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GLOBAL ATTRACTIVITY OF DELAY DIFFERENCE EQUATIONS

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In this paper, a sufficient condition on the global attractivity of difference equation

$$x_{n+1} = \alpha x_n + f(x_n, x_{n-k_1}, ..., x_{n-k_m})$$

is obtained.

Key Words : Global Attractivity; Nonlinear and Autonomous Difference Equation

1. INTRODUCTION

In this paper we investigate the global attractivity of the equilibrium of the following nonlinear autonomous delay difference equation

$$x_{n+1} = \alpha x_n + f(x_n, x_{n-k_1}, ..., x_{n-k_m}), \qquad ... (1.1)$$

where $\alpha \in (0, 1)$, $k_i \in N (i = 1, 2, ..., m)$ and $f \in C[R^{m+1}, R]$, there exists $r_0 (r_0 \geq 0)$, for every $u_0, u_1, ..., u_m > r_0$, $f(u_0, u_1, ..., u_m)$ is nonincreasing in each of its arguments $u_0, u_1, ..., u_m$.

Let x^* is an equilibrium of (1.1). In Section 3, we will establish sufficient conditions for x^* to be a global attractor of all positive solutions of (1.1).

This paper was motivated by the problem below which was proposed by Kocic and Ladas [1].

Problem — Study the global attractivity of the difference equation

$$N_{n+1} - N_n = -\alpha N_n + \frac{\beta N_{n-k}}{1 + N_{n-k}^m}, \qquad ... (1.2)$$

where $\alpha \in (0, 1)$, $\frac{\beta}{\alpha} > 1$, $m \in N$, and

$$N_{n+1} - N_n = -\alpha N_n + \beta e^{-\gamma N_{n-k}}, \qquad ... (1.3)$$

where $\alpha \in (0, 1), \beta > 0, \gamma > 0$ and $k \in \{0, 1, ...\}$.

In the next section we will introduce a Theorem in [3] which will be applied in several mathematical biologic modes. For example,

$$N_{n+1} - N_n = N_n(a + bN_{n-k} - cN_{n-k}^2), \quad \text{... (1.4)}$$

where $a, c \in (0, \infty), b \in (-\infty, \infty)$ and $k \in \{0, 1, ...\}$.

Eq. (1.4) is viewed as a discrete analogue of the delay differential equation

$$N'(t) = N(t)\,[a + bN(t - \tau) - cN^2(t - \tau)] \quad (t \geq 0),$$

where $a, c \in (0, \infty), b \in (-\infty, \infty)$ and $\tau \in [0, \infty)$, which is a model of single species with a quadratic percapita growth rate, and was investigated by Gopalsamy and Ladas[4].

A more general case of (1.4) is a model

$$N_{n+1} - N_n = N_n\,(a + bN_{n-k} - cN_{n-l}^2), \quad \text{... (1.5)}$$

where $a, c \in (0, \infty), b \in (-\infty, \infty)$ and $k, l \in \{0, 1, ...\}$.

This theorem will be also applied to the delay difference equation

$$x_{n+1} = Ax_n\left(1 - \sum_{i=0}^{m} B_i x_{n-i}\right), \quad \text{... (1.6)}$$

where $A \in (1, \infty)$, $m \in \{0, 1, ...\}$ and $B_i \in (0, \infty)$ $(i = 1, 2, ..., m)$

In fact, the global attractivity of (1.2)-(1.6) is research projects in [1].

2. PRELIMINARY

The following theorem is extracted from [3].

Theorem A — *Assume that x^* is the unique positive equilibrium of delay difference equation*

$$x_{n+1} = x_n f(x_n, x_{n-k_1}, x_{n-k_2}, ..., x_{n-k_m}), \quad \text{... (2.1)}$$

where $k_i \in N (i = 1, 2, ..., m)$ and $f \in C[R^{m+1}, R]$, there exists $r_0 (r_0 \geq 0)$, for every $u_0, u_1, ..., u_m > r_0$, $f(u_0, u_1, ..., u_m)$ is nonincreasing in each of its arguments $u_0, u_1, ..., u_m$.

Set

$$F(u) = [f(u, u, ..., u)]^{k+1}$$

and satisfies

$$F(r_0) > 1 \text{ and } F(Q_0) > 0.$$

Finally, assume that the solution of the IVP

$$\left.\begin{aligned} Q_{n+1} &= x^* F(Q_n) \\ Q_0 &= x^* F(r_0) \end{aligned}\right] \qquad \ldots (2.2)$$

tends to x^* as n tends to ∞. Then x^* is a global attractor of all positive solutions of (2.1).

We introduce a lemma in [2] which provides a sufficient condition for the solution of the form of IVP (2.2) to tend to the equilibrium.

Lemma — Consider the difference equation

$$y_{n+1} = h(y_n), \qquad \ldots (2.3)$$

where $h \in C^1 [(l_1, l_0), (l_1, R^+)]$ with $l_1 \geq 0$, and $n = 0, 1, \ldots$.

Assume that y^* is the unique fixed point of h and that h is a nonincreasing function. Suppose also that

$$h'(y)\, h'(h(y)) < 1 \text{ either for } l_1 < y < y^* \text{ or for } y^* < y < h_0.$$

Then the solution $\{y_n\}$ of (2.3) with $y_0 = h(l_1)$ tends to y^* as n tends to ∞.

3. THEOREM AND APPLICATIONS

The following theorem is the main result in this paper.

Theorem 3.1 — *Assume that x^* is the unique positive equilibrium of (1.1). Suppose that*

$$Q_0 > x^* \quad \text{and} \quad Q_1 > 0$$

and assume that the solution of the IVP

$$\left.\begin{aligned} Q_{n+1} &= F(Q_n) \\ Q_0 &= \frac{F(r_0)}{1-\alpha} \end{aligned}\right] \qquad \ldots (3.1)$$

tends to x^* as n tends to ∞, where $F(u) = f(u, u, \ldots, u)$. Then x^* is a global attractor of all positive solutions of (1.1).

PROOF : Assume that x_n is a positive solution of (1.1). Let

$$\lambda = \limsup_{n \to \infty} x_n$$

and

$$\mu = \liminf_{n \to \infty} x_n.$$

Since $f(u_0, u_1, \ldots, u_m)$ is nonincreasing in each of its arguments $u_0, u_1, \ldots, u_m > r_0$, we see that from (1.1)

$$x_{n+1} \leq \alpha x_n + F(r_0).$$

By taking the superior limits on both sides of the above inequality, we obtain that

$$\lambda \leq \frac{F(r_0)}{1-\alpha} = Q_0.$$

In a similar way, we obtain that

$$\mu \geq \frac{F(Q_0)}{1-\alpha}.$$

It is easy to prove by induction that for $n = 1, 2, \ldots,$

$$Q_1 \leq Q_3 \leq \ldots \leq Q_{2n+1} \leq \mu \leq x_n \leq \lambda \leq Q_{2n} \leq \ldots \leq Q_2 \leq Q_0.$$

Since $\{Q_n\}$ tends to x^* as n tends to ∞ from (3.1), x_n also tends to x^* as n tends to ∞. The proof is complete.

Next, we apply Theorem 3.1 to obtain sufficient conditions for global attractivity of (1.2) and (1.3).

Corollary 3.1 — Assume that N is the unique positive equilibrium of (1.2). Suppose that

$$m > 1, \quad \frac{\beta Q_0}{\alpha(1+Q_0^m)} > r_0$$

and

$$-\frac{(m-1)^2}{4m}\,\frac{\alpha+(1-m)(\beta-\alpha)}{\alpha} < 1, \qquad \ldots (3.2)$$

where

$$Q_0 = \frac{\beta r_0}{\alpha(1+r_0^m)} \quad \text{and} \quad r_0 = \sqrt[m]{\frac{1}{m-1}}.$$

Then N is a global attractor of all positive solutions of (1.2).

PROOF : Clearly, if we show that the solution of the IVP

$$\left.\begin{aligned} Q_{n+1} &= \frac{\beta Q_n}{\alpha(1+Q_n^m)} \\ Q_0 &= \frac{\beta r_n}{\alpha(1+r_n^m)} \end{aligned}\right] \qquad \ldots (3.3)$$

tends to the equilibrium N as n tends to ∞, then all the hypothesis of Theorem 3.1 will be satisfied and the proof will be complete.

To this end, set

$$h(u) = \frac{\beta u}{\alpha(1+u^m)}.$$

Observe that

$$h'(u) = \frac{\beta}{\alpha} \frac{1+(1-m)\,u^m}{(1+u^m)^2}$$

and so $h(u)$ is decreasing for $u > Q_1$. We have

$$h'(u)\,h'(h(u)) = \left(\frac{\beta}{\alpha}\right)^2 \frac{1+(1-m)\,u^m}{(1+u^m)^2} \frac{1+(1-m)\,h^m(u)}{(1+h^m(u))^2}.$$

We claim that

$$h'(u)\,h'(h(u) < 1 \quad \text{for} \quad Q_1 < u < N. \qquad \ldots (3.4)$$

Set

$$g(s) = \frac{1+(1-m)\,s^m}{(1+s^m)^2}$$

and obtain that

$$g'(s) = \frac{ms^{m-1}[-1-m-(1-m)s^m]}{(1+s^m)^3}.$$

So we see that

$$g'(s_0) = 0 \quad \text{for} \quad s_0 = \sqrt[m]{\frac{m+1}{m-1}}.$$

Observe that

$$g(s) < 0 \quad \text{for} \quad Q_1 < s < Q_0$$

and $g(s)$ is decreasing for $Q_1 < s < s_0$ and increasing for $s_0 < s < Q_0$. Then it follows that

$$g(s) \geq g(s_0) = -\frac{m(m-1)^2}{4m^2};$$

also as $h(u) > h(N) = N$ and either $N < s_0$ or $N > s_0$ we find

$$g(u)\,g(h(u)) \leq g(N)\,g(s_0) = -\frac{\alpha^2(m-1)^2}{4m\beta^2} \frac{\alpha+(1+m)\,(\beta-\alpha)}{\alpha}. \qquad \ldots (3.5)$$

Thus it follows from (3.2) and (3.5) that (3.4) holds. Hence by Lemma, the solution of (3.3) tends N as n tends to ∞. Therefore by Theorem 3.1, every solution of (1.2) tends to N as n tends to ∞. The proof is complete.

Corollary 3.2 — Assume that N is the unique positive equilibrium of (1.3). Suppose that

$$\frac{\beta^2\gamma^2}{\alpha^2} e^{-\gamma(Q_1+N)} < 1,$$

where

$$Q_1 = \frac{\beta}{\alpha} e^{-\beta\gamma/\alpha}.$$

Then N is a global attractor of all positive solutions of (1.3).

PROOF : It is easy to see that all the hypotheses of Theorem 3.1 will be satisfied provided the solution of the IVP

$$\left.\begin{aligned} Q_{n+1} &= \frac{\beta}{\alpha} e^{-\gamma Q_n} \\ Q_0 &= \frac{\beta}{\gamma} \end{aligned}\right]$$

tends to N as n tends to ∞ whose proof is similar Corollary 3.1. The proof is complete.

In the following corollaries, we obtain sufficient conditions for global attractivity of (1.4)-(1.6) by using Theorem A.

Let N denote the unique positive equilibrium of (1.4). Then

$$N = \frac{b + \sqrt{b^2 + 4ac}}{2c}.$$

By applying Theorem A and Lemma we obtain the following results.

Corollary 3.3 — Assume that N is the unique equilibrium of (1.4), and suppose that

$$Q_1 > r_0, \qquad F(Q_0) > 0$$

and

$$N^2(k+1)^2 \sqrt[2]{(b^2+4ac)(b^2-D)} \left(1 + a + \frac{D}{4c}\right)^k < 1, \qquad \text{... (3.6)}$$

where

$$\left.\begin{aligned} Q_{n+1} &= NF^{k+1}(Q_n) \\ Q_0 &= NF^{k+1}(r_0) \end{aligned}\right] \qquad \text{... (3.7)}$$

$$r_0 = \max\left\{0, \frac{b}{2c}\right\}, \quad F(u) = 1 + a + bu - cu^2$$

and

$$D = \frac{2kb^2 - 4c(1+a)}{2k+1}. \qquad \ldots (3.8)$$

Then N is a global attractor of all positive solutions of (1.4).

PROOF : First, let N_n be a nonoscillatory positive solution of (1.4) about N. We assume that $N_n > N$ for $n \geq n_0$. The case $N_n < N (n > n_0)$ is similar and will be omitted.

Since $N_n > N$ for $n > n_0$, we see that for $n \geq n_0 + k$ and

$$N_{n+1} = N_n (a + bN_{n-k} - cN_{n-k}^2)$$

$$\leq N_n(1 + a + bN - cN^2) = N_n.$$

Hence $\{N_n\}$ is monotonic for $n \geq n_0 + k$. Let

$$l = \lim_{n \to \infty} N_n$$

and for the sake contradiction, assume that $l > N$. Then by taking limits on both sides of (1.4), we obtain that

$$a + bl - cl^2 = 1$$

which contradicts the hypothesis that N is the unique positive solution of (1.4).

Next, let N_n be oscillation solution. To this end, let

$$\{N_{p_i}, N_{p_i+2}, \ldots, N_{q_i}\}$$

be the i^{th} positive semicycle of $\{N_n\}$ follows by the i^{th} negative semicycle

$$\{N_{q_i+1}, N_{q_i+2}, \ldots, N_{p_i+1}\}.$$

Let N_{M_i} and N_{m_i} be the extreme values in these two semicycle, respectively with the smallest possible indices M_i and m_i. Then we claim that

$$M_i - p_i \leq k_m + 1 \quad \text{and} \quad m_i - q_i \leq k_m + 1. \qquad \ldots (3.9)$$

We will prove (3.9) for positive semicycles. The proof for negative semicycles is similar and will be omitted. Assume for the sake of contradiction that the first inequality in (3.9) is not true. Then $M_i - p_i > k + 1$ and the terms $N_{M_i-1}, \ldots, N_{M_i-k-1}$ are in a positive semicycle, from (1.4) we obtain

$$N_{M_i} = N_{M_i-1}(1 + a + bN_{M_i-1} - cN_{M_i-1}^2)$$

$$\leq N_{M_i-1}(1+a+bN-cN^2)=N_{M_i-1}$$

which is a contradiction.

Let

$$\lambda=\limsup_{n\to\infty} N_n=\limsup_{i\to\infty} M_i$$

$$\mu=\liminf_{n\to\infty} N_n=\liminf_{i\to\infty} m_i .$$

By multiplying both sides of (1.4) from $n = p_i$ to $n = M_i$, for i sufficiently large, we obtain that

$$N_{M_i}=N_{p_i}\prod_{j=p_i}^{M_i-1}(1+a+bN_{j-k}-cN_{j-k}^2)$$

Since $N_{p_i}\leq N$ and $F(u)$ is nonincreasing in $u > r_0$, we obtain that

$$N_{M_i}\leq N(1+a+br_0-cr_0^2)^{k+1}=NF^{k+1}(r_0)=Q_0.$$

That is

$$N_n\leq Q_0.$$

By multiplying both sides of (1.4) from $n = q_i$ to $n = m_i$, for i sufficiently large, we obtain

$$N_{m_i}=N_{q_i}\prod_{j=q_i}^{m_i-1}(1+a+bN_{j-k}-cN_{j-k}^2)$$

Since $N_{q_i}\geq N$ and $F(u)$ is nonincreasing in $u > r_0$, we obtain that

$$N_{m_i}\geq NF^{k+1}(Q_0)=Q_1.$$

That is,

$$Q_1\leq N_n.$$

It is easy to prove by induction that $n = 1, 2, \ldots$

$$Q_1\leq Q_3\leq\ldots\leq Q_{2n+1}\leq\mu\leq N_n\leq\lambda\leq Q_{2n}\leq\ldots\leq Q_2\leq Q_0.$$

Clearly, if we show that the solution of the IVP (3.7) tends to the equilibrium N as n tends to ∞, then all the hypotheses of Theorem A will be satisfied and the proof will be complete.

To this end, set

$$h(u) = N(1 + a + bu - cu^2)^{k+1} \quad \text{for } u > Q_1.$$

Observe that

$$h'(u) = N(k+1)(1 + a + bu - cu^2)^k (b - 2cu)$$

and so $h(u)$ is decreasing for $u > Q_1$. We have

$$h'(u)\, h'(h(u)) = N^2(k+1)^2 (1 + a + bu - cu^2)^k (b - 2cu)$$

$$(1 + a + bh(u) - ch^2(u))^k (b - 2ch(u)).$$

We claim that

$$h'(u)\, h'(h(u)) < 1 \quad \text{for} \quad Q_1 < u < N. \qquad \ldots (3.10)$$

Set

$$g(s) = (1 + a + bs - cs^2)^k (b - 2cs) \quad \text{for} \quad s > Q_1$$

and obtain that for $s > Q_1$

$$g'(s) = 2c(1 + a + bs - cs^2)^{k-1} \left[c(2k+1)s^2 - b(2k+1)s + \frac{kb^2}{2c} - (1 + a) \right].$$

Then $g'(s_0) = 0$ for $s_0 = \dfrac{b + \sqrt{b^2 - D}}{2c}$, where D is defined in (3.8).

Furthermore, we observe that $g(s)$ is decreasing for $Q_1 < s < S_0$ and increasing for $s_0 < s < Q_0$. Then it follows that for $Q_1 < s < Q_0$,

$$g(s) \geq g(s_0) = -\sqrt[2]{b^2 - D} \left(1 + a + \frac{D}{4c} \right)^k$$

also $h(u) > g(N) = N$ and either $N < s_0$ or $N > s_0$ we find

$$g(u)\, g(h(u)) \leq \sqrt{(b^2 + 4ac)(b^2 - D)} \left(1 + a + \frac{D}{4c} \right)^k. \qquad \ldots (3.11)$$

Thus it follows from (3.6) and (3.11) that (3.10) holds. Hence by Lemma, the solution of (3.7) tends N as n tends to ∞. Therefore by Theorem A, every solution of (1.4) tends to N as n tends to ∞. The proof is complete.

Corollary 3.4 — Assume that N is the unique equilibrium of (1.5).

Assume that

$$m = \max\{k, l\}, \quad b < 0, \quad F(Q_0) > 0$$

and

$$N^2(m+1)^2 \sqrt[2]{(b^2+4ac)(b^2-D)}\left(1+a+\frac{D}{4c}\right)^m < 1,$$

where

$$Q_0 = N(1+a)^{m+1}, \quad F(u) = 1+a+bu-cu^2$$

and

$$D = \frac{2mb^2 - 4c(1+a)}{2m+1}.$$

Then N is a global attractor of all positive solutions of (1.5).

The proof is a similar Theorem 3 and omitted.

Corollary 3.5 — Assume that x^* is the unique positive equilibrium of (1.6). Suppose that

$$A < 2 \quad \text{and} \quad (m+1)^2(A-1)^2\left(\frac{2}{A}-1\right)^m < 1.$$

Then x^* is a global attractor of all positive solutions of (1.6).

PROOF : Set

$$f(u_0, u_1, \ldots, u_m) = A\left(1 - \sum_{i=0}^{m} B_i u_i\right).$$

It is easy to see that all the hypotheses of Theorem 1 will be satisfied provided the solution of the IVP

$$\left.\begin{aligned} Q_{n+1} &= Ax^*(1-BQ_n)^{m+1} \\ Q_0 &= Ax^* \end{aligned}\right]$$

tends to zero as n tends to ∞ whose proof is similar Corollary 3.3, where $B = \sum_{i=0}^{m} B_i$. The proof is complete.

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