Indian J. pure appl. Math., 30(4) : 407-418, April 1999
© Printed in India.

ON SUBHARMONIC BEHAVIOUR OF FUNCTIONS AND THEIR TANGENTIAL DERIVATIVES ON THE UNIT BALL IN $\mathbb{C}^n$

MIROLJUB JEVTIĆ AND MIROSLAV PAVLOVIĆ

Matematički Fakultet, Studentski TRG 16, 11 000 *Beograde Serbia*

(*Received* 29 *May* 1997; *after Revision* 9 *November* 1997;
Accepted 23 *November* 1998)

We define the class K of C^1-functions on the unit of ball in $\mathbb{C}^n$, that contains the generalized (α, β)-harmonic functions, and we show that the functions of the class K behave twice as well in the complex tangential directions. As an application, we deal with several characterizations of the Dirichlet spaces $\mathcal{D}_p^s(\alpha, \beta)$ of (α, β)-harmonic functions.

Key Words : Functions — Harmonic & Subharmonic; Class *K*-function; Dirichlet spaces; Holomorphic; Laplacian; Lebesgue measure; Differential Operators

1. INTRODUCTION

It is a well-known general principle that holomorphic functions behave twice as well in the complex tangential directions. In this paper we define the class K of C^1-functions on the unit ball in $\mathbb{C}^n$, that contains the generalized (α, β)-harmonic functions, and we show that the functions of the class K also behave twice as well in the complex tangential directions.

Our second goal is to make precise how this principle applies to the Dirichlet spaces $\mathcal{D}_p^s(\alpha, \beta)$ of (α, β)-harmonic functions. More precisely, we give several characterizations of the Dirichlet spaces $\mathcal{D}_p^s(\alpha, \beta)$.

Let $B := B_n$ be the open unit ball in $\mathbb{C}^n$ and $S = \partial B$ the unit sphere in $\mathbb{C}^n$. We denote by v the normalized Lebesgue measure on B and by σ the rotation invariant probability measure on S.

Let $\tilde{\Delta}$ be the invariant Laplacian on B. That is, $\tilde{\Delta} f(z) = \Delta(f \circ \varphi_z)(0)$, $f \in C^2(B)$, where Δ is the ordinary laplacian and φ_z the standard automorphism of B, $\varphi_z \in Aut(B)$, taking 0 to z (see [12]).

Geller[4] introduced a family of differential operators

$$\Delta_{\alpha\beta} = (1 - |z|^2)\left\{\sum_{i,j} (\delta_{ij} - z_i \bar{z}_j) D_i \bar{D}_j + \alpha R + \beta \bar{R} - \alpha\beta\right\},$$

where $D_i = \frac{\partial}{\partial z_i}$, and R is the radial derivative given by $R = \sum_{i=1}^{n} z_i D_i$. If $\alpha = \beta = 0$, Δ_{00} is the invariant or Bergman laplacian $\tilde{\Delta}$. The functions annihilated by $\tilde{\Delta}$ are called invariantly harmonic or $\mathcal{M}$-harmonic (see [12, chapter 4] and [13] for general properties of these functions). We will call (α, β)-harmonic the functions f such that $\Delta_{\alpha,\beta} f = 0$ (we write $f \in \mathcal{M}_{\alpha\beta}$; $\mathcal{M}_{00} = \mathcal{M}$).

For $f \in C^1(B)$, $Df = \left(\frac{\partial f}{\partial z_1}, \ldots, \frac{\partial f}{\partial z_n} \right)$, denotes the complex gradient of f, $\nabla f = \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_{2n}} \right)$, $z_k = z_{2k-1} + ix_{2k}$, $k = 1, 2, \ldots, n$, denotes the real gradient of f.

For $f \in C^1(B)$ let $\tilde{D} f(z) = D(f \circ \varphi_z)(0)$, $z \in B$, and $\tilde{\nabla} f(z) = \nabla(f \circ \varphi_z)(0)$, $z \in B$, be the invariant complex gradient of f and the invariant real gradient of f respectively.

For $z \in B$ and r, $0 < r < 1$, $E_r(z) = \{w \in B : |\varphi_z(w)| < r\}$.

A function $f \in C^1(B)$ is said of class K_L, $0 < L < \infty$, if

$$|\tilde{\nabla} f(z)| \le L\varepsilon^{-1} \sup_{w \in E_\varepsilon(z)} |f(w)|, \text{ for all } z \in B, 0 < \varepsilon < 1, \qquad \ldots (1.1)$$

and of class K if $f \in K_L$, for some $L > 0$.

We are now ready to state our first result.

Theorem 1.1 — *If $f \in K$ and $0 < p < \infty$ then*

(*i*) $|f(a)|^p \le C\delta^{-2n} \int_{E_\delta(a)} |f(z)|^p \, d\tau(z)$, $a \in B$, $0 < \delta < 1$,

(*ii*) $|T_{ij} f(a)|^p \le C\delta^{-2n-p} (1 - |a|^2)^{-p/2} \int_{E_\delta(a)} |f(z)|^p \, d\tau(z)$, $a \in B$,

and $0 < \delta < 1$, $1 \le i < j \le n$,

(*iii*) $|Rf(a)|^p \le C\delta^{-2n-p} (1 - |a|^2)^{-p} \int_{E_\delta(a)} |f(z)|^p \, d\tau(z)$, $a \in B$, $0 < \delta < 1$.

Analogous estimates hold for $\bar{T}_{ij} f$ and $\bar{R} f$. Here $d\tau(z) = (1 - |z|^2)^{-n-1} \, dv(z)$ is the measure on B that is invariant under the group $\mathrm{Aut}(B)$ and $T_{ij} f(z) = \bar{z}_i \frac{\partial f}{\partial z_j}(z) - \bar{z}_j \frac{\partial f}{\partial z_i}(z)$ are tangential derivatives of f.

In this paper constants are denoted by $C, C_1, C_2, \ldots, L_1, L_2, \ldots$ which may indicate different constant from one occurrence to the next.

$\mathcal{M}$-subharmonic behaviour of $|f|^p$ (the estimate (i)) for $f \in \mathcal{M}_{\alpha\beta}$ is obtained in [2] by a different method (an application of the representation formulas for the functions in $\mathcal{M}_{\alpha\beta}$ obtained in [1]). Since $\mathcal{M}_{\alpha\beta} \subset K$ (see section 3, Lemma 3.1 and Theorem 3.2) our result is a generalization of that of Ahern, Bruna and Cascante.

We note that the special case of the estimate (ii), when $f \in \mathcal{M}$, is obtained in [9].

Definition 1.2 — For $0 < p < \infty$, and $s \in \mathbb{R}$, the Dirichlet space $\mathcal{D}_p^s$ is defined as the space of C^1-functions f on B for which

$$\|f\|_{p,s} = \left(\int_B |\tilde{\nabla} f(z)|^p (1-|z|^2)^s \, d\tau(z) \right)^{1/p} < \infty.$$

Theorem 1.3 — *Let $0 < p < \infty$, $s > n - p/2$ and $f \in \mathcal{M}_{\alpha\beta}$. Then the following statements are equivalent :*

$$(i)\ f \in \mathcal{D}_p^s, \qquad (ii) \int_B |\nabla f(z)|^p (1-|z|^2)^{s+p} \, d\tau(z) < \infty$$

and

$$(iii) \int_B (|Rf(z)| + |\bar{R} f(z)|)^p (1-|z|^2)^{s+p} \, d\tau(z) < \infty.$$

As a consequence of our results we have :

1. If $s > n$, $\mathcal{D}_p^s(\alpha, \beta) = \mathcal{D}_p^s \cap \mathcal{M}_{\alpha\beta} = \mathcal{A}_p^s \cap \mathcal{M}_{\alpha\beta}$, where $\mathcal{A}_p^s$ is the weighted Bergman space defined as the space of measurable functions on B for which

$$\|f\|_{\mathcal{A}_p^s} = \left[\int_B |f(z)|^p (1-|z|^2)^s \, d\tau(z) \right]^{1/p} < \infty.$$

2. For $-p < s < 0$ the functions in the space $\mathcal{D}_p^s(\alpha, \beta)$ have Lipschitz continuity of order $-s/p$ and thus extend continuously to the closed unit ball.

We note that in [13] it is shown that $\mathcal{D}_p^s \cap \mathcal{M} = \mathcal{A}_p^s \cap \mathcal{M}$, for $1 \le p < \infty$, $s > n$ and in [7] for $0 < p < 1$ and $s > n$.

In [5] authors considered the boundary behaviour in the Besov spaces $\mathcal{B}_p^s$ of $\mathcal{M}$-harmonic functions and it is shown that 2) holds for functions in $\mathcal{B}_p^s$ (Theorem 1.4) in the case $1 < p < \infty$. We note that $\mathcal{B}_p^s = \mathcal{D}_p^s \cap \mathcal{M}$ for all $s \in \mathbb{R}$ and $1 < p < \infty$. Therefore 2) is a generalization of Theorem 1.4 ([5]).

2. Subharmonic Behaviour of $|f|^p$

Proof of Theorem 1.1 — (*i*) By Lemma 2.4[11] the proof reduces to the case $p = 1$. In [11] it is proved that

$$|\tilde{\nabla} f(z)|^2 = 2(|\tilde{D} f(z)|^2 + |\tilde{D}\bar{f}(z)|^2)$$

$$= 2(1-|z|^2)(|Df(z)|^2 - |Rf(z)|^2 + |D\bar{f}(z)|^2 - |R\bar{f}(z)|^2). \quad \ldots (2.1)$$

An application of Cauchy-Schwarz inequality shows that

$$|\tilde{\nabla} f(z)| \geq (1-|z|^2)\,|\nabla f(z)|. \quad \ldots (2.2)$$

Let $0 < t < 1/2$ and $z \in E_t(0)$. Then using (1.1) and (2.2) we find that

$$|f(0)| \leq |f(z)| + |f(z) - f(0)| \leq |f(z)| + \int_0^1 \left| \frac{d}{d\rho} f(\rho z) \right| d\rho \leq |f(z)|$$

$$+ |z| \int_0^1 |\nabla f(\rho z)|\, d\rho \leq |f(z)| + t \int_0^1 \frac{|\tilde{\nabla} f(\rho z)|}{1-\rho|z|}\, d\rho \leq |f(z)|$$

$$+ t \sup_{0<\rho<1} |\tilde{\nabla} f(\rho z)| \int_0^1 \frac{d\rho}{1-\rho|z|} \leq |f(z)| + C\frac{t}{r} \sup_{w \in E_{t+r}(0)} |f(w)|.$$

Since the class K is invariant under the group Aut (B), for arbitrary $a \in B$ we consider the function $f \circ \varphi_a(z)$. This gives

$$|f(a)| \leq C\frac{t}{r} \sup_{w \in E_{t+r}(a)} |f(w)| + |f(z)|,\ z \in E_t(a) \text{ and } 0 < r, t < 1/2. \quad \ldots (2.3)$$

Let now $0 < \delta < 1$. Choose $a \in \delta B$ so that the function $|f(z)|(\delta - |z|)^{2n}$, $|z| \leq \delta$, attains its maximum for $z = a$. Then we have

$$|f(w)| \leq \left(\frac{\delta - |a|}{\delta - |w|} \right)^{2n} |f(a)| \leq 2^{2n} |f(a)|, \quad \text{for} \quad w \in E_\varepsilon(a),\ \varepsilon = \frac{\delta - |a|}{2}.$$

From this and (2.3) we get $|f(a)| \leq |f(z)| + C_1 \frac{t}{r} |f(a)|$ whenever r, $t > 0$ are such that $r + t = \varepsilon$ and $z \in E_t(a)$. Now we choose r, $t > 0$ so that $r + t = \varepsilon$ and $C_1 t/r = 1/2$. Then we have $|f(a)| \leq 2|f(z)|$, $z \in E_t(a)$. Integrating this inequality over $E_t(a)$ we obtain

$$\tau(E_t(a))\,|f(a)| \leq 2 \int_{E_t(a)} |f(z)|\, d\tau(z) \leq C \int_{E_\delta(0)} |f(z)|\, d\tau(z).$$

Since $t = C(\delta - |a|)$ (C = constant), we finally get

$$|f(0)|\delta^{2n} \le |f(a)|(\delta - |a|)^{2n} \le C \int_{E_\delta(0)} |f(z)|\, d\tau(z).$$

(*ii*) It is easy to check that

$$|z|^2 |Df(z)|^2 = |Rf(z)|^2 + \sum_{i<j} |T_{ij} f(z)|^2.$$

Using this and (2.1) we find that

$$|z|^2 |\tilde{\nabla} f(z)|^2 = 2(1-|z|^2)\left[(1-|z|^2)(|Rf(z)|^2 + |R\bar{f}(z)|^2) + \sum_{i<j} |T_{ij} f(z)|^2 + \sum_{i<j} |T_{ij}\bar{f}(z)|^2\right]. \quad \ldots (2.4)$$

Now (ii) and (iii) are consequences of (2.4) and (i).

3. CLASS $\mathcal{M}_{\alpha\beta}$

A function $f \in C^2(B)$ is said to be of class K_{L_1, L_2}, $0 < L_1, L_2 < \infty$ if

$$|\tilde{\Delta} f(z)| \le L_1 |\tilde{\nabla} f(z)| + L_2 |f(z)|, \quad z \in B,$$

and of class K_0 if $f \in K_{L_1, L_2}$ for some $0 < L_1, L_2 < \infty$.

Lemma 3.1 — If $\Delta_{\alpha\beta} f = \lambda f$, for some $\lambda \in \mathbb{C}$, then $f \in K_0$.

PROOF : Since $\Delta_{\alpha\beta} f = \lambda f$, we have

$$|\tilde{\Delta} f(z)| = |(1-|z|^2)(-\alpha Rf(z) - \beta \bar{R} f(z) + \alpha\beta f(z) + \lambda f(z)|$$

$$\le C_1 (1-|z|^2)(|Rf(z)| + |\bar{R} f(z)|) + C_2 |f(z)|$$

$$\le C_1 (1-|z|^2) |\nabla f(z)| + C_2 |f(z)|$$

$$\le C_1 |\tilde{\nabla} f(z)| + C_2 |f(z)|, \quad \text{by (2.2).}$$

Theorem 3.2 — $K_0 \subset K$.

PROOF : Let $f \in K_0$. Since K_0 is also invariant under the group Aut(B), it suffices to prove (1.1) for $a = 0$.

The proof of Lemma 2.3 ([8, p.373]) shows that if $f \in C^2(B)$ then there exist constants C_1 and C_2 such that

$$|\tilde{\nabla} f(a)| \le C_1 \varepsilon^{-2n-1} \int_{E_\varepsilon(a)} |f(z)|\, d\tau(z) + C_2 \varepsilon \sup_{z \in E_\varepsilon(a)} |\tilde{\Delta} f(z)|, \qquad \ldots (3.1)$$

$a \in B$, $0 < \varepsilon < 1$.

By (3.1) we have

$$|\tilde{\nabla} f(a)| \le C_1 \varepsilon^{-1} \sup_{z \in E_\varepsilon(a)} |f(z)| + C_2 \varepsilon \sup_{z \in E_\varepsilon(a)} |\tilde{\nabla} f(z)| + C_3 \varepsilon \sup_{z \in E_\varepsilon(a)} |f(z)|$$

$$\le C_4 \varepsilon^{-1} \sup_{z \in E_\varepsilon(a)} |f(z)| + C_5 \varepsilon \sup_{z \in E_\varepsilon(a)} |\tilde{\nabla} f(z)|, \quad a \in B, 0 < \varepsilon < 1. \qquad \ldots (3.2)$$

Let $0 < 2C_5\delta < 1$. Choose $a \in \delta B = E_\delta(0)$ so that $|\tilde{\nabla} f(z)|(\delta - |z|) \le |\tilde{\nabla} f(a)|(\delta - |a|)$ for all $z \in \delta B$. Let $\varepsilon = (\delta - |a|)/2$. Then, if $z \in E_\varepsilon(a)$, we have

$$|z| = |\varphi_z(0)| \le |\varphi_z(a)| + |\varphi_a(0)| < \frac{\delta - |a|}{2} + |a| = \frac{\delta + |a|}{2},$$

and hence $\delta - |z| (\delta - |a|)/2$.

Therefore $|\tilde{\nabla} f(z)| \le \dfrac{\delta - |a|}{\delta - |z|} |\tilde{\nabla} f(a)| \le 2 |\tilde{\nabla} f(a)|$, for $z \in E_\varepsilon(a)$.

Applying now (3.2) we get

$$|\tilde{\nabla} f(a)| \le C_4 \varepsilon^{-1} \sup_{z \in E_\varepsilon(a)} |f(z)| + 2C_5 \varepsilon |\tilde{\nabla} f(a)|.$$

Since $2\varepsilon < \delta$ we have $2C_5\varepsilon < C_5\delta < 1/2$ so

$$|\tilde{\nabla} f(a)| \le 2C_4 (\delta - |a|)^{-1} \sup_{z \in E_\delta(0)} |f(z)| + \frac{1}{2} |\tilde{\nabla} f(a)|$$

and hence

$$|\tilde{\nabla} f(a)| \le 4C_4 (\delta - |a|)^{-1} \sup_{z \in E_\delta(0)} |f(z)|$$

i.e.,

$$|\tilde{\nabla} f(0)|\, \delta \le |\tilde{\nabla} f(a)|(\delta - |a|) \le 4C_4 \sup_{z \in E_\delta(0)} |f(z)|,$$

which was to be proved.

Remarks : 1) The proof of Lemma 2.3 shows[8] that if $f \in C^2$ and

$$|\tilde{\Delta} f(a)| \le L_1 \varepsilon^{-2} \sup_{z \in E_\varepsilon(a)} |f(z)|, \quad a \in B, \ 0 < \varepsilon < 1, \qquad \ldots (3.3)$$

then $|f|^p$ has $\mathcal{M}$-harmonic behaviour.

By (3.1), we see that, for C^2 functions on B (3.3) implies (1.1).

(2) Using (3.1) and the above proofs one shows that (3.3) is implied by

$$|\tilde{\Delta} f(a)| \leq L_2 \varepsilon^{-2} \sup_{z \in E_\varepsilon(a)} |\tilde{\nabla} f(z)| + L_3 \varepsilon^{-2} \sup_{z \in E_\varepsilon(a)} |f(z)|, \quad a \in B, \ 0 < \varepsilon < 1.$$

4. On Dirichlet Space $\mathcal{D}_p^s(\alpha, \beta)$

We begin with four known lemmas.

Lemma 4.1 — If $\alpha > 1$, then

$$\int_0^1 \frac{dt}{|1 - t\langle z, w\rangle|^\alpha} \leq \frac{C}{|1 - \langle z, w\rangle|^{\alpha - 1}}, \quad z, w \in B.$$

Lemma 4.2 ([120, p.17) — If $0 < s < t$, then

$$\int_B \frac{(1 - |\xi|^2)^{s-1} \, d\nu(\xi)}{|1 - \langle z, \xi\rangle|^{n+t}} = 0\left(\frac{1}{(1 - |z|)^{t-s}}\right), \quad z \in B.$$

Lemma 4.3 ([3]) — Let $1 \leq p < \infty$, $0 < q < \infty$, $t \in R$ and $rp > q + pt > 0$. There exists a constant $C > 0$ such that if $f \in \mathcal{A}_p^{q+tp+n}$ then

$$\left(\int_B \left(\int_B \frac{|f(w)|(1 - |w|^2)^{r-1} \, d\nu(w)}{|1 - \langle z, w\rangle|^{n+r-t}}\right)^p (1 - |z|^2)^{q-1} \, d\nu(z)\right)^{1/p} \leq C \|f\|_{\mathcal{A}_p^{q+tp+n}}.$$

Lemma 4.4 ([3]) — For $1 < p < r < \infty$, $0 < q < \infty$ and a measurable f on B we have

$$\left(\int_B \left(\int_B \frac{|f(w)|(1 - |w|^2)^{q-1} \, d\nu(w)}{|1 - \langle z, w\rangle|^{n+q}}\right)^r (1 - |z|^2)^{r\left(\frac{n+q}{p} - \frac{n}{r}\right) - 1} \, d\nu(z)\right)^{1/r} \leq C \|f\|_{\mathcal{A}_p^{q+n}}.$$

PROOF OF THEOREM 1.3 : From (2.2) it follows that (*i*) $\Rightarrow$ (*ii*).

(*ii*) $\Rightarrow$ (*iii*) It is easy to see that if (ii) holds then

$$\int_B (1 - |z|^2)^{s+p} \left|\frac{\partial f}{\partial z_j}(z)\right|^p d\tau(z) < \infty, \quad 1 \leq j \leq n,$$

and

$$\int_B (1 - |z|^2)^{s+p} \left|\frac{\partial f}{\partial \bar{z}_j}(z)\right|^p d\tau(z) < \infty, \quad 1 \leq j \leq n,$$

which in turn implies that

$$\int_B (1-|z|^2)^{s+p}\,|Rf(z)|^p\,d\tau(z)<\infty \quad \text{and} \quad \int_B (1-|z|^2)^{s+p}\,|\overline{R}f(z)|^p\,d\tau(z)<\infty.$$

Thus, $(ii) \Rightarrow (iii)$.

$(iii) \Rightarrow (i)$ Assume, now, that

$$\int_B (1-|z|^2)^{s+p}\,(|Rf(z)|+|\overline{R}f(z)|)^p\,d\tau(z)<\infty.$$

Hence, by (2.4), to show that $f\in \mathcal{D}^s_p(\alpha,\beta)$ it is sufficient to show that

$$\int_B (1-|z|^2)^{s+p/2}\,(|T_{ij}f(z)|^p+|T_{ij}\bar{f}(z)|^p)\,d\tau(z)<\infty, \quad 1\le i<j\le n.$$

An integration by parts show that

$$f(z)=\int_0^1 [Rf(tz)+\overline{R}f(tz)+f(tz)]\,dt.$$

From this we conclude that it is sufficient to prove that

$$\int_B (1-|z|^2)^{s+p/2}\left(\int_0^1 |T_{ij}u(tz)|\,dt\right)^p d\tau(z)<\infty, \quad 1\le i<j\le n,$$

where u is Rf, $\overline{R}f$, $R\bar{f}$, $\overline{R}\bar{f}$ or f.

We will show that, for fixed $1\le i<j\le n$,

$$I=\int_B (1-|z|^2)^{s+p/2}\left(\int_0^1 |T_{ij}Rf(tz)|\,dt\right)^p d\tau(z)<\infty.$$

The remaining cases may be treated analogously.

Since $\Delta_{\alpha\beta}f=0$, we have $\Delta_{\alpha,\beta-1}(Rf-\beta f)=0$, by Lemma 1.1 ([1]). Hence, using Theorem 3.2 and Theorem 1.1 we find that

$$|T_{ij}Rf(w)|\le C\left(\int_B \frac{|Rf(z)-\beta f(z)|\,(1-|z|^2)^a\,dv(z)}{|1-\langle z,w\rangle|^{n+a+3/2}}+\int_B \frac{|f(z)|\,(1-|z|^2)^a\,dv(z)}{|1-\langle z,w\rangle|^{n+a+3/2}}\right),$$

for any $a>0$.

Using Lemma 4.1 and Fubini's theorem we obtain

$$\int_0^1 |T_{ij} Rf(tw)|\, dt \le C\left(\int_B \frac{|Rf(z)-\beta f(z)|\,(1-|z|^2)^a}{|1-\langle z,w\rangle|^{n+a+1/2}}\, d\nu(z) + \int_B \frac{|f(z)|\,(1-|z|^2)^a\, d\nu(z)}{|1-\langle z,w\rangle|^{n+a+1/2}} \right). \quad \ldots (4.1)$$

Hence, $I \le C(I_1 + I_2)$, where

$$I_1 = \int_B (1-|z|^2)^{s+p/2} \left(\int_B \frac{|Rf(w)|\,(1-|w|^2)^a\, d\nu(w)}{|1-\langle z,w\rangle|^{n+a+1/2}} \right)^p d\tau(z),$$

$$I_2 = \int_B (1-|z|^2)^{s+p/2} \left(\int_B \frac{|f(w)|\,(1-|w|^2)^a\, d\nu(w)}{|1-\langle z,w\rangle|^{n+a+1/2}} \right)^p d\tau(z).$$

Assume now $1 \le p < \infty$ and $a > \max\left\{0, \frac{s-n}{p}\right\}$.

By applying Lemma 4.3 and replacing r, t and q by $a+1$, $1/2$ and $s + p/2 - n$ respectively we obtain

$$I_1 \le C \int_B (1-|z|^2)^{s+p} |Rf(z)|^p\, d\tau(z)$$

and

$$I_2 \le C \int_B (1-|z|^2)^{s+p} |f(z)|^p\, d\tau(z).$$

It is easy to see that the last integral is at most

$$C \int_B (1-|z|^2)^{s+p} (|Rf(z)| + |\bar{R} f(z)|)^p\, d\tau(z).$$

Thus, $I < \infty$, for $1 \le p < \infty$.

Case $0 < p < 1$ — Using $\mathcal{M}$-subharmonic behaviour of the function $(Rf(w) - \beta f(w))(1-\langle w,z\rangle)^{-n-a-1/2}$ (z-fixed) we obtain

$$\int_B \frac{|Rf(w)-\beta f(w)|\,(1-|w|^2)^a\, d\nu(w)}{|1-\langle z,w\rangle|^{a+n+1/2}}$$

$$\le C \int_B \left(\int_B \frac{|Rf(\xi)-\beta f(\xi)|^{p/2}\,(1-|\xi|^2)^{p(a+n+1)-n-1}\, d\nu(\xi)}{|1-\langle z,\xi\rangle|^{p(n+a+1/2)/2}\,|1-\langle w,\xi\rangle|^{p(n+a+1)}} \right)^{2/p} (1-|w|^2)^a\, d\nu(w).$$

(We may suppose now that a is a positive number satisfying $p(a + n) > s$.) An analogous estimate holds for

$$\int_B \frac{|f(w)|(1-|w|^2)^a}{|1-\langle z, w\rangle|^{n+a+1/2}}\, dv(w).$$

Applying now Lemma 4.4 and replacing p, r, q by 2, $2/p$, $p(a + n + 1) - n$ respectively and using Lemma 4.2 and (4.1) we find that

$$\begin{aligned} I &\le C \int_B (1-|z|^2)^{s+p/2} \\ &\quad \left(\int_B \frac{|Rf(w)|^p + |f(w)|^p)\ (1-|w|^2)^{p(a+n+1)-n-1}\, dv(w)}{|1-\langle z, w\rangle|^{p(n+a+1/2)}} \right) d\tau(z) \\ &= C \int_B (1-|w|^2)^{p(a+n+1)-n-1}\, (|Rf(w)|^p + |f(w)|^p) \\ &\quad \left(\int_B \frac{(1-|z|^2)^{s+p/2-n-1}\, dv(z)}{|1-\langle z, w\rangle|^{p(a+n+1/2)}} \right) dv(w) \\ &\le C \int_B (1-|w|^2)^{s+p}\, (|Rf(w)| + |\overline{R} f(w)|)^p\, d\tau(w) < \infty. \end{aligned}$$

Remark : Let $\mathcal{B}(\alpha, \beta)$ be the Bloch space of (α, β)-harmonic functions on B; that is $B(\alpha, \beta)$ is the space of all (α, β)-harmonic functions such that $\sup \{|\tilde{\nabla} f(z)| : z \in B\} < \infty$. Using the above results one shows that the following is true.

Theorem 4.5 — *Let $f \in \mathcal{M}_{\alpha\beta}$ Then the following statements are equivalent :*

(i) $f \in \mathcal{B}(\alpha, \beta)$, *(ii)* $\sup_{z \in B} (1-|z|)\, |\nabla f(z)| < \infty$ *and*

(iii) $\sup_{z \in B} (1-|z|)\, (|Rf(z)| + |\overline{R} f(z)|) < \infty$.

The validity of Theorem 4.5 was observed in [9] in the case $\alpha = \beta = 0$.

5. BERGMAN AND LIPSCHITZ SPACES

In this section we give an application of the results and techniques of the previous sections to the Bergman and Lipschitz spaces

Theorem 5.1 — *Let $h \in \mathcal{M}_{\alpha\beta}$*

(i) For all p, $0 < p < \infty$ and $s \in \mathbb{R}$, there exists a positive constant C, independent of h, such that

$$\int_B (1-|z|^2)^s\, |\tilde{\nabla} h(z)|^p\, d\tau(z) \le C \int_B (1-|z|^2)^s\, |h(z)|^p\, d\tau(z).$$

(*ii*) *For all* p, $0 < p < \infty$ *and* $s > n$, *there exists a positive constant* C, *independent of* h, *such that*

$$\int_B (1-|z|^2)^s |h(z)|^p \, d\tau(z) \le C\left(|h(0)|^p + \int_B (1-|z|^2)^{s+p} |\nabla h(z)|^p \, d\tau(z) \right) \quad \ldots (5.1)$$

PROOF : Item (i) follows from the estimate $|\tilde{\nabla} h(z)|^p \le C \int_{E_r(z)} |h(w)|^p \, d\tau(w)$ (a corollary of Theorem 1.1, Lemma 3.1 and Theorem 3.2).

(*ii*) An easy computation shows that $\Delta_{\alpha,\beta-1}\left(\frac{\partial h}{\partial z_j}\right) \equiv 0$, $1 \le j \le n$, and $\Delta_{\alpha-1,\beta}\left(\frac{\partial h}{\partial \bar{z}_j}\right) = 0$, $1 \le j \le n$. Hence $|\nabla h(z)|$ has $\mathcal{M}$-subharmonic behaviour, i.e.

$$|\nabla h(z)|^p \le C \int_{E_r(z)} |\nabla h(w)|^p \, d\tau(w), \quad z \in B, \ 0 < r < 1. \quad \ldots (5.2)$$

From this it follows that

$$|h(z)|^p \le C\left[|h(0)|^p + \left(\int_B \frac{|\nabla h(w)| (1-|w|^2)^a}{|1-\langle z, w\rangle|^{n+a}} \, dv(w) \right)^p \right].$$

Now we proceed as in the proof of Theorem 1.3 to conclude that (5.1) holds.

Corollary 5.3 — For all p, $0 < p < \infty$, and $s > n$, we have $\mathcal{A}_p^s \cap \mathcal{M}_{\alpha\beta} = \mathcal{D}_p^s \cap \mathcal{M}_{\alpha\beta}$.

Theorem 5.3 — *Let* $0 < p < \infty$ *and* $-p < s < 0$. *Then there exists a positive constant* C *such that*

$$|f(z) - f(0)| \le \frac{C}{|z-a|^{s/p}} \|f\|_{p,s}, \quad a, z \in B, \quad \ldots (5.3)$$

for all (α, β)*-harmonic functions* f *on* B. *In particular* f *in* $\mathcal{D}_p^s(\alpha, \beta)$ *satisfies the Lipschitz conditions of order* $-s/p$.

PROOF : By (5.2) we have

$$|\nabla f(z)|^p \le C \int_{E_r(z)} |\nabla f(w)|^p \, d\tau(w)$$

$$\le \frac{C}{(1-|z|)^{s+p}} \int_{E_r(z)} (1-|w|^{2)^{s+p}} |\nabla f(w)|^p \, d\tau(w).$$

Hence,

$$|\nabla f(z)| \leq \frac{C}{(1-|z|)^{1+s/p}} \|f\|_{p,s}, \text{ by Theorem 1.3.}$$

Now if $-p < s < 0$, this and Lemma 6 in [12] imply that f satisfies (5.3).

ACKNOWLEDGEMENT

The authors are grateful to the referee for useful suggestions.

REFERENCES

1. P. Ahern and C. Cascante, *Pacific J. Math.*, **153** (1992), 1-15.
2. P. Ahern, J. Bruna and C. Cascante, *Indiana Univ. Math.*, **45** (1996), 103-35.
3. F. Beatrous and J. Burbea, *Dissert. Math.*, **270** (1989), 1-57.
4. D. Geller, *Duke Math. J.*, **47** (1980), 365-91.
5. K. Hahn and E. Youssfi, *J. math. Anal. Appl.*, **175** (1993), 206-21.
6. K. Hahn, E. Youssfi, *Manuscripta Math.*, **71** (1991), 67-81.
7. M. Jevtic, *Publ. Inst. Maths.*, **58** (72), (1995), 35-42.
8. M. Jectic, M. Pavlovic, *Arch. Math.*, **61** (1993), 367-76.
9. M. Jevtic, M. Pavlovic, *Proc. Amer. Math. Soc.*, **123** (1995), 1385-93.
10. M. Pavlovic, *On Subharmonic Behaviour and Oscilattion of Functions on Ball in* R^n, *Publ. Inst. Maths.*, **55** (69) (1994), 18-22.
11. M. Pavlovic, *Indag. Math,*. **2** (1991), 89-98.
12. W. Rudin, *Function Theory in the Unit Ball of* C^n, Springer-Verlag, New York, 1980.
13. M. Stoll, *Invariant Potential Theory in the Unit Ball of* C^n, Cambridge University Press, Cambridge, 1994.