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# COMMON SOLUTIONS OF VECTOR VARIATIONAL INEQUALITIES IN ORDERED BANACH SPACES*

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The concept of $\tilde{C}$-monotone pair of mappings in ordered Banach spaces is introduced and the existence of solutions for a system of vector variational inequalities involving $\tilde{C}$-monotone pair of mappings is proved.

**Key Words : $\tilde{C}$-Monotone Mapping; Vector Variational Inequality; KKM Mapping; Ordered Banach Space**

## 1. INTRODUCTION

It is well known that the vector variational inequality is an important tool in the vector optimization[2,3,5 &7]. Giannessi[5] introduced the vector variational inequality in finite dimensional spaces and showed its applications. Chen and Yang[2,3 & 7] proved the existence of solutions for vector variational inequalities in infinite dimensional spaces, and discussed the relationship between the vector variational inequality and the vector complementarity problem.

In this paper, we introduce the concept of $\tilde{C}$-monotone pair of mappings in ordered Banach spaces, and prove the existence of solutions for a system of vector variational inequalities involving $\tilde{C}$-monotone pair of mappings. Our theorems unify and generalize the main results of [7 and 8].

## 2. DEFINITIONS AND PRELIMINARY RESULTS

Throughout this paper, $X$ and $Y$ are assumed to be real Banach spaces. Let $D$ be a nonempty, closed and convex subset of $X$. We denote by $\{C(x) : x \in D\}$ a family of closed, pointed and convex cones of $Y$ with int $C(x) \neq \phi$, $\forall x \in D$, where int$C(x)$ denotes the interior of $C(x)$.

Let $P$ be a closed and convex cone of $Y$. An ordering in $Y$ with respect to $P$ is defined as follows : $\forall x, y \in Y$, $x \leq_P y \Leftrightarrow y - x \in P$; if int $P \neq \phi$, then a weak ordering in $Y$ with respect to $P$ is defined by $x \lneq y \Leftrightarrow y - x \notin$ int $P$.

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*Definition* 1 — A mapping $f: D \to Y$ is called $P$-convex if for any $x, y \in D$, $0 < t < 1$, $tf(x) + (1 - t) f(y) \in f(tx + (1-t)y) + P$; or equivalently, $f(tx + (1-t)y) \leq_p tf(x) + (1-t)f(y)$. Term $f$ is called $P$-concave if $-f$ is $P$-convex.

*Definition* 2 — Let $\varphi, \psi: D \times D \to Y$ and $\tilde{C} := \bigcap_{x \in D} C(x) \neq \phi$. Term $\varphi$ and $\psi$ is said to be a $\tilde{C}$-monotone pair, if $\forall x, y \in D$, $\varphi(x, y) + \psi(y, x) \in \tilde{C}$ and $\varphi(x, x) = \psi(x, x) = 0$; if $\psi = \varphi$, then $\varphi$ is said to be $\tilde{C}$-monotone.

Let $L(X, Y)$ be the space of all linear continuous operators from $X$ into $Y$. Denote by $\langle S, x \rangle$ the value of $S \in L(X, Y)$ at $x \in X$.

*Definition* 3 — Let $T: D \to L(X, Y)$. $T$ is called $\tilde{C}$-monotone if $\langle T(x) - T(y), x - y \rangle \in \tilde{C}$, $\forall x, y \in D$.

Let $\varphi: D \times D \to Y$ be given. The General Vector Variational Inequality (in short, GVVI) consists in finding $\bar{x} \in D$ such that

(GVVI) $\varphi(\bar{x}, y) \notin \text{int } C(\bar{x})$, $\forall y \in D$.

Let $T: D \to L(X, Y)$ be given. Choose $\varphi(x, y) = \langle T(x), x - y \rangle$, $\forall x, y \in D$. Then we obtain the vector variational inequality :

(VVI) find $\bar{x} \in D$ such that $\langle T(\bar{x}), \bar{x} - y \rangle \notin \text{int } C(\bar{x})$, $\forall y \in D$, which is considered in Chen[2].

If $Y = \mathbb{R}$, $X = \mathbb{R}^n$ and for each $x \in D$, $C(x) := \mathbb{R}_+$, and $f: D \to \mathbb{R}^n$ is a given function, then the above problem collapses to the classic scalar variational inequality :

(VI) find $\bar{x} \in D$ such that $\langle f(\bar{x}), \bar{x} - y \rangle \leq 0$, $\forall y \in D$.

*Definition* 4 — Let $F: D \to 2^Y$ be a set-valued mapping. $F$ is called upper semicontinuous at $x \in D$ if $\{x_n\} \subset D, x_n \to x$ (in norm), $y_n \in F(x_n)$ and $y_n \to y$ (in norm) imply $y \in F(x)$.

*Definition* 5 — A mapping $\varphi: D \times D \to Y$ is called hemicontinuous if $\forall x, y \in D$, the mapping $t \to \varphi(x + t(y - x), y)$, $0 < t < 1$, is continuous at 0.

*Definition* 6 — Let $G$ be an arbitrary subset of a vector space $E$. A set-valued mapping $F: G \to 2^E$ is said to be a KKM mapping if $\text{co}(x_1, \ldots, x_n) \subset \bigcup_{i=1}^{n} F(x_i)$ for every finite subset $\{x_1, \ldots, x_n\}$ of $G$, where $\text{co}(B)$ denotes the convex hull of a set $B$.

The following is the well-known Ky Fan-Knaster-Kuratowski-Mazurkiewicz (in short, FKKM) theorem, which is our important tool.

***Theorem*** 4 — *Let $G$ be an arbitrary subset of a topological vector space $E$, and $F: G \to 2^E$ a KKM mapping. If all the sets $F(x)$ are closed in $E$ and if at least one is compact, then* $\bigcap_{x \in G} F(x) \neq \phi$.

## 3. MAIN RESULTS

*Lemma*[2] — Let $(V, P)$ be an ordered topological vector space with a closed, pointed and convex cone $P$, $\text{int}P \neq \phi$. Then, $\forall y, z \in V$, we have

(*i*) $y - z \in \text{int } P$ and $y \notin \text{int } P$ imply $z \notin \text{int } P$;

(*ii*) $y - z \in P$ and $y \notin \text{int } P$ imply $z \notin \text{int } P$;

(*iii*) $y - z \in -\text{int } P$ and $y \notin -\text{int } P$ imply $z \notin -\text{int } P$; and

(*iv*) $y - z \in -P$ and $y \notin -int\ P$ imply $z \notin -\text{int } P$.

Let $\varphi, \psi : D \times D \to Y$ be given. For each fixed $y \in D$, let

$$F_1(y) := \{x \in D : \varphi(x, y) \notin \text{int } C(x)\};$$

$$F_2(y) := \{x \in D : \psi(y, x) \notin -\text{int } C(x)\};$$

$$F_3(y) := \{x \in D : \psi(x, y) \notin \text{int } C(x)\};$$

$$F_4(y) := \{x \in D : \varphi(y, x) \notin -\text{int } C(x)\},$$

and $$F_i := \bigcap_{y \in D} F_i(y),\ i = 1, 2, 3 \text{ and } 4.$$

First, we extend the linearization lemma on monotone operators to the vector case.

***Theorem*** 1 — *Let $\varphi, \psi : D \times D \to Y$ be a $\tilde{C}$-monotone pair of hemicontinuous mappings. If for each fixed $x \in D$, $\varphi(x, y)$ and $\psi(x, y)$ are $\tilde{C}$-concave in $y$. Then $F_1 = F_2 = F_3 = F_4$.*

PROOF : If $\bar{x} \in F_1$, then $\forall y \in D$, $\varphi(\bar{x}, y) \notin \text{int } C(\bar{x})$. Since $(\varphi, \psi)$ are $\tilde{C}$-monotone, we have $\varphi(\bar{x}, y) + \psi(y, \bar{x}) \in \tilde{C} \subset C(\bar{x})$, $\forall y \in D$. For the ordered Banach space $(Y, C(\bar{x}))$, by Lemma $(ii)$, $-\psi(y, \bar{x}) \notin \text{int } C(\bar{x})$, $\forall y \in D$. Hence, $\bar{x} \in F_2$. Thus we have $F_1 \subset F_2$. Similarly, $F_3 \subset F_4$. Let $\bar{x} \in F_4$. Then we have $\varphi(y, \bar{x}) \notin -\text{int } C(\bar{x})$, $\forall y \in D$. For any $y \in D$, let $x_t := ty + (1 - t)\bar{x} = \bar{x} + t(y - \bar{x}) \in D$, $0 < t < 1$. Since $\varphi(x, y)$ is $\tilde{C}$-concave in $y$, we have

$$0 = \varphi(x_t, x_t) \geq_{\tilde{C}} t\varphi(x_t, y) + (1 - t)\,\varphi(x_t, \bar{x}).$$

Hence, $-[t\varphi(x_t, y) + (1 - t)\varphi(x_t, \bar{x})] \in \tilde{C} \subset C(\bar{x})$. By Lemma $(ii)$, it follows from $-\varphi(x_t, \bar{x}) \notin \text{int } C(\bar{x})$ that $\varphi(x_t, y) \notin \text{int } C(\bar{x})$, i.e., $\varphi(x_t, y) \in Y \setminus int\ C(\bar{x})$. Since $\varphi$ is hemicontinuous and $Y \setminus int\ C(\bar{x})$ is closed, letting $t \to 0$, we have $\varphi(\bar{x}, y) \in Y \setminus int\ C(\bar{x})$. Hence, $\bar{x} \in F_1$. Thus $F_4 \subset F_1$. Similarly, $F_2 \subset F_3$. This completes the proof.

***Theorem*** 2 — *Let $\varphi$ and $\psi$ be as in Theorem 1. In addition, for fixed $x \in D$, let $\varphi(x, y)$ and $\psi(x, y)$ be continuous in $y$. Let the set-valued mapping $x \to Y \setminus \text{int } C(x)$ be upper semicontinuous. Suppose that one of the following conditions holds :*

*(A) $D$ is compact;*

*and (B) there exist a compact subset $K$ of $X$ and $y_0 \in D \cap K$ such that*

$$-\psi(y_0, x) \in int\ C(x),\quad \forall x \in D \setminus K.$$

*Then there exists an* $\bar{x} \in D$ *such that for all* $y \in D$

$$(\Delta)\qquad \begin{cases} \varphi(\bar{x}, y) \notin \text{int } C(\bar{x}), & \psi(\bar{x}, y) \notin \text{int } C(\bar{x}); \\ \psi(y, \bar{x}) \notin -\text{int } C(\bar{x}), & \varphi(y, \bar{x}) \notin -\text{int } C(\bar{x}). \end{cases}$$

PROOF : For any $y \in D$, let $F_1(y) = \{x \in D:\ \varphi(x, y) \notin \text{int } C(x)\}$. We claim that $F_1$ is a KKM mapping. If the assertion were false, then there would exist $\{z_1, z_2, \ldots, z_n\} \subset D$ and $t_i \geq 0$, $\sum_{i=1}^{n} t_i = 1$, and $z = \sum_{i=1}^{n} t_i z_i \notin \bigcup_{i=1}^{n} F_1(z_i)$. Hence, for all $i$, $z \notin F_1(z_i)$, i.e., $\varphi(z, z_i) \in$ int $C(z)$. Thus, $\sum_{i=1}^{n} t_i \varphi(z, z_i) \in$ int $C(z)$. ... (1)

Since $\varphi(x, y)$ is $\tilde{C}$-concave in $y$, we have

$$0 = \varphi(z, z) \geq_{\tilde{C}} \sum_{i=1}^{n} t_i \varphi(z, z_i),$$

i.e., $0 - \sum_{i=1}^{n} t_i \varphi(z, z_i) \in \tilde{C} \subset C(z)$. By Lemma (*ii*), it follows from $0 \notin$ int $C(z)$ that $\sum_{i=1}^{n} t_i \varphi(z, z_i) \notin$ int $C(z)$. This contradicts (1). So that $F_1$ is a KKM mapping.

Let $F_2(y) := \{x \in D:\ \psi(y, x) \notin -\text{int } C(x)\}$ for each $y \in D$.

From the proof of Theorem 1, we have $F_1(y) \subset F_2(y)$, $\forall y \in D$. Hence, $F_2$ is also a KKM mapping. Next, we show that for any fixed $y \in D$, $F_2(y)$ is closed in $X$. Indeed, let $\{x_n\} \subset F(y)$, $x_n \to x$ (in norm). and $-\psi(y, x_n) \in Y \setminus int\ C(x_n)$. Since $\psi(y, \cdot)$ is continuous and the set-valued mapping $x \to Y \setminus \text{int } C(x)$ is upper semicontinuous, we have $-\psi(y, x) \in Y \setminus int\ C(x)$, i.e., $\psi(y, x) \notin -int\ C(x)$. Thus $x \in F_2(y)$. So that $F_2(y)$ is closed.

For the case (*A*), $D$ is compact in $X$ $\forall y \in D$, $F_2(y)$ is a compact subset of $D$. By the FKKM Theorem, $F_2 = \bigcap_{y \in D} F_2(y) \neq \phi$. Let $\bar{x} \in F_2$. By Theorem 1, $\bar{x} \in F_2 = F_1 = F_3 = F_4$, i.e., $\bar{x}$ is a solution to the system ($\Delta$) of vector variational inequalities.

For the case (*B*), we have $F_2(y_0) \subset K$. Then $F_2(y_0)$ is a compact subset of $K$. The FKKM Theorem and Theorem 1 yield the conclusion. This completes the proof.

*Corollary* 1 — Let $T : D \to L(X, Y)$ be hemi-continuous and $\tilde{C}$-monotone, and the mapping $x \to Y \setminus int\ C(x)$ upper semi-continuous. Suppose that one of the following conditions holds :

(*A*) $D$ is compact;

(*B*) There exist a compact subset $K$ of $X$ and $y_0 \in D \cap K$ such that

$$\langle T(y_0), x - y_0 \rangle \in \text{int } C(x), \quad \forall x \in D \setminus K.$$

Then the VVI has solutions in D.

PROOF : For any $x, y \in D$, let $\psi(x, y) = \varphi(x, y) = \langle T(x), x - y \rangle$. For each fixed $x \in D$, since $T(x)$ is a linear continuous operator, then $\varphi(x, y)$ is continuous in $y$. All the assumptions in Theorem 2 hold. Theorem 2 yields the conclusion.

***Theorem*** 3 — *Let* $\varphi : D \times D \to Y$ *satisfy*

*(i) for any* $x \in D$, $\varphi(x, x) = 0$;

*(ii) for any fixed* $y \in D$, $\{x \in D : \varphi(x, y) \notin \text{int } C(x)\}$ *is closed; and*

*(iii) for any fixed* $x \in D$, $\varphi(x, y)$ *is* $C(x)$*-concave in* $y$.

*Supose that one of the following conditions hold :*

*(A) D is compact ;*

*(B) there exist a compact subset* $K$ *of* $X$ *and* $y_0 \in D \cap K$ *such that* $\varphi(x, y_0) \in$ *-int* $C(x)$, $\forall x \in D \setminus K$;

*(C) D is locally compact, and there is an* $r > 0$ *and* $x_0 \in D, \| x_0 \| < r$, *such that for all* $y \in D, \| y \| = r$, $\varphi(y, x_0) \in C(y)$.

*Then there exists an* $\bar{x} \in D$ *such that* $\varphi(\bar{x}, y) \notin$ *int* $C(\bar{x})$, $\forall y \in D$.

PROOF : For each $y \in D$, let $F(y) := \{x \in D : \varphi(x, y) \notin -\text{ int } C(x)\}$. By the same argument as in Theorem 2, we can show that $F$ is a KKM mapping. It follows from the condition (ii) that $F(y)$ is closed.

For cases (*A*) and (*B*), we proceed as in the proof of Theorem 2 to obtain the conclusion.

For the case (*C*), let $D_r = \{x \in D : \| x \| \le r\}$. Since $D$ is locally compact, $D_r$ is compact. It follows from the above case (*A*) of this Theorem that there exists an $\bar{x} \in D_r$ such that $\varphi(\bar{x}, y) \notin$ int $C(\bar{x})$, $\forall y \in D_r$. Then $\bar{x}$ is the desired solution of the GVVI.

Indeed, (*i*) If $\| \bar{x} \| = r$, by the condition (C), we have

$$\varphi(\bar{x}, x_0) \in C(\bar{x}). \qquad \ldots (2)$$

For any $y \in D$, choose $t \in (0, 1)$ such that $z = ty + (1 - t)x_0 \in D_r$.

Then $\varphi(\bar{x}, z) \notin$ int $C(\bar{x})$. Since $\varphi(\bar{x}, y)$ is $C(\bar{x})$-concave in $y$ and (2), we have $\varphi(\bar{x}, z) \ge_{c(\bar{x})} t\varphi(\bar{x}, y) + (1 - t)\, \varphi(\bar{x}, x_0) \ge_{c(\bar{x})} t\varphi(\bar{x}, y)$, i.e. $\varphi(\bar{x}, z) - t\varphi(\bar{x}, y) \in C(\bar{x})$. By Lemma (*ii*) and $\varphi(\bar{x}, z) \notin$ int $C(\bar{x})$, we have $\varphi(\bar{x}, y) \notin$ int $C(\bar{x})$, $\forall y \in D$ .

(*ii*) If $\| \bar{x} \| < r$, for any $y \in D$, choose $t \in (0, 1)$ such that $z = ty + (1 - t)\bar{x} \in D_r$. Then $\varphi(\bar{x}, z) \notin$ *int* $C(\bar{x})$. Since $\varphi(\bar{x}, y)$ is $C(\bar{x})$-concave in $y$, we have $\varphi(\bar{x}, z) \ge_{c(\bar{x})} t\varphi(\bar{x}, y) + (1 - t)\, \varphi(\bar{x}, \bar{x}) = t\varphi(\bar{x}, y)$, i.e., $\varphi(\bar{x}, z) - t\varphi(\bar{x}, y) \in C(\bar{x})$. By Lemma (ii) and $\varphi(\bar{x}, z) \notin$ *int* $C(\bar{x})$, we have $\varphi(\bar{x}, y) \notin$ *int* $C(\bar{x})$, $\forall y \in D$. The proof is completed.

*Corollary* 2 — Let $C$ be a closed, pointed and convex cone of $Y$ with int $C \neq \phi$. Let $\varphi : D \times D \to Y$ satisfy

(*i*) for any $x \in D$, $\varphi(x, x) = 0$;

(*ii*) for any fixed $y \in D$ $\{x \in D : \varphi(x, y) \notin \text{int } C\}$ is closed; and

(*iii*) for any fixed $x \in D$, $\varphi(x, y)$ is $C$-concave in $y$.

Suppose that one of the following conditions holds :

(*A*) $D$ is compact;

(*B*) there exist a compact subset $K$ of $X$ and $y_0 \in D \cap K$ such that

$$\varphi(x, y_0) \in \text{int } C, \quad \forall x \in D \setminus K ;$$

(*C*) $D$ is locally compact, and there is an $r > 0$ and $x_0 \in D$, $\| x_0 \| < r$, such that for all

$$y \in D, \ \| y \| = r, \ \varphi(y, x_0) \leq 0.$$

Then there exists an $\bar{x} \in D$ such that $\varphi(\bar{x}, y) \not> 0$, $\forall y \in D$.

PROOF : In Theorem 3, let $C(X) := C$ for each $x \in D$ and define a weak ordering $\lhd$ on $Y$ by the cone $C$. Then, Theorem 3 yields the conclusion.

*Remark* 1 : Let $T : D \to L(X, Y)$ be a given continuous mapping.

Define $\varphi(x, y) := \langle T(x), y - x \rangle$, $\forall x, y \in D$. Since $T$ is continuous and $Y \setminus \text{int } C$ is closed, it is easy to check that for any fixed $y \in D$, the set $\{x \in D : \langle T(x), y - x \rangle \not> 0\}$ is closed in $X$. From the above Corollary 2, we derive Theorem 2.2 of [3] and Theorem 1 of [7].

*Remark* 2 : Let $Y = \mathbb{R}$, $C = \mathbb{R}_+$ and $D$ be a nonempty closed, convex and bounded subset of $X = \mathbb{R}^n$. Let $f : D \to X$ be given. Define $\varphi(x, y) := \langle f(x), y - x \rangle$. Then, we derive Theorem 3.1 and 3.6 of [8] from the above Corollary 2.

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