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ON POLYNOMIALS HAVING ZEROS IN CLOSED EXTERIOR OR CLOSED INTERIOR OF A CIRCLE

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Let $f(z)$ be an entire function and $M(f, r) = \max_{|z|=r} |f(z)|$.

For a polynomial $p(z)$ of degree n, having all its zeros in $|z| \le k$, $k > 0$, it is known that

$$M(p, R) \ge \left(\frac{R+k}{1+k}\right)^n M(p, 1), k \le 1 \ \& \ R > 1 \text{ or } k > 1 \ \& \ R \ge k^2,$$

whereas, for polynomials having no zeros in $|z| < k$, and $p(z) = c_0 + \sum_{\nu=\mu}^{n} c_\nu z^\nu$

$$M(p', 1) \le \frac{n}{J + k^\mu} M(p, 1), \quad k \ge 1.$$

Using $m = \min_{|z|=k} |p(z)|$, we have refined the above inequalities as

$$M(p, R) \ge \left(\frac{R+k}{1+k}\right)^n \left[M(p, 1) + \left\{ 1 - \left(\frac{R+k}{1+k}\right)^{-n} \right\} m \right], k \le 1 \ \& \ R > 1 \text{ or } k > J \ \& \ R \ge k^2, \quad \ldots (1)$$

$M(p, R) \ge R^s \left(\frac{R+k}{J+k}\right) M(p, 1), k > 1 \ \& \ k < R < k^2$ with S being the order of a possible zero of $p(z)$ at $z = 0$,

$$M(p', 1) \le \frac{n}{1 + k^\mu} \left[M(p, 1) - \frac{m}{k^{n-\mu}} \right], \quad k \ge 1.$$

Inequality (*i*) is sharp.

Key Words : Polynomials; Zeros; Closed Exterior & Inferior of a circle; Inequality

1. INTRODUCTION AND STATEMENT OF RESULTS

Let $f(z)$ be an entire function and $M(f, r) = \max_{|z|=r} |f(z)|$. If $p(z)$ is a polynomial of degree n, then we have, as a simple deduction from maximum modulus principle [6, p. 158, problem III 267 & 269] :

$$M(p, r) \geq r^n M(p, 1), r < 1.$$

Here, equality holds iff $p(z) = \alpha z^n$.

For polynomials not vanishing in $|z| < 1$, Rivlin[7] obtained stronger inequality

$$M(p, r) \geq \left(\frac{1+r}{2}\right)^n M(p, 1), \quad r < 1. \qquad \ldots (1.1)$$

Here, equality holds for $p(z) = (\alpha + \beta z)^n, |\alpha| = |\beta|$.

Aziz[1] obtained the following generalisation of inequality (1.1), for polynomials not vanishing in $|z| < k, k > 0$.

Theorem *A — If $p(z)$ is a polynomial of degree n, having no zeros in $|z| < k, k > 0$, then*

$$M(p, r) \geq \left(\frac{r+k}{1+k}\right)^n M(p, 1), k \geq 1 \ \& \ r < 1 \text{ or } k < 1 \ \& \ r \leq k^2.$$

Here equality holds for $p(z) = (z + k)^n$.

On applying Theorem A to the polynomial $z^n p(1/z)$, Aziz[1] obtained

Theorem *A′ — If $p(z)$ is a polynomial of degree n, having all its zeros in $|z| \leq k, k > 0$ then*

$$M(p, r) \geq \left(\frac{R+k}{1+k}\right)^n M(p, 1), k \leq 1 \ \& \ R > 1 \text{ or } k > 1 \ \& \ R \geq k^2.$$

Here equality holds for the polynomial $p(z) = (z + k)^n$.

We have obtained a refinement of Theorem A′. More precisely, we prove

Theorem 1 — *Let $p(z)$ be a polynomial of degree n, having all its zeros in $|z| \leq k, k > 0$. Then*

$$M(p, R) \geq \left(\frac{R+k}{1+k}\right)^n \left[M(p, 1) + \left\{ 1 - \left(\frac{R+k}{1+k}\right)^{-n} \right\} m \right], \quad k \leq 1 \ \& \ R > 1$$

$$\text{or } k > 1 \ \& \ R \geq k^2, \qquad \ldots (1.2)$$

and

$$M(p, R) \geq R^s \left(\frac{R+k}{1+k}\right) M(p, 1), k > 1 \ \& \ k < R < k^2, \qquad \ldots (1.3)$$

where

$$m = \min_{|z| = k} |p(z)|,$$

S = *Order of a possible zero of $p(z)$ at $z = 0$.*

Here equality holds in (1.2) for the polynomial $p(z) = (z + k)^n$.

On the other hand, concerning the estimate of $|p'(z)|$ on the unit disc $|z| \leq 1$, we have the following famous result known as Bernstein's inequality[8]

$$M(p', 1) \leq n M(p, 1).$$

Here equality holds for $p(z) = \alpha z^n$.

For polynomials not vanishing in $|z| < 1$, Erdös conjectured and Lax[4] proved a stronger inequality

$$M(p', 1) \leq \frac{n}{2} M(p, 1). \qquad \ldots (1.4)$$

In (1.4) equality holds for all polynomials having their zeros on unit circle $|z| = 1$.

Chan and Malik[3] (see also (5)) obtained the generalisation of inequality (1.4) for polynomials not vanishing in $|z| < k, k \geq 1$, and proved

Theorem *D — Let*

$$p(z) = c_0 + \sum_{\nu=\mu}^{n} c_\nu z^\nu$$

be a polynomial of degree n, having no zeros in $|z| < k, k \geq 1$. *Then*

$$M(p', 1) \leq \frac{n}{1 + k^\mu} M(p, 1). \qquad \ldots (1.5)$$

Aziz and Dawood[2] obtained certain modification of inequality (1.4) and proved

Theorem *E — If p(z) is a polynomial of degree n, having no zeros in* $|z| < 1$, *then*

$$M(p', 1) \leq \frac{n}{2} [M(p, 1) - m_1], \qquad \ldots (1.6)$$

where

$$m_1 = \min_{|z|=1} |p(z)|.$$

In (1.6), equality holds for $p(z) = \alpha + \beta z^n, |\alpha| \geq |\beta|$.

We have obtained similar type of modification of inequality (1.5).

To be precise, we have

Theorem 2 — *Let*

$$p(z) = c_0 + \sum_{\nu=\mu}^{n} c_\nu z^\nu$$

be a polynomial of degree n, having no zeros in $|z| < k, k \geq 1$. *Then*

$$M(p', 1) \leq \frac{n}{1+k^{\mu}}\left[M(p, 1) - \frac{m}{k^{n-\mu}} \right],$$

where m is as in Theorem 1. Here equality holds for $p(z) = (z^{\mu} + k^{\mu})^{n/\mu}$, where n is a multiple of μ.

2. LEMMAS

For the proofs of the theorems, we require the following lemma :

Lemma 1 — Let $p(z)$ be a polynomial of degree n, having all its zeros in $|z| \geq k, k > 0$. Then

$$\frac{M(p, r)}{r^n + k r^{n-1}} \geq \frac{M(p, R)}{R^n + kR^{n-1}}, \quad r \leq k \ \& \ R \geq k.$$

PROOF OF LEMMA 1 — Let $r \leq k, R \geq k$. Obviously the polynomial $T(z) = p(rz)$ has no zeros in $|z| < k/r$. As $k/r \geq 1$, we have, by (1.5)

$$M(T', 1) \leq \frac{n}{1+k/r} M(T, 1),$$

i.e.,

$$rM(p', r) \leq \frac{nr}{k+r} M(p, r),$$

i.e.,

$$M(p', r) \leq \frac{n}{k+r} M(p, r). \qquad \ldots (2.1)$$

As $p'(z)$ is a polynomial of degree $(n-1)$, we have by maximum modulus principle [6, p. 158, problem III 269]

$$\frac{M(p', t)}{t^{n-1}} \leq \frac{M(p', r)}{r^{n-1}}, \quad t \geq r. \qquad \ldots (2.2)$$

Combining (2.1) and (2.2), we get

$$M(p', t) \leq \frac{nt^{n-1}}{r^n + kr^{n-1}} M(p, r), \quad t \geq r. \qquad \ldots (2.3)$$

Now we have, for $0 \leq \theta \leq 2\pi$

$$p(Re^{i\theta}) - p(re^{i\theta}) = \int_r^R p'(te^{i\theta})\, e^{i\theta}\, dt,$$

which implies

$$\left| p(Re^{i\theta}) - p(re^{i\theta}) \right| \leq \int_r^R \left| p'(te^{i\theta}) \right| dt,$$

$$\leq \int_r^R \frac{nt^{n-1}}{r^n + kr^{n-1}} M(p, r)dt, \qquad \text{(by (2.3))}$$

$$= \frac{R^n - r^n}{r^n + kr^{n-1}} M(p, r).$$

Hence,

$$| p(Re^{i\theta}) | \leq | p(re^{i\theta}) | + \frac{R^n - r^n}{r^n + kr^{n-1}} M(p, r),$$

which implies

$$M(p, R) \leq \frac{R^n + kr^{n-1}}{r^n + kr^{n-1}} M(p, r),$$

$$\leq \frac{R^n + kR^{n-1}}{r^n + kr^{n-1}} M(p, r),$$

i.e.,

$$\frac{M(p, R)}{R^n + kR^{n-1}} \leq \frac{M(p, r)}{r^n + kr^{n-1}},$$

and this completes the proof of Lemma 1.

3. Proofs of the Theorems

Proof of Theorem 1 — By Rouche's theorem, the polynomial

$$p(z) + \alpha m, \ | \alpha | < 1,$$

has all its zeros in $| z | \leq k$. On applying Theorem A′ to the polynomial

$$p(z) + \alpha m, \ | \alpha | < 1,$$

we have

$$\max_{|z|=R} | p(z) + \alpha m | \geq \left(\frac{R + k}{1 + k} \right)^n \max_{|z|=1} | p(z) + \alpha m |, \ k \leq 1 \ \& \ R > 1 \text{ or } k > 1 \ \& \ R \geq k^2,$$

$$M(p, R) + |\alpha| m \geq \left(\frac{R+k}{1+k}\right)^n |p(z_1) + \alpha m|, k \leq 1 \ \& \ R > 1 \text{ or } k > 1 \ \& \ R > k^2, \ \ldots (3.1)$$

where

$$|p(z_1)| = M(p, 1).$$

Now choosing the argument of α suitably in (3.1) and then letting $|\alpha| \to 1$, inequality (1.2) follows:

Further, the polynomial

$$q(z) = z^n \overline{p(1/\bar{z})},$$

has all its zeros in $|z| \geq 1/k, 1/k < 1$, and is of degree $n - s$. On applying Lemma 1 to the polynomial $q(z)$, with $R = 1$, we have

$$\frac{M(q, r)}{r^{n-s} + \dfrac{r^{n-s-1}}{k}} \geq \frac{M(q, 1)}{1 + \dfrac{1}{k}}, \quad \frac{1}{k^2} < r < \frac{1}{k}$$

i.e.,

$$M(q, r) \geq r^{n-s-1}\left(\frac{r + 1/k}{1 + 1/k}\right) M(q, 1), \quad \frac{1}{k^2} < r < \frac{1}{k},$$

i.e.,

$$M(p, 1/r) \geq \frac{1}{r^{s+1}}\left(\frac{r + 1/k}{1 + 1/k}\right) M(p, 1) < \frac{1}{k^2} < r < \frac{1}{k},$$

and the inequality (1.3) follows, on replacing r by $1/R$. This completes the proof of Theorem 1.

PROOF OF THEOREM 2 : By Rouche's theorem, the polynomial

$$p(z) + m\beta z^n, \qquad |\beta| < \frac{1}{k^n},$$

will have no zeros in $|z| < k$. So, on applying Theorem D to the polynomial

$$p(z) + m\beta z^n, \qquad |\beta| < \frac{1}{k^n},$$

we have

$$\max_{|z|=1} |p'(z) + mn\beta z^{n-1}| \leq \frac{n}{1 + k^{\mu}} \max_{|z|=1} |p(z) + \beta m z^n|,$$

i.e.

$$| p'(z_0) + mn \beta z_0^{n-1} | \leq \frac{n}{1+k^{\mu}} [M(p, 1) + m | \beta |], \quad \ldots (3.2)$$

where

$$| p'(z_0) | = M(p', 1).$$

Now choosing the argument of β suitably in (3.2) and finally letting $| \beta | \to \frac{1}{k^n}$, Theorem 2 follows.

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