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BOUNDEDNESS AND PERIODICITY RESULTS FOR A CERTAIN SYSTEM OF THIRD ORDER NONLINEAR DIFFERENTIAL EQUATIONS

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This paper studies the problems of the boundedness and periodicity of the third order system of nonlinear differential equations

$$\dddot{X} + A\ddot{X} + G(\dot{X}) + H(X) = P(t, X, \dot{X}, \ddot{X}), \qquad \ldots (*)$$

where A is a real $n \times n$ constant symmetric matrix and $X \in R_n$. We obtain some sufficient conditions which ensure that all the solutions of eq. (*) are ultimately bounded and we also give some sufficient conditions which guarantee that there exists at least one periodic solution of eq (*). Our results improve and include those results obtained by previous authors.

Key Words : Boundedness; Periodicity; Third Order Nonlinear Differential Equations; Euclidean *n*-Dimensional space

1. INTRODUCTION

Let R_n denote the real Euclidean n-dimensional space equipped with the usual Euclidean norm which will be represented throughout by $\|\cdot\|$. Let R^+ denote the real interval $0 \le t < \infty$. We shall consider here systems of real differential equations of the form

$$\dddot{X} + A\ddot{X} + G(\dot{X}) + H(X) = P(t, X, \dot{X}, \ddot{X}), \qquad \ldots (1.1)$$

where $X \in R_n$, A is a constant $n \times n$ matrix, and $G : R_n \to R_n$, $H : R_n \to R_n$, $P : R^+ \times R_n \times R_n \to R_n$ are such that, for any $X_0, Y_0, Z_0 \in R_n$, there is a uniquely defined solution $X = X(t, X_0, Y_0, Z_0)$ of (1.1), continuous in t, X_0, Y_0, Z_0 satisfying the condition $X(t_0) = X_0$, $\dot{X}(t_0) = Y_0$, $\ddot{X}(t_0) = Z_0$. The dots here, as elsewhere, stand for differentiation with respect to the independent variable t. We shall assume as basic throughout what follows that G and H are both of class C^1 in R_n.

Equations of the form (1.1) and the special case $G(\dot{X}) = B\dot{X}$ have been studied by various authors[1, 3-5 & 8]. They have obtained some results related to the boundedness, periodicity and stability properties of solutions. In the case $n = 1$, (1.1) and various other third order differential equations have received a considerable amount of attention during the past forty years. See, for example [6],

[7], [11] and 12]. Many of these results are summarised in [10]. Motivation for the study of (1.1) comes to me from the work of Ezeilo[3]. To prove boundedness for the solutions of (1.1) Ezeilo assumed that

$$\| P(t, X, Y, Z) \| \leq \theta_1 (t) + \theta_2 (t) (\| Y \|^2 + \| Z \|^2)^{1/2}$$

for all t. In our case we shall assume that $P(t, X, Y, Z)$ satisfies

$$\| P(t, X, Y, Z) \| \leq \theta_1 (t) + \theta_2 (t) (\| X \|^2 + \| Y^2 \| + \| Z \|^2)^{\alpha/2}$$

$$+ \delta_0 (\| X \|^2 + \| Y \|^2 + \| Z \|^2)^{1/2} \qquad \ldots (1.2)$$

for all t. This was left open in [3] except for the special case $G(\dot{X}) = B\dot{X}$ treated in [4].

Furthermore the boundedness result considered in [3] is of the type in which the bounding constant depends on the solution in question. But we shall examine also here the ultimate boundedness result and the existence of periodic solutions of (1.1).

Afuwape[1] also studied the ultimate boundedness of the solution for the special case $G(\dot{X}) = B\dot{X}$ of (1.1). Later Meng[8] improved those results obtained by Afuwape. Recently, Chunhua[2] dealt with periodicity of solutions for third order systems with periodic coefficients. One of the differences between our results and those of the above mentioned authors is that we have weakened the condition on P at the expense of strengthening the condition on H.

2. Statement of the Results

Throughout the sequel $\boldsymbol{J_h(X), J_g(Y)}$ are the Jacobian matrices $\frac{\partial h_i}{\partial x_j}, \frac{\partial g_i}{\partial y_j}$ corresponding to the vector functions $H(X)$, $G(Y)$ respectively. One of the main results of this paper is the followng theorem which gives an estimate for solutions of (1.1) :

Theorem 1 — *Assume, given the equation (1.1), that G(0) = 0 = H(0) and that the matrix* $\boldsymbol{A}$ *as well as the Jacobian matrices* $\boldsymbol{J_g(Y), J_h(X)}$ *are all symmetric. Suppose also that*

i) there exist a symmetric matrix $\boldsymbol{B}$ *and constants* $\delta_2 > 0, \delta_3 > 0$ *such that* $\boldsymbol{J_g(Y) - B}$ *is positive semidefinite and*

$$\delta_3 \leq \lambda_i (J_h(X)) \leq \delta_2 \text{ for all } X, \qquad \ldots (2.1)$$

$$\delta_1 \leq \lambda_i (J_g(Y)) \leq \delta_b + 4\delta_3 \delta_b^{-1} \text{ for all } Y, \qquad \ldots (2.2)$$

where $\delta_1 \delta_a > \delta_2$ *and* $\delta_a > 0, \delta_b > 0$ *are the least eigenvalues of* $\boldsymbol{A}$ *and* $\boldsymbol{B}$, *respectively.*

ii) The matrices, $\boldsymbol{A, B, J_g(Y)}$ *and* $\boldsymbol{J_h(X)}$ *are pairwise commutative for arbitrary X, Y in* R_n *and furthermore for each* X_1, X_2 *in* R_n, $\boldsymbol{J_h(X)}$ *satisfies*

$$\boldsymbol{J_h(X_1) (J_n(X_2) = J_h(X_2) J_n(X_1)}.$$

iii) *The vector* $\boldsymbol{P}$ *satisfies (1.2) for all* $t \geq 0$, *uniformly in X, Y and Z where* $\alpha (0 \leq \alpha < 1)$ *and* $\delta_0 \geq 0$ *are constants and the functions* $\theta_1(t)$, $\theta_2(t)$ *are continuous in t.*

Then there are constants $\Delta_0, \Delta_1, \Delta_2, \Delta_3$ such that if $\delta_0 \leq \Delta_0$, then for every solution $X(t)$ of (1.1) with

$$X(t_0) = X_0, \dot{X}(t_0) = Y_0, \ddot{X}(t_0) = Z_0 \qquad \ldots (2.3)$$

and for any constant k whatever in the range $\frac{1}{2} \leq k \leq 1$, the inequality

$$\left\{ \| X(t) \|^2 + \| \dot{X}(t) \|^2 + \| \ddot{X}(t) \|^2 \right\}^k \leq \Delta_1 \left(\Delta_3 e^{-\Delta_2 (t - t_0)} + \psi(t) \right) \qquad \ldots (2.4)$$

holds for all $t \geq t_0$ where $\Delta_3 = \Delta_3 (X_0, Y_0, Z_0)$ and

$$\psi(t) = \int_{t_0}^{t} e^{-\Delta_2 (t - \tau)} \left\{ \theta_1^{2k} (\tau) + \theta_2^{2k(1-\alpha)^{-1}} (\tau) \right\} d\tau.$$

When specialised to the case $n = 1$ with $\boldsymbol{P}$ depending only on t the above estimate (2.4) reduces to the estimate in [6].

In the case $\boldsymbol{P} = 0$ we obtain the following stability result :

Corollary 1 — If $\boldsymbol{P} \equiv 0$ and if all the conditions of Theorem 1 hold, then every solution $X(t)$ of (1.1) satisfies

$$\| X(t) \|^2 + \| \dot{X}(t) \|^2 + \| \ddot{X}(t) \|^2 \to 0 \qquad \ldots (2.5)$$

as $t \to \infty$ provided that $\delta_0 \leq \Delta_0$.

Indeed by setting $\theta_1(t) = 0 = \theta_2(t)$ in (2.4) we have that if $\delta_2 \leq \Delta_0$ then

$$\left\{ \| X(t) \|^2 + \| \dot{X}(t) \|^2 + \| \ddot{X}(t) \|^2 \right\}^k \leq \Delta_1 \left(\Delta_3 e^{-\Delta_2 (t - t_0)}, t \geq t_0 \right.$$

from which (2.5) follows on letting $t \to \infty$.

Another interesting result which can be deduced from Theorem 1 is the following boundedness result.

Theorem 2 — *Assume that the conditions of Theorem 1 hold and let* $\delta_0 \leq \Delta_0$. *Suppose further that each of the functions* $\theta_1(t)$, $\theta_2(t)$ *satisfy*

$$\max_{0 \leq t < \infty} \theta_1(t) < \infty, \quad \max_{0 \leq t < \infty} \theta_2(t) < \infty. \qquad \ldots (2.6)$$

Then there exists a constant Δ_4 such that every solution $X(t)$ of (1.1) ultimately satisfies

$$\| X(t) \|^2 + \| \dot{X}(t) \|^2 + \| \ddot{X}(t) \|^2 \leq \Delta_4. \qquad \ldots (2.7)$$

Finally from the Theorem 2 we have a result about the existence of peirodic solutions of eq. (1.1) which can be proved in the same way as the proof of Theorem 3 given in [8].

Theorem 3 — *In the differential equation (1.1), let* $\boldsymbol{P}$ *satisfy*

$$\boldsymbol{P}(t+w, X, Y, Z) = \boldsymbol{P}(t, X, Y, Z)$$

uniformly for all X, Y, Z in R_n. *Suppose also that all the conditions of Theorem 2 are satisfied. Then there exists a periodic solution X(t) of (1.1) with a period w.*

3. SOME PRELIMINARY RESULTS

We dispose of some algebraic results which will be required in the proofs. The first of these is quite a standard one :

Lemma 1^9 — Let $\boldsymbol{D}$ be a real symmetric $n \times n$ matrix. Then for any $x \in R_n$

$$\lambda_0 \| X \|^2 \leq \langle \boldsymbol{D}X, X \rangle \leq \wedge_0 \| X \|^2 \qquad \ldots (3.1)$$

where λ_0, $\wedge_0$ are respectively the least and greatest eigenvalues of D.

We also require

Lemma 2^1 — Let $\boldsymbol{Q}$, $\boldsymbol{D}$ be any two real $n \times n$ commuting symmetric matrices. Then

i) The eigenvalues $\lambda_i(\boldsymbol{QD})$ (i = 1, 2, ..., n) of the product matrix $\boldsymbol{QD}$ are all real and satisfy

$$\max_{1 \leq j, k \leq n} \lambda_j(\boldsymbol{Q})\lambda_k(\boldsymbol{D}) \geq \lambda_i(\boldsymbol{QD}) \geq \min_{1 \geq j, k \leq n} \lambda_j(\boldsymbol{Q})\lambda_k(\boldsymbol{D}). \qquad \ldots (3.2)$$

ii) The eigenvalues $\lambda_i(\boldsymbol{Q}+\boldsymbol{D})$ (i = 1, 2, ..., n) of the sum of matrices $\boldsymbol{Q}$ and $\boldsymbol{D}$ are real and satisfy

$$\left\{ \max_{1 \leq j \leq n} \lambda_j(\boldsymbol{Q}) + \max_{1 \leq k \leq n} \lambda_k(\boldsymbol{D}) \right\} \geq \lambda_i(\boldsymbol{Q}+\boldsymbol{D}) \geq \left\{ \min_{1 \leq j \leq n} \lambda_j(\boldsymbol{Q}) + \min_{1 \leq k \leq n} \lambda_k(\boldsymbol{D}) \right\} \qquad \ldots (3.3)$$

where $\lambda_i(\boldsymbol{Q})$ and $\lambda_k(\boldsymbol{D})$ are respectively the eigenvalues of $\boldsymbol{Q}$ and $\boldsymbol{D}$.

In what follows, $\gamma_1 > 0$, $\gamma_2 > 0$ are constants which are fixed so that

$$\delta_a^{-1} < \gamma_1 < \delta_b \delta_2^{-1}, \qquad \ldots (3.4)$$

and

$$\delta_a \gamma_2 < \Delta_a \gamma_2 < \delta_b \delta_a - \delta_2, \qquad \ldots (3.5)$$

where Δ_a is the greatest eigenvalue of the matrix A and δ_2, δ_a, δ_b are as defined in Theorem 1. The choice of γ_1, γ_2 satisfying (3.4) and (3.5) respectively, is guaranteed by the hypothesis (*i*) in the statement of Theorem 1.

Let us, for convenience, replace eq. (1.1) by the equivalent from

$$\dot{X} = Y, \qquad \ldots (3.6)$$

$$\dot{Y} = Z$$

and $$\dot{Z} = -AZ - G(Y) - H(X) + P(t, X, Y, Z).$$

In the proofs of the theorems, the main tool is the function $V = V(X, Y, Z)$ defined by

$$V = V_1 + V_2 \qquad \ldots (3.7)$$

where V_1, V_2 are given by

$$2V_1 = 2\int_0^1 \langle H(\alpha X), X\rangle\, d\alpha + \langle AY, Y\rangle$$

$$+ 2\gamma_1 \int_0^1 \langle G(\alpha Y), Y\rangle\, d\alpha + \gamma_1 \langle Z, Z\rangle + 2\langle Y, Z\rangle + 2\gamma_1 \langle H(X), Y\rangle, \qquad \ldots (3.8)$$

$$2V_2 = \gamma_2 \langle BX, X\rangle + 2\int_0^1 \langle AH(\alpha X), X\rangle\, d\alpha + \langle \boldsymbol{AY}, \boldsymbol{AY}\rangle - \gamma_2 \langle Y, Y\rangle$$

$$+ 2\int_0^1 \langle G(\alpha Y), Y\rangle\, d\alpha + \langle Z, Z\rangle + 2\gamma_2 \langle \boldsymbol{AX}, Y\rangle + 2\gamma_2 \langle X, Z\rangle$$

$$+ 2\langle \boldsymbol{AY}, Z\rangle + 2\langle Y, H(X)\rangle. \qquad \ldots (3.9)$$

Note that the function V is an n-dimensional analogue of the function V used in [6]. Its properties which will be most relevant to the present proofs are contained in the two lemmas which follow.

Lemma 3 — Assume that all the conditions on A, G and H in Theorem 1 are satisfied. Then there are constants $D_1 > 0$ and $D_2 > 0$ such that

$$D_1 (\| X \|^2 + \| Y \|^2 + \| Z^2 \|) \le V(X, Y, Z) \le D_2 (\| X \|^2 + \| Y \|^2 + \| Z \|^2) \qquad \ldots (3.10)$$

for arbitrary X, Y, Z in R_n.

PROOF : First we establish separate estimates for the components V_1, V_2 of V. The expression (3.8) may be reset in the form

$$2V_1 = \gamma_1 \| Z + \gamma_1^{-1} Y \|^2 + \langle AY, Y\rangle - \gamma_1^{-1} \| Y \|^2$$

$$+ 2\left\{\int_0^1 \langle H(\alpha X), X\rangle\, d\alpha + \gamma_1 \langle Y, H(X)\rangle + \gamma_1 \int_0^1 \langle G(\alpha Y), Y\rangle\, d\alpha\right\}. \quad \ldots (3.11)$$

Because the terms making up $2V_1$ are of the same form as in [3], a lower bound can be estimated for $2V_1$ in the same way as in [3]. Hence, we have that

$$2V_1 \geq \gamma_1 \| Z + \gamma_1^{-1} Y\|^2 + (\delta_a - \gamma_1^{-1}) \| Y\|^2 + \gamma_1 \beta_1 \| X\|^2 \quad \ldots (3.12)$$

with $\beta_1 = \delta_3 (\gamma_1^{-1} - \delta_2 \delta_1^{-1}) > 0$ and $\delta_a - \gamma_1^{-1} > 0$ in view of (3.4).

We shall next obtain a lower bound estimate for V_2. We observe that A, being symmetric and positive definite, has a symmetric non-singular square root $A^{1/2}$. With the aid of this matrix $A^{1/2}$, rewrite the expression for $2V_2$ in (3.9) thus :

$$2V_2 = \gamma_2 \langle (B - \gamma_2)X, X\rangle + \| Z + AY + \gamma_2 X\|^2 + \delta_2 \| A^{-1/2} Y + \delta_2^{-1} A^{1/2} H(X)\|^2$$

$$+ 2 \int_0^1 \langle G(\alpha Y), Y\rangle\, d\alpha - \delta_2 \langle A^{-1} Y, Y\rangle - \gamma_2 \langle Y, Y\rangle$$

$$+ 2 \int_0^1 \langle AH(\alpha X), X\rangle\, d\alpha - \delta_2^{-1} \| A^{1/2} H(X)\|^2 \quad \ldots (3.13)$$

for all $X, Y, Z \in R_n$.

By Lemma 1 we have

$$\gamma_2 \langle (B - \gamma_2) X, X\rangle \geq \beta_2 \| X\|^2$$

with $\beta_2 = \gamma_2(\delta_b - \gamma_2) > 0$ in view of (3.5). Set

$$\psi_1 = 2 \int_0^1 \langle G(\alpha Y), Y\rangle\, d\alpha - \delta_2 \langle A^{-1} Y, Y\rangle - Y_2 \langle Y, Y\rangle.$$

The terms in ψ_1 can be combined as follows :

$$\psi_1 = 2 \int_0^1 \int_0^1 \alpha_1 \langle \{J_g(\alpha_1 \alpha_2 Y) - \delta_2 A^{-1} - \gamma_2 I\}\ Y, Y\rangle\, d\alpha_1\, d\alpha_2,$$

so that by Lemma 1, we can also show that

$$\langle \{J_g(\alpha_1\, \alpha_2 Y) - \delta_2 A^{-1} - \gamma_2 I\} Y_1 Y\rangle \geq \beta_3 \| Y\|^2,$$

with $\beta_3 = \delta_a^{-1}(\delta_1\delta_a - \gamma_2\delta_a - \delta_2)$ positive, since

$$\delta_a\gamma_2 \le \Delta_a\gamma_2 < \delta_1\delta_a - \delta_2$$

by (3.5).

To complete the proof of the left half of the inequality (3.10) it remains now to establish a lower bound estimate for the last two terms on the right hand side of (3.13). First note that the two terms may indeed be combined into a single double integral as follows :

$$\psi_2 = 2\int_0^1 \langle AH(\alpha X), X\rangle\, d\alpha - \delta_2^{-1} \| A^{1/2} H(X)\|^2$$

$$= 2\delta_2^{-1}\int_0^1\int_0^1 \alpha_1 \langle A\{\delta_2 I - \boldsymbol{J_h}(\alpha_1 X)\}\boldsymbol{J_h}(\alpha_2\alpha_1 X)X, X\rangle\, d\alpha_1\, d\alpha_2. \quad \ldots (3.14)$$

This rearrangement is facilitated by the fact $A^{1/2}$ being symmetric,

$$\| A^{1/2} H(X)\|^2 = 2\int_0^1 \alpha_1 \langle A\, J_h(\alpha_1 X)X, H(\alpha_1 X)\rangle\, d\alpha_1,$$

where in the last step we have used the additional hypothesis that the symmetric matrices A and J commute. From (3.14) we have that $\psi_2 \ge 0$, by (2.1) and Lemma 2.

Thus on combining our estimates of the various components of the expression on the right-hand side of (3.13) we have that

$$2V_2 \ge \| Z + AY + \gamma_2 X\|^2 + \beta_2 \| X\|^2 + \beta_3 \| Y\|^2. \quad \ldots (3.15)$$

Hence the left hand inequality in (3.10) follows at once from (3.15) and (3.12).

The right half of the inequality in (3.10) is much easier to establish. Estimates of the upper bounds of the terms

$$\langle \boldsymbol{J_h}(\alpha X)X, X\rangle, \langle AY, Y\rangle, \langle \boldsymbol{J_g}(\alpha Y)Y, Y\rangle, \langle BX, X\rangle, \langle AJ_h(\alpha X)X, X\rangle, \langle AY, AY\rangle,$$

which appear in (3.8) and (3.9) can be handled quite readily with the aid of (3.1), since the matrices involved are all symmetric. For the terms $\langle Y, Z\rangle, \langle X, Z\rangle, \langle AY, Z\rangle, \langle AX, Y\rangle$ and $\langle H(X), Y\rangle$ we have the following estimates.

$$|\langle Y, Z\rangle| \le \| Y\|\,\| Z\| \le \frac{1}{2}(\| Y\|^2 + \| Z\|^2)$$

$$|\langle X, Z\rangle| \le \frac{1}{2}(\| X\|^2 + \| Z\|^2)$$

$$|\langle AY, Z\rangle| \le \frac{1}{2}(\Delta_a^2 \| Y\|^2 + \| Z\|^2)$$

$$| \langle AX, Y \rangle | \leq \frac{1}{2} (\Delta_a^2 \| X \|^2 + \| Y \|^2)$$

and

$$| \langle H(X), Y \rangle | \leq \frac{1}{2} (\delta_2^2 \| X \|^2 + \| Y \|^2).$$

On substituting these various estimates in (3.8) and (3.9) and then combining with (3.7) we arrive quite readily at the right-hand inequality in (3.10) so that the proof of the Lemma 3 is now complete.

We also require the following lemma

Lemma 4 — Assume that all the conditions of Theorem 1 hold. Let $X = X(t)$ be any solution of (1.1) and let $V(t) = V(X(t), \dot{X}(t), \ddot{X}(t))$. Then there exist constants Δ_0, D_3 and D_4 such that if the constant δ_0 in (1.2) satisfies $\delta_0 \leq \Delta_0$ then

$$V(t) \leq - D_3 \psi^2 + D_4 (\theta_1(t) \psi + \theta_2(t) \psi^{1+\alpha}), \qquad \ldots (3.16)$$

$$\psi \equiv (\| X(t) \|^2 + \| \dot{X}(t) \|^2 + \| \ddot{X}(t) \|^2)^{1/2}.$$

PROOF : Let $(X(t), Y(t), Z(t))$ be any solution of (3.6). Then the terms in $V(t)$ arising from the integral terms in (3.7) can be shown, exactly as in [3], to have the following derivatives

$$\frac{d}{dt} \int_0^1 \langle AH(\alpha X), X \rangle \, d\alpha = \langle AH(X), \dot{X} \rangle$$

$$\frac{d}{dt} \int_0^1 \langle G(\alpha \dot{X}), \dot{X} \rangle \, d\alpha = \langle G(\dot{X}), \ddot{X} \rangle.$$

The differentiation of the other terms presents no difficulty whatever. The various results, when combined with the fact that

$$\dddot{X} = - A\ddot{X} - G(\dot{X}) - H(X) + P(t, X, \dot{X}, \ddot{X})$$

lead, after due simplification, to the following estimate :

$$\dot{V}(t) = - U_1 - U_2 - U_3 + U_4$$

where

$$U_1 = \langle (B - \gamma_1 J_h(X))Y, Y \rangle + \langle AB - \gamma_2 A - J_h(X)Y, \dot{Y} \rangle$$

$$U_2 = \langle (\gamma_1 A - I)Z, Z \rangle + \langle G(Y), AY \rangle - \langle ABY, Y \rangle$$

$$U_3 = - \langle Y, G(Y) \rangle - \langle BY, \dot{Y} \rangle + \gamma_2 \langle X, G(Y) \rangle - \gamma_2 \langle BX, Y \rangle + \gamma_2 \langle X, H(X) \rangle$$

and

$$U_4 = \langle (\gamma_1 + I)Z + \gamma_2 IX + (A + I)Y, P(t, X, Y, Z) \rangle.$$

From (2.1), Lemma 1 and Lemma 2 we have that.

$$U_1 \geq (\delta_b - \gamma_1 \delta_2) \| Y \|^2 + (\delta_a \delta_b - \gamma_2 \Delta_a - \delta_2) \| Y \|^2 + (\gamma_1 \delta_a - 1) \| Z \|^2$$

which is positive by (3.4) and (3.5). Now (3.4) implies that

$$\langle (\gamma_1 A - I)Z, Z \rangle > 0.$$

Since

$$\langle AG(Y), Y \rangle = \int_0^1 \langle AJ_g(\alpha Y), Y \rangle \, d\alpha,$$

hypothesis (i) of Theorem 1 implies that $U_2 \geq 0$. Next we give an estimate for U_3. Rewrite U_3 in the form

$$U_3 = \int_0^1 \{ \langle (J_g(\alpha Y) - B)Y, Y \rangle + \gamma_2 \langle (J_g(\alpha Y) - B)Y, X \rangle \} \, d\alpha$$

$$+ \gamma_2 \int_0^1 \langle J_h(\alpha X)X, X \rangle \, d\alpha. \qquad \ldots (3.17)$$

Since $(J_g(\alpha Y) - B)$ is symmetric and non-negative it obviously has a square root $(J_g(\alpha Y) - B)^{1/2}$ which is symmetric for all $Y \in R_n$. With the help of this matrix, the expression inside the curly brackets in U_3 can be rearranged thus :

$$\langle (J_g - B)Y, Y \rangle + \gamma_2 \langle (J_g - B)Y, X \rangle = \left\| \left\| (J_g - B)^{1/2} Y + \frac{\gamma_2}{2} (J_g - B)^{1/2} X \right\| \right\|^2$$

$$- \frac{\gamma_2^2}{4} \| (J_g - B)^{1/2} X \|^2$$

where J_g here stands for $J_g(\alpha Y)$. From this and the last integral in (3.17), we obtain

$$U_3 \geq \gamma_2 \int_0^1 \langle \left\{ J_h(\alpha_1 X) - \frac{\gamma_2}{4} J_g(\alpha_1 Y) + \frac{\gamma_2}{4} B \right\} X, X \rangle \, d\alpha_1.$$

In view of Lemma 1 and (2.1), (2.2) we have that

$$U_3 \geq \gamma_2 \beta_4 \| X \|^2$$

where $\beta_4 = \delta_3 \delta_b^{-1} (\delta_b - \gamma_2)$ is positive by (3.5).

Finally, we are left with U_4. Since $P(t, X, Y, Z)$ satisfies (1.2) we obtain

$$| U_4 | \leq \delta_{10} \psi(\theta_1(t) + \theta_2(t) \psi^{\alpha}) + \delta_{10}\delta_0 \psi^2,$$

where $\delta_{10} = \max \{\gamma_2, 1 + \Delta_a, 1 + \gamma_1\}$.

On combining this last estimate with the other estimates we have that

$$V \leq -\gamma_2\delta_3(\delta_b - \gamma_2) \| X \|^2 - (\delta_b - \gamma_1\delta_2) \| Y \|^2 - (\delta_a\delta_b - \gamma_2 \Delta_a - \delta_2) \| Y \|^2$$

$$- (\gamma_1\delta_a - 1) \| Z \|^2 + \delta_{10}\delta_0\psi^2 + \delta_{10}\theta_1(t)\psi + \delta_{10}\theta_2(t)\psi^{\alpha+1}.$$

Let Δ_0 be now fixed as

$$\Delta_0 = \frac{1}{2} \delta_{10}^{-1} \min \{\gamma_2\delta_3(\delta_b - \gamma_2), \delta_b - \gamma_1\delta_2, \gamma_1\delta_a - 1, \delta_a\delta_b - \gamma_2\Delta_a - \delta_2\} > 0.$$

Then for $\delta_0 \leq \Delta_0$, we shall have from the above inequality for V that

$$V \leq -\delta_{11}\psi^2 + \delta_{10} \{\phi_1(t) \psi + \phi_2(t) \psi^{\alpha+1}\}$$

where $\delta_{11} = \delta_0\Delta_0$

so that (3.16) is true for the choice of Δ_0 given above.

4. Proof of Theorem 1

Let $X(t)$ be any solution of (1.1) satisfying the initial conditions (2.3), where X_0, Y_0, Z_0 are arbitrary vectors in R_n, and let k be any constant in the range $\frac{1}{2} \leq k \leq 1$. Set

$$V(t) = V(X(t), \dot{X}(t), \ddot{X}(t)).$$

We shall show that if $\delta_0 \leq \Delta_0$, Δ_0 here being the constant in Lemma 4, then

$$V^k(t) \leq V^k(t_0) e^{-\Delta_2(t - t_0)} + \Delta_1 \psi(t), \qquad t \geq t_0 \qquad \ldots (4.1)$$

for some constants $\Delta_1 > 0$, $\Delta_2 > 0$. In view of the estimate (3.10) in Lemma 3 this will imply (2.4), so that the theorem will be proved.

We have from (3.16) that

$$\dot{V} \leq -\frac{\delta_{11}}{2}\psi^2(t) + \psi_{11}(t) + \psi_{12},$$

where

$$\psi_{11} \equiv -\frac{\delta_{11}}{4}\psi^2(t) + \delta_{10}\theta_1(t)\,\psi,\ \ \psi_{12} \equiv -\frac{\delta_{11}}{4}\psi^2(t) + \delta_{10}\theta_1(t)\,\psi^{1+\alpha}.$$

It is a simple matter to verify that

$$\psi_{11} \leq \delta_{12}\,\psi^{2(1-k)}\,\theta_1(t)^{2k},\ \ \psi_{12} \leq \delta_{12}\,\psi^{2(1-k)}\,\theta_2(t)^{\frac{2k}{1-\alpha}}$$

for some constant $\delta_{12} > 0$. Then, on substituting these in our previous inequality for $\dot{V}$ we have that

$$\dot{V} \leq -\frac{\delta_{11}}{2}\psi^2(t) + \delta_{12}\,(\theta_1^{2k}(t) + \theta_2^{\frac{2k}{1-\alpha}}(t))\,\psi^{2(1-k)}. \qquad \ldots (4.2)$$

It is clear from (3.10) that (4.2) implies also that there are two constants $\delta_{13} > 0$ and $\delta_{14} > 0$ such that

$$\dot{V}(t) + \delta_{13}\,V(t) \leq \delta_{14}(\theta_1^{2k}(t) + \theta_2^{\frac{2k}{1-\alpha}}(t))\,V^{1-k}(t).$$

Hence, as a result of this inequality, we obtain (4.1), and Theorem 1 then follows as pointed out earlier.

5. Proof of Theorem 2

According to (2.6) there are constants β_1, β_2 such that

$$0 \leq \theta_1(t) \leq \beta_1,\ 0 \leq \theta_2(t) \leq \beta_2,\ t \geq 0. \qquad \ldots (5.1)$$

If the vector $P(t, X, Y, Z)$ satisfies inequality (1.2) then by using (5.1) we have that

$$\dot{V} \leq -\delta_{11}\psi^2 + \delta_{15}\,(\psi + \psi^{\alpha+1}), \qquad \ldots (5.2)$$

where $\delta_{15} = \delta_{10}\beta_3$ and $\beta_3 = \max\,\{\beta_1, \beta_2\}$.

If we choose $\psi \geq \delta_{16}\left(2\delta_{11}^{-1}\,\delta_{15}\right)^{\frac{1}{1-\alpha}}$, inequality (5.2) implies that

$$\dot{V} \leq -\frac{\delta_{11}}{2}(\|X\|^2 + \|Y\|^2 + \|Z\|^2). \qquad \ldots (5.3)$$

In fact, we can obtain

$$\dot{V} \leq -1$$

if we choose $\psi \geq \max\left\{\left(\frac{\delta_{11}}{2}\right)^{-1/2}, \delta_{16}\right\}$.

Now, using Yoshizawa's approach in [13] we can prove that for any solution $(X(t), Y(t), Z(t))$ of system (3.6) we ultimately have

$$(\| X \|^2 + \| Y \|^2 + \| Z \|^2) \leq \Delta_4.$$

That is, for any solution $(X(t), Y(t), Z(t))$ of (3.6) we cannot have

$$(\| X(t) \|^2 + \| Y(t) \|^2 + \| Z(t) \|^2) \geq \delta_{16}^2 \qquad \ldots (5.4)$$

for all $t \geq 0$.

This is quite standard argument now.

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REFERENCES

1. A. U. Afuwape, *J. math. Anal. Appl.* **97** (1983), 140-50.
2. F. Chunhua, *Ann. Diff. Eqs.* **11**(3), (1995), 264-69.
3. J. O. C. Ezeilo, *J. math. Anal. Appl.* **18** (1967) 395-416.
4. J. O. C. Ezeilo and H. O. Tejumola, *Ann. Mat. Pura. Appl.* **IV** (74), (1966), 283-316.
5. J. O. C. Ezeilo and H. O. Tejumola, *Atti. Acad. Lincei-Rend. Sc. fis. mat.e. nat.* **LVIII** (1975), 143-51.
6. M. Harrow, *Proc. Camb. phil. Soc.*, (1967), 63-147.
7. M. Harrow, *J. London math. Soc.*, **43** (1968), 587-92.
8. F. W. Meng, *J. math. Anal. Appl.*, **177** (1993), 496-509.
9. L. Mirsky, *Introduction to Linear Algebra*, Oxford Univ. Press, 1955.
10. R. Reissig, G. Sansone and R. Conti, *Nonlinear Differential Equations of Higher Order*, Noordhoff, Groningen, 1974.
11. H. O. Tejumola, *Ann. Mat. Pura Appl., (IV)*, **83** (1969), 195
12. H. O. Tejumola, *Ann. Mat. Pura Appl., (V)*, **92** (1972), 65-75.