, hence, $A_1 A_1 u = A_1 Su = SA_1 u = SSu$. Similarly, compatibility of A_k and T implies that $A_k A_k z = A_k Tz = TA_k z = TTz$. If $A_1 u \ne A_1 A_1 u$, using (*iii*) we get

$$d(A_1 u, A_1 A_1 u) = d(A_1 A_1 u, A_k z) < M_{1k}(A_1 u, z) = d(A_1 A_1 u, A_k z),$$

a contradiction. Hence, $A_1 u = A_1 A_1 u$ and $A_1 u = A_1 A_1 u = SA_1 u$, that is, $A_1 u$ is a common fixed point of A_1 and S. Similarly, using (*iii*) we find that $A_k z\ (= A_1 u)$ is a common fixed point of A_k and T.

Moreover, if $A_k z \neq A_i A_k z$ for some $i > 1$, using (*iii*) we get

$$d(A_k z, A_i A_k z) = d(A_1 u, A_i A_k z) < M_{1i}(u, A_k z) = d(A_1 u, A_i A_k z),$$

a contradiction. Hence, $A_k z\ (= A_1 u)$ is a common fixed point of T and A_i for every $i > 1$. Uniqueness of the common fixed point follows easily. The proof is similar when A_1 and S are assumed reciprocally continuous.

We now give an example to illustrate the above theorem

Example — Let $X = [2, 20]$ and d be the usual metric on X. Define $\{A_i\}$, S and $T : X \to X$ by

$$S2 = 2, \quad Sx = 6 \quad \text{if } x > 2$$

$$T2 = 2, \quad Tx = 12 \quad \text{if } 2 < x \leq 5, \quad Tx = x - 3 \quad \text{if } x > 5$$

$$A_1 2 = 2, \quad A_1 x = 3 \quad \text{if } x > 2$$

$$A_2 x = 2 \quad \text{if } x = 2 \text{ or } > 5, \qquad A_2 x = 6 \quad \text{if } 2 < x \leq 5$$

and for $i > 2$

$$A_i x = 2 \quad \text{if } x \leq 2 + (1/i) \text{ or } > 5, \qquad A_i x = 6 \quad \text{if } 2 + (1/i) < x \leq 5.$$

Then $\{A_i\}$, S and T satisfy all the conditions of the above theorem and have a unique common fixed point $x = 2$. It may be noted in this example that A_1 and S are reciprocally continuous and compatible maps. But neither A_1 nor S is continuous, not even at their common fixed point $x = 2$. It may also be verified that T and A_i are compatible when $i > 2$. However, T and A_2 are noncompatible. To see that T and A_2 are noncompatible, let us consider a decreasing sequence $\{x_n\}$ such that $x_n \to 5$. Then $Tx_n = x_n - 3 \to 2, A_2 x_n = 2, TA_2 x_n = 2, A_2 T x_n = A_2(x_n - 3) = 6$. Hence, T and A_2 are noncompatible. It is also clear that all the mappings involved in this example are discontinuous.

As a corollary of Theorem 1 we get the following :

Theorem 1′ — *Let A_1, A_2, S and T be selfmappings of a complete metric space (X, d) such that*

(i)′ $A_1 X \subseteq TX$ *and* $A_2 X \subseteq SX$

(ii)′ $d(A_1 x, A_2 y) \leq \phi(M_{12}(x, y))$.

Let (A_1, S) and (A_2, T) be compatible pairs. If the mappings in one of the compatible pairs are reciprocally continuous then A_1, A_2, S and T have a unique common fixed point.

The proof follows directly from the proof of the main theorem and requires no additional argument. To consider an example for this theorem, we may define (X, d), A_1, A_2 and S as in the above example and T may be defined by

$$Tx = 2 \quad \text{if} \quad x = 2, \qquad Tx = 12 \quad \text{if} \quad 2 < x \leq 5, \quad Tx = x - (5/2) \quad \text{if} \quad x > 5.$$

It may, however, be noted that all the mappings involved in this example are discontinuous at the fixed point.

DISCUSSION AND AUXILIARY RESULTS

All the analogous results concerning four mappings satisfying compatibility conditions or similar results involving a sequence of mappings require one of the mappings in a compatible pair to be continuous. For example, Theorem 3.3 of Jachymski[2] assumes one of the mappings to be continuous while Theorem 5.1 of the same paper assumes S or T to be continuous. The main theorem of Rhoades *et al.*[9] and Jungck *et al.*[4] requires S or T to be continuous. Likewise, the theorems of Fisher[1] and Pant[5, 6, 7] assume one of the mappings to be continuous. From the example given above, it is obvious that the present theorems do not require one of the mappings in a compatible pair or any other mapping to be continuous. On the other hand, we now proceed to show that under the contractive condition (*ii*) continuity of one of the mappings in a compatible pair implies their reciprocal continuity.

So, in the above theorem let us assume that A_1 and S are compatible and S is continuous. Let $\{x_n\}$ be a sequence such that $A_1x_n \to t$ and $Sx_n \to t$ for some t in X. Since S is continuous, we get $SSx_n \to St$ and $SA_1x_n \to St$. Compatibility of A_1 and S implies that $\lim_n d(A_1Sx_n, SA_1x_n) = 0$, that is, $\lim_n A_1Sx_n \to St$. Since $A_1X \subseteq TX$, for each x_n there exists y_n in X such that $A_1Sx_n = Ty_n$. Then $SSx_n \to St$, $SA_1x_n \to St$, $A_1Sx_n \to St$ and $Ty_n \to St$ where $Ty_n = A_1Sx_n$. We claim that $A_2y_n \to St$. If not, then there exist a subsequence $\{A_2y_m\}$ or $\{A_2y_n\}$, a number $r > 0$ and a positive integer M such that for each $m \geq M$ we have $d(A_1Sx_m, A_2y_m) \geq r$, $d(A_2y_m, St) \geq r$ and

$$d(A_1Sx_m, A_2y_m) \leq \phi(M_{12}(Sx_m, y_m)) = \phi(d(A_2y_m, Ty_m)) < d(A_2y_m, A_1Sx_m),$$

a contradiction. Hence, $A_2y_n \to St$. If $A_1t \neq St$, then by (*ii*) we get

$$d(A_1t, A_2y_n) \leq \phi(M_{12}(t, y_n)).$$

On letting $n \to \infty$ and in view of upper semicontinuity of ϕ the above inequality yields $d(A_1t, St) \leq \phi(d(A_1t, St)) < d(A_1t, St)$, a contradiction. Hence, $A_1t = St$. Thus by assuming $A_1x_n \to t$, $Sx_n \to t$ and continuity of S we obtain that $SA_1x_n \to St$ and $A_1Sx_n \to A_1t\ (= St)$, that is, A_1 and S are found to be reciprocally continuous if they are assumed compatible and S is continuous.We arrive at the same conclusion when A_1 and S are compatible and A_1 is assumed continuous. The same conclusion can be drawn when the contractive condition (*ii*) is replaced by the Meir-Keeler type (ε, δ) contractive condition

$$(iv) \qquad \varepsilon \leq M_{12}(x, y) < \varepsilon + \delta \Rightarrow d(A_1x, A_2y) < \varepsilon$$

and δ is assumed to be nondecreasing e.g., in Pant[5,6] or lower semicontinuous e.g., in Jungck *et al.*[4] This is so because, as shown in Jachymski[2], the (ε, δ) contractive condition implies contractive (*ii*) whenever δ is nondecreasing or lower semicontinuous.

Thus our theorem improves the common fixed point theorems for compatible maps satisfying contractive conditions (*ii*) or (*iv*) in two respects. Firstly, it replaces the condition of continuity of one of the mappings by reciprocal continuity which, by virtue of contractive condition (*ii*) or by virtue of lower semicontinuity or nondecreasing character of δ in (*iv*), is a weaker condition than the continuity of one mapping. Secondly, as compared to analogous results, our theorem uses fewer compatibility conditions and weaker contractive conditions. For example, Theorem 5.1 of Jachymski[2] has been proved by assuming T to be compatible with A_i for each $i > 1$ and by using the contractive

condition $d(A_1x, A_iy) < \phi_i(M_{1i}(x, y))$ for each $i > 1$, where ϕ_i has been assumed to be upper semicontinuous. The present theorem assumes T to be compatible with one mapping only and uses the plain contractive condition $d(A_1x, A_iy) < M_{1i}(x, y)$ for each $i > 2$. In the example given above, T is noncompatible with A_2. Similarly, the main theorem in Jungck *et al.*[4] and in Rhoades *et al.*[9] has been proved by assuming each A_i to be compatible with both S and T and by using Meir-Keeler type (ε, δ) contractive condition for each pair of mappings A_i, A_1. Likewise, the theorem of Pant[7] requires T to be compatible with A_2 in place of assuming it compatible with A_k for some $k > 1$. Among the results which can be obtained as special cases of the present theorem we mention only those due to Fisher[1], Jachymski[2], Jungck *et al.*[4], Pant[5,6,7] and Rhoades *et al.*[9]

Remark : There is an interesting possibility of applying the notion of reciprocal continuity to the problem of finding common fixed points of four mappings A, B, S and T satisfying the Meir-Keeler type contractive condition :

given $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\varepsilon \le M(x, y) < \varepsilon + \delta \Rightarrow d(Ax, By) < \varepsilon. \quad \ldots (4)$$

It is known, as was shown by Rao and Rao[8], that the condition (4) and continuity of one of the mappings do not guarantee the existence of a common fixed point unless δ satisfies some additional condition. For example, Jungck *et al.*[4] assume δ to be lower semi continuous while Pant[5,6] assumes δ to be nondecreasing. We make the conjecture that condition (4) implies the existence of a common fixed point if one of the pairs (A, S) or (B, T) is assumed compatible and reciprocally continuous. A detailed study for this alongwith the study of some other consequences of the notion of reciprocal continuity will be undertaken in a proposed paper.

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