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SOME GENERALIZATIONS OF CARLSON'S INEQUALITIES

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Some generalizations of Carlson's inequality are established in this paper.

Key Words : Carlson's Inequality; Schwarz Inequality; Hölder Inequality

1. INTRODUCTION

In 1934, Carlson[4] proved the following curious inequalities:

$$\sum_{n=1}^{\infty} |a_n| < \sqrt{\pi}\left(\sum_{n=1}^{\infty} a_n^2\right)^{1/4}\left(\sum_{n=1}^{\infty} n^2 a_n^2\right)^{1/4}, \qquad \ldots (1)$$

and

$$\int_0^{\infty} |f(x)|\,dx \le \sqrt{\pi}\left(\int_0^{\infty} f^2(x)dx\right)^{1/4}\left(\int_0^{\infty} x^2 f^2(x)dx\right)^{1/4}, \qquad \ldots (2)$$

where $\sqrt{\pi}$ is the best possible constant in both cases.

Recently, Barza and Popa[2] pointed out that the inequalities (1), (2) and their generalizations have applications to some moment problems (see [6]), in the interpolation theory (see [5], [7], [8]) and optimal reconstruction of a sampling signal (see [3]). They also proved the following interesting generalization of inequality (2) :

Theorem A — *Let $f, g : [0, \infty) \to R$ be Lebesgue measurable functions and g be differentiable. If $g(0) = 0$, $\lim_{x \to \infty} g(x) = \infty$ and $0 < m = \inf_{x \in [0,\infty)} g'(x) < \infty$, then*

$$\left(\int_0^\infty |f(x)|\,dx\right)^4 \le \left(\frac{\pi}{m}\right)^2 \left(\int_0^\infty f^2(x)dx\right)\left(\int_0^\infty g^2(x)f^2(x)dx\right). \qquad \ldots (3)$$

The aim of this paper is to give further generalizations of (1), (2) and (3).

2. Discrete Case

Theorem 2.1 — *Let* $\{a_n\}_{n=1}^\infty$ *be a real sequence. If* $0<\alpha\le 1$, *then*

$$\sum_{n=1}^\infty |a_n| < \sqrt{\frac{\pi}{\alpha}} \left(\sum_{n=1}^\infty n^{1-\alpha} a_n^2\right)^{\frac{1}{4}} \left(\sum_{n=1}^\infty n^{1+\alpha} a_n^2\right)^{\frac{1}{4}}. \qquad \ldots (4)$$

PROOF : Let $A=\sum_{n=1}^\infty n^{1-\alpha}a_n^2$, $B=\sum_{n=1}^\infty n^{1+\alpha}a_n^2$ and $\lambda=\sqrt{\frac{B}{A}}$. If $A=0$ or ∞, or $B=\infty$, then (4) holds trivially so we may without loss of generality assume that $0<A<\infty$ and $B<\infty$.

By using the Schwarz inequality we have

$$\left(\sum_{n=1}^\infty |a_n|\right)^2 = \left(\sum_{n=1}^\infty |a_n| \sqrt{\lambda n^{1-\alpha}+\frac{1}{\lambda}n^{1+\alpha}} \cdot \frac{1}{\sqrt{\lambda n^{1-\alpha}+\frac{1}{\lambda}n^{1+\alpha}}}\right)^2$$

$$\le \left(\sum_{n=1}^\infty \left(\lambda n^{1-\alpha}+\frac{1}{\lambda}n^{1+\alpha}\right)a_n^2\right)\left(\sum_{n=1}^\infty \left(\lambda n^{1-\alpha}+\frac{1}{\lambda}n^{1+\alpha}\right)^{-1}\right)$$

$$= \left(\lambda A+\frac{1}{\lambda}B\right)\left(\sum_{n=1}^\infty \left(\lambda n^{1-\alpha}+\frac{1}{\lambda}n^{1+\alpha}\right)^{-1}\right).$$

Since $0<\alpha\le 1$, the function $f(x)=\left(\lambda x^{1-\alpha}+\frac{1}{\lambda}x^{1+\alpha}\right)^{-1}$ is strictly decreasing on $(0,\infty)$, so that

$$\sum_{n=1}^\infty \left(\lambda n^{1-\alpha}+\frac{1}{\lambda}n^{1+\alpha}\right)^{-1} < \int_0^\infty \frac{1}{\lambda x^{1-\alpha}+\frac{1}{\lambda}x^{1+\alpha}}\,dx = \frac{\pi}{2\alpha}.$$

Hence

$$\left(\sum_{n=1}^{\infty} |a_n| \right)^2 < \left(\frac{\pi}{2\alpha} \right) \left(\lambda A + \frac{1}{\lambda} B \right) = \frac{\pi}{\alpha} \sqrt{AB}.$$

This completes the proof.

Remark : We note that the inequality (1) is the special case of (4) when $\alpha = 1$.

Theorem 2.2 — *Let* $\{a_n\}_{n=1}^{\infty}$ *be a real sequence and let* $g : [0, \infty) \to \boldsymbol{R}$ *be a continuous differentiable function with* $g(0) = 0$, $\lim_{x \to \infty} g(x) = \infty$ *and* $0 < m = \inf_{x \in [0, \infty)} g'(x) < \infty$. *If* $0 < \alpha \leq 1$, *then*

$$\sum_{n=1}^{\infty} |a_n| < \sqrt{\frac{\pi}{\alpha m}} \left(\sum_{n=1}^{\infty} g^{1-\alpha}(n)\, a_n^2 \right)^{\frac{1}{4}} \left(\sum_{n=1}^{\infty} g^{1+\alpha}(n) a_n^2 \right)^{\frac{1}{4}}. \qquad \ldots (5)$$

PROOF : Let $A = \sum_{n=1}^{\infty} g^{1-\alpha}(n) a_n^2$, $B = \sum_{n=1}^{\infty} g^{1+\alpha}(n)\, a_n^2$ and $\lambda = \sqrt{\frac{B}{A}}$.

By using the Schwarz inequality we have

$$\left(\sum_{n=1}^{\infty} |a_n| \right)^2 = \left(\sum_{n=1}^{\infty} |a_n| \sqrt{\lambda g^{1-\alpha}(n) + \frac{1}{\lambda} g^{1+\alpha}(n)} \; \frac{1}{\sqrt{\lambda g^{1-\alpha}(n) + \frac{1}{\lambda} g^{1+\alpha}(n)}} \right)^2$$

$$\leq \left[\sum_{n=1}^{\infty} \left(\lambda g^{1-\alpha}(n) + \frac{1}{\lambda} g^{1+\alpha}(n) \right) a_n^2 \right] \left[\sum_{n=1}^{\infty} \left(\lambda g^{1-\alpha}(n) + \frac{1}{\lambda} g^{1+\alpha}(n) \right)^{-1} \right].$$

Since $0 < \alpha \leq 1$, the function $f(x) = \left(\lambda g^{1-\alpha}(x) + \frac{1}{\lambda} g^{1+\alpha}(x) \right)^{-1}$ is strictly decreasing on $(0, \infty)$, so that

$$\sum_{n=1}^{\infty} \left(\lambda g^{1-\alpha}(n) + \frac{1}{\lambda} g^{1+\alpha}(n) \right)^{-1} < \int_0^{\infty} \frac{1}{\lambda g^{1-\alpha}(x) + \frac{1}{\lambda} g^{1+\alpha}(x)} \, dx$$

$$\leq \frac{1}{m} \int_0^{\infty} \frac{g'(x)}{\lambda g^{1-\alpha}(x) + \frac{1}{\lambda} g^{1+\alpha}(x)} \, dx = \frac{\pi}{2\alpha m}.$$

Hence,

$$\left(\sum_{n=1}^{\infty} |a_n| \right)^2 < \frac{\pi}{2\alpha m} \left(\lambda A + \frac{1}{\lambda} B \right) = \frac{\pi}{\alpha m} \sqrt{AB}.$$

This completes the proof.

Remark : The inequality (4) is a special case of (5) when $g(x) = x$.

Theorem 2.3 — *Let* $\{a_n\}_{n=1}^{\infty}$ *be a real sequence and let* $g : [0, \infty) \to R$ *be a continuous differentiable function with* $g(0) = 0$, $\lim_{x \to \infty} g(x) = \infty$ *and* $0 < m = \inf_{x \in [0, \infty)} g'(x) < \infty$. *Suppose* p, α *and* r *are real numbers such that* $p > 1$ *and* $0 < \alpha \leq 1$. *Then*

$$\left(\sum_{n=1}^{\infty} |a_n| \right)^{2p} < \left(\frac{\pi}{\alpha m} \right)^2 \left(\sum_{n=1}^{\infty} g^{1-\alpha}(n) |a_n|^{p(1+2r-rp)} \right) \cdot \left(\sum_{n=1}^{\infty} g^{1+\alpha}(n) |a_n|^{p(1+2r-rp)} \right) \left(\sum_{n=1}^{\infty} |a_n|^{rp} \right)^{2(p-2)} \quad \ldots (6)$$

PROOF : Let $A = \sum_{n=1}^{\infty} g^{1-\alpha}(n) |a_n|^{p(1+2r-rp)}$, $B = \sum_{n=1}^{\infty} g^{1+\alpha}(n) |a_n|^{p(1+2r-rp)}$ and $\lambda = \sqrt{\frac{B}{A}}$.

By using Hölder inequality with indices p and q, where $p > q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, we have

$$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} |a_n| \left(\lambda g^{1-\alpha}(n) + \frac{1}{\lambda} g^{1+\alpha}(n) \right)^{\frac{1}{p}} \left(\frac{1}{\lambda g^{1-\alpha}(n) + \frac{1}{\lambda} g^{1+\alpha}(n)} \right)^{\frac{1}{p}}$$

$$\leq \left[\sum_{n=1}^{\infty} |a_n|^q \left(\lambda g^{1-\alpha}(n) + \frac{1}{\lambda} g^{1+\alpha}(n) \right)^{\frac{q}{p}} \right]^{\frac{1}{q}} \left[\sum_{n=1}^{\infty} \left(\lambda g^{1-\alpha}(n) + \frac{1}{\lambda} g^{1+\alpha}(n) \right)^{-1} \right]^{\frac{1}{p}}.$$

Since $0 < \alpha \leq 1$, the function $f(x) = \left(\lambda g^{1-\alpha}(x) + \frac{1}{\lambda} g^{1+\alpha}(x) \right)^{-1}$ is strictly decreasing on $(0, \infty)$, we have

$$\sum_{n=1}^{\infty}\left(\lambda g^{1-\alpha}(n)+\frac{1}{\lambda}g^{1+\alpha}(n)\right)^{-1}<\int_0^{\infty}\frac{1}{\lambda g^{1-\alpha}(x)+\frac{1}{\lambda}g^{1+\alpha}(x)}\,dx$$

$$\leq\frac{1}{m}\int_0^{\infty}\frac{g'(x)}{\lambda g^{1-\alpha}(x)+\frac{1}{\lambda}g^{1+\alpha}(x)}\,dx$$

$$=\frac{\pi}{2\alpha m}.$$

Hence, using Hölder inequality with indices $\frac{p}{q}$ and $\frac{p}{p-q}$, we have

$$\sum_{n=1}^{\infty}|a_n|$$

$$<\left(\frac{\pi}{2\alpha m}\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty}|a_n|^q\left(\lambda g^{1-\alpha}(n)+\frac{1}{\lambda}g^{1+\alpha}(n)\right)^{\frac{q}{p}}\right)^{\frac{1}{q}}$$

$$=\left(\frac{\pi}{2\alpha m}\right)^{\frac{1}{p}}\left[\sum_{n=1}^{\infty}|a_n|^{q-r(p-q)}\left(\lambda g^{1-\alpha}(n)+\frac{1}{\lambda}g^{1+\alpha}(n)\right)^{\frac{q}{p}}|a_n|^{r(p-q)}\right]^{\frac{1}{q}}$$

$$\leq\left(\frac{\pi}{2\alpha m}\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty}|a_n|^{\frac{p(q-r(p-q))}{q}}\left(\lambda g^{1-\alpha}(n)+\frac{1}{\lambda}g^{1+\alpha}(n)\right)\right)^{\frac{1}{p}}$$

$$\left[\sum_{n=1}^{\infty}(|a_n|^{r(p-q)})^{\frac{p}{p-q}}\right]^{\frac{p-q}{pq}}$$

$$=\left(\frac{\pi}{2\alpha m}\right)^{\frac{1}{p}}\left[\sum_{n=1}^{\infty}|a_n|^{p(1+2r-rp)}\left(\lambda g^{1-\alpha}(n)+\frac{1}{\lambda}g^{1+\alpha}(n)\right)\right]^{\frac{1}{p}}\left[\sum_{n=1}^{\infty}|a_n|^{rp}\right]^{\frac{p-2}{p}}$$

$$=\left(\frac{\pi}{2\alpha m}\right)^{\frac{1}{p}}\left(\lambda A+\frac{1}{\lambda}B\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty}|a_n|^{rp}\right)^{\frac{p-2}{p}}$$

$$= \left(\frac{\pi}{\alpha m}\right)^{\frac{1}{p}} (AB)^{\frac{1}{2p}} \left(\sum_{n=1}^{\infty} |a_n|^{rp}\right)^{\frac{p-2}{p}}.$$

This completes the proof.

Remarks :

(*i*) We note that the inequality (5) is a special case of the inequality (6) when $p = 2$.

(*ii*) Let $\alpha = 1$ and $g(x) \doteq x$ in the inequality (6). Then

$$\left(\sum_{n=1}^{\infty} |a_n|\right)^{2p}$$

$$< \pi^2 \left(\sum_{n=1}^{\infty} |a_n|^{p(1+2r-rp)}\right)\left(\sum_{n=1}^{\infty} n^2 |a_n|^{p(1+2r-rp)}\right)\left(\sum_{n=1}^{\infty} |a_n|^{rp}\right)^{2(p-2)} \quad \ldots (7)$$

We note that the inequality (7) reduces to the inequality (1) when $rp = 1$.

(*iii*) Let $r = \frac{1}{p-1}$ in the inequality (7). Then

$$\left(\sum_{n=1}^{\infty} |a_n|\right)^{2p} < \pi^2 \left(\sum_{n=1}^{\infty} |a_n|^{\frac{p}{p-1}}\right)^{2p-3} \left(\sum_{n=1}^{\infty} n^2 |a_n|^{\frac{p}{p-1}}\right), \quad \ldots (8)$$

which reduces to the inequality (1) when $p = 2$.

3. CONTINUOUS CASE

Theorem 3.1 — *Let $f: [0, \infty) \to \boldsymbol{R}$ be Lebesgue measurable function and $\alpha > 0$. Then*

$$\left(\int_0^{\infty} |f(x)|\, dx\right)^4 \leq \left(\frac{\pi}{\alpha}\right)^2 \left(\int_0^{\infty} x^{1-\alpha} f^2(x) dx\right)\left(\int_0^{\infty} x^{1+\alpha} f^2(x) dx\right). \quad \ldots (9)$$

PROOF : Let $A = \int_0^{\infty} x^{1-\alpha} f^2(x) dx$, $B = \int_0^{\infty} x^{1+\alpha} f^2(x)\, dx$ and $\lambda = \sqrt{\frac{B}{A}}$.

By using the Schwarz inequality we have

$$\left(\int_0^\infty |f(x)|\,dx\right)^2 = \left(\int_0^\infty |f(x)| \sqrt{\lambda x^{1-\alpha} + \frac{1}{\lambda} x^{1+\alpha}} \cdot \frac{1}{\sqrt{\lambda x^{1-\alpha} + \frac{1}{\lambda} x^{1+\alpha}}}\,dx\right)^2$$

$$\leq \left(\int_0^\infty f^2(x)\left(\lambda x^{1-\alpha} + \frac{1}{\lambda} x^{1+\alpha}\right)dx\right)\left(\int_0^\infty \frac{1}{\lambda x^{1-\alpha} + \frac{1}{\lambda} x^{1+\alpha}}\,dx\right)$$

$$= \left(\frac{\pi}{2\alpha}\right)\left(\lambda A + \frac{1}{\lambda} B\right)$$

$$= \left(\frac{\pi}{\alpha}\right)\sqrt{AB}.$$

This completes the proof.

Remark : We note that the inequality (2) is a special case of (9) when $\alpha = 1$.

Theorem 3.2 — *Let $f, g : [0, \infty) \to \mathbf{R}$ be Lebesgue measurable functions and g be continuous differentiable with $g(0) = 0$, $\lim_{x \to \infty} g(x) = \infty$ and $0 < m = \inf_{x \in [0,\infty)} g'(x) < \infty$. If $\alpha > 0$, then*

$$\left(\int_0^\infty |f(x)|\,dx\right)^4 \leq \left(\frac{\pi}{\alpha m}\right)^2 \left(\int_0^\infty g^{1-\alpha}(x) f^2(x)dx\right)\left(\int_0^\infty g^{1+\alpha}(x) f^2(x)dx\right). \quad \ldots (10)$$

PROOF : Let $A = \int_0^\infty g^{1-\alpha}(x) f^2(x)dx$, $B = \int_0^\infty g^{1+\alpha}(x) f^2(x)dx$ and $\lambda = \sqrt{\frac{B}{A}}$.

By using the Schwarz inequality we have

$$\left(\int_0^\infty |f(x)|\,dx\right)^2$$

$$= \left(\int_0^\infty |f(x)| \sqrt{\lambda g^{1-\alpha}(x) + \frac{1}{\lambda} g^{1+\alpha}(x)} \cdot \frac{1}{\sqrt{\lambda g^{1-\alpha}(x) + \frac{1}{\lambda} g^{1+\alpha}(x)}}\,dx\right)^2$$

$$\leq \left(\int_0^\infty f^2(x)\left(\lambda g^{1-\alpha}(x) + \frac{1}{\lambda} g^{1+\alpha}(x)\right)dx\right)\left(\int_0^\infty \frac{1}{\lambda^{1-\alpha}(x) + \frac{1}{\lambda} g^{1+\alpha}(x)}\,dx\right)$$

$$\leq \left(\lambda \int_0^\infty g^{1-\alpha}(x) f^2(x)dx + \frac{1}{\lambda} \int_0^\infty g^{1+\alpha}(x) f^2(x)dx \right) \left(\frac{1}{m} \int_0^\infty \frac{g'(x)}{\lambda g^{1-\alpha}(x) + \frac{1}{\lambda} g^{1+\alpha}(x)} dx \right)$$

$$= \frac{\pi}{2\alpha m} \left(\lambda A + \frac{1}{\lambda} B \right)$$

$$= \frac{\pi}{\alpha m} \sqrt{AB}$$

This completes the proof.

Remarks :

(*i*) The inequality (3) is a special case of (10) when $\alpha = 1$.

(*ii*) The inequality (9) is a special case of (10) when $g(x) = x$.

Theorem 3.3 — *Let $f, g : [0, \infty) \to R$ be Lebesgue measurable functions and g be continuous differentiable with $g(0) = 0$, $\lim_{x \to \infty} g(x) = \infty$ and $0 < m = \inf_{x \in [0, \infty)} g'(x) < \infty$. Suppose p, q, α and r are real numbers such that $\frac{1}{p} + \frac{1}{q} = 1, p > q > 1$, and $\alpha > 0$. Then*

$$\left(\int_0^\infty |f(x)|\, dx \right)^{2p} \leq \left(\frac{\pi}{\alpha m} \right)^2 \left(\int_0^\infty g^{1-\alpha}(x) |f(x)|^{p(1+2r-rp)} dx \right) \cdot$$

$$\left(\int_0^\infty g^{1+\alpha}(x) |f(x)|^{p(1+2r-rp)} dx \right) \left(\int_0^\infty |f(x)|^{rp} dx \right)^{2(p-2)} \quad \ldots (11)$$

PROOF : Let $A = \int_0^\infty g^{1-\alpha}(x) |f(x)|^{p(1+2r-rp)} dx$, $B = \int_0^\infty g^{1+\alpha}(x) |f(x)|^{p(1+2r-rp)} dx$ and $\lambda = \sqrt{\frac{B}{A}}$.

By using Hölder inequality with indices p, q and $\frac{p}{q}, \frac{p}{p-q}$, respectively, we have

$$\int_0^\infty |f(x)|\, dx$$

$$= \int_0^\infty |f(x)| \left(\lambda g^{1-\alpha}(x) + \frac{1}{\lambda} g^{1+\alpha}(x) \right)^{\frac{1}{p}} \left(\frac{1}{\lambda g^{1-\alpha}(x) + \frac{1}{\lambda} g^{1+\alpha}(x)} \right)^{\frac{1}{p}} dx$$

$$\leq \left[\int_0^\infty |f(x)|^q \left(\lambda g^{1-\alpha}(x) + \frac{1}{\lambda} g^{1+\alpha}(x) \right)^{\frac{q}{p}} dx \right]^{\frac{1}{q}} \left[\int_0^\infty \frac{1}{\lambda g^{1-\alpha}(x) + \frac{1}{\lambda} g^{1+\alpha}(x)} dx \right]^{\frac{1}{p}}$$

$$= \left(\frac{\pi}{2\alpha m} \right)^{\frac{1}{p}} \left[\int_0^\infty |f(x)|^{q-r(p-q)} \left(\lambda g^{1-\alpha}(x) + \frac{1}{\lambda} g^{1+\alpha}(x) \right)^{\frac{q}{p}} |f(x)|^{r(p-q)} dx \right]^{\frac{1}{q}}$$

$$< \left[\frac{\pi}{2\alpha m} \int_0^\infty |f(x)|^{p(1+2r-rp)} \left(\lambda g^{1-\alpha}(x) + \frac{1}{\lambda} g^{1+\alpha}(x) \right) dx \right]^{\frac{1}{p}} \left[\int_0^\infty |f(x)|^{rp} dx \right]^{\frac{p-2}{p}}$$

$$= \left(\frac{\pi}{2\alpha m} \right)^{\frac{1}{p}} \left(\lambda A + \frac{1}{\lambda} B \right)^{\frac{1}{p}} \left(\int_0^\infty |f(x)|^{rp} dx \right)^{\frac{p-2}{p}}$$

$$= \left(\frac{\pi}{\alpha m} \right)^{\frac{1}{p}} (AB)^{\frac{1}{2p}} \left(\int_0^\infty |f(x)|^{rp} dx \right)^{\frac{p-2}{p}}.$$

This completes the proof.

Remarks :

(*i*) We note that the inequality (10) is a special case of the inequality (11) when $p = 2$.

(*ii*) Let $\alpha = 1$ and $g(x) = x$ in the inequality (11). Then

$$\left(\int_0^\infty |f(x)|\, dx \right)^{2p} \leq \pi^2 \left(\int_0^\infty |f(x)|^{p(1+2r-rp)} dx \right) \left(\int_0^\infty x^2 |f(x)|^{p(1+2r-rp)} dx \right) \left(\int_0^\infty |f(x)|^{rp} dx \right)^{2(p-2)}, \quad \ldots (12)$$

which reduces to the inequality (2) when $rp = 1$.

(*iii*) Let $r = \frac{1}{p-1}$ in the inequality (12). Then

$$\left(\int_0^\infty |f(x)|\, dx \right)^{2p} \leq \pi^2 \left(\int_0^\infty |f(x)|^{\frac{p}{p-1}} dx \right)^{2p-3} \left(\int_0^\infty x^2 |f(x)|^{\frac{p}{p-1}} dx \right), \quad \ldots (13)$$

which reduces to the inequality (2) when $p = 2$.

(*iv*) Some other generalizations of Carlson's inequalities can found in [1] and [9] and the references given there. e

REFERENCES

1. E. F. Beckenbach and R. Bellman, *Inequalities*, 2nd ed., Springer-Verlag, Berlin-New York, 1965.
2. S. Barza and E. C. Popa, *Tamkang J. Math.* **29** : 1 (1998), 59-64.
3. J. Bergh, *J. math. Anal. Appl.* **115** (1986), 574-577.
4. F. Carlson, *Ark. Mat. Astr. och Fysik* **25B** (1934), 1-5.
5. J. Gustavsson and J. Peetre, *Studia Math.* **60** (1977), 33-59.
6. B. Kjellberg, *C. R. Dixiéme Congrés des Mathematiciens Scandinaves* (1946), Jul, Gjellerups Forlag, Copenhagen (1946), 333-340.
7. N. Ya. Kruglyak, L. Maligranda and L. E. Persson, *Studia Math.* **104** (1993), 161-180.
8. L. Maligranda and L. E. Persson, Inequalities and Interpolations, *Collect. Math.* **44** (1993), 183-201.
9. D. S. Mitrinović, J. E. Pečarić and A. M. Fink, *Classical and New Inequalities in Analysis*, Kluwer Acad. Publ., Dordrecht/Boston/Londen, (1993).