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THE LINEAR DIOPHANTINE EQUATION WITH n-UNKNOWS IN $Q(\sqrt{5})$

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The solution of the linear diophantine equation

$$a_1x_1 + a_2x_2 + \ldots + a_nx_n = b \qquad \ldots (1)$$

in rational integers is very well known, where $a_1, a_2, \ldots, a_n, b$ are integers. The purpose of this note is to discuss the solution of this equation when $a_1, a_2, \ldots, a_n, b$ are integers in $Q(\sqrt{5})$ and the solution are integers in $Q(\sqrt{5})$.

Key Words : Continued Fraction; Diophantine Equations; Quadratic Field

1 INTRODUCTION

Recall in [1], that the continued fraction $[r_1, r_2, \ldots, r_N]$ is called the Nth convergent of the continued fraction $[r_1, r_2, \ldots, r_n]$, where N is a nonnegative integer less than or equal to n . Furthermore, we have,

(i) $\dfrac{p_n}{q_n} - \dfrac{p_{n-1}}{q_{n-1}} = \dfrac{(-1)^{n+1}}{q_nq_{n-1}}$, where $\dfrac{p_{n-1}}{q_{n-1}}$ and $\dfrac{p_n}{q_n}$ are $(n-1)$th and nth convergents of the continued fraction $[r_1, r_2, \ldots, r_n]$, respectively; and

(ii) there are integer solutions of equation (1) if and only if $(a_1, a_2, \ldots a_n) \mid b$.

In [4], it has been shown that the integers of the $Q(\sqrt{5})$, have the form $r + \lambda s$, where $r, s \in Z$ and $\lambda = \dfrac{1+\sqrt{5}}{2}$, $\lambda^2 = \left(\dfrac{1+\sqrt{5}}{2}\right)^2 = 1 + \lambda$.

It was shown in [2], that the element of $Q(\sqrt{5})$ have a unique finite continued fraction representation and every other real number has a unique infinite continued fraction representation. The continued fractions will be referred to as λ_5-fractions and every finite λ_q-fraction is an element in the algebraic number field $Q(\lambda q)$. Leutbecher[6] showed that only in the case $q = 5$, every element in $Q(\sqrt{5}) = Q(\lambda_5)$ has a finite λ_5-fraction, and a real number is an element of $Q(\sqrt{5})$ if and only if it has a finite λ_5-continued fraction representation.

Definition —A λ_5-fraction is an expression of the form

$$r_0\lambda + \cfrac{\varepsilon_1}{r_1\lambda + \cfrac{\varepsilon_2}{r_2\lambda + }}$$

where $\varepsilon_i = \pm 1, r_i \in Z^+, i \geq 1, r_0 \in Z$ and $\lambda = \frac{1+\sqrt{5}}{2}$. The continued fraction will be written as $[r_0\lambda, \varepsilon_1/r_1\lambda, \varepsilon_2/r_2\lambda, \ldots]$.

Let η be any arbitrary real number and $\{\}$ the nearest integer to η/λ for which $-\frac{1}{2} \leq r_0 - (\eta/\lambda) < \frac{1}{2}$. It follows that $(\eta/\lambda) = r_0$ and if $\eta_1 = \varepsilon_1/(\eta - r_0\lambda)$ then $\eta = r_0\lambda + (\varepsilon_1/\eta_1)$, where $\varepsilon_1 > 0$ if $r_0\lambda < \eta$ and $\varepsilon_1 < 0$ if $r_0\dot{\lambda} > \eta$. Similarly, $r_1 = (\eta_1/\lambda)$ and by continuing the above process one obtains $r_{n-1} = (\eta_{n-1}/\lambda)$. The nearest integer algorithm yields a unique λ_5-fraction[2].

Theorem 1 — *Let $p, q, r \in Z(\sqrt{5})$, and suppose that, except for units, p, q, r are relatively prime. Then the diophantine equation $px + qy = r$ has integer solution in $Q(\sqrt{5})$. If x_0, y_0 is a particular solution, then any other solution has the form $x = x_0 + qt, y = y_0 - pt$. If $(p, q) = d$ and $d \mid r$, then*

$$(p / d)x + (q / d)y = (r / d)$$

is solvable in $Q(\sqrt{5})$[3].

Theorem 2 — *If the elements p, q, r, s which are not units, in $Z(\sqrt{5})$ and $(q, r) = 1$, then the diophantine equation*

$$px + qy + rz = s \qquad \ldots (2)$$

has integer solutions in $Q(\sqrt{5})$. If x_0, y_0, z_0 is a particular solution then any other solution has the form

$$x = x_0 + Av$$

$$y = y_0 + Bu + Cv$$

$$z = z_0 + Du + Ev$$

where A, B, C, D, E are known integers in $Q(\sqrt{5})$ determined by the algorithm and u, v are arbitrary integers in $Q(\sqrt{5})$[5].

2. Main Result

Theorem — *Let $a_i (3 < i \leq n)$ be integers different from a unit and zero in $Z(\sqrt{5})$ such that $(a_i, a_j) = 1$. Then the linear diophantine equation*

$$a_1x_1 + a_2x_2 + \ldots + a_nx_n = b \qquad \ldots (3)$$

has an integer solution in $Q(\sqrt{5})$.

PROOF : Let

$$x_{n-1} = \alpha s + \beta t \qquad \ldots (4)$$

and

$$x_n = \gamma s + \delta t,$$

where α, β, γ and δ are integers in $Q(\sqrt{5})$ chosen so that $\alpha\delta - \beta\gamma = 1$. Then it follows that $s = \delta x_{n-1} - \beta x_{n-1}$ and $t = -\gamma x_{n-1} + \alpha x_n$. Note that s, t are integers in $Q(\sqrt{5})$ if and only if x_{n-1}, x_n are also integers in $Q(\sqrt{5})$. If we take

$$\beta = \frac{-a_n}{(a_n, a_n - 1)}, \quad \delta = \frac{a_{n-1}}{(a_n, a_{n-1})}, \qquad \ldots (5)$$

where in (3) $(a_n, a_{n-1}) = 1$ by hypothesis, then $(\beta, \delta) = 1$ and since

$$\alpha\delta - \alpha\beta = 1, \qquad \ldots (6)$$

we can find α and γ. Thus substituting (5) in (4) and (4) in (3) we get

$$a_1x_1 + a_2x_2 + \ldots + (a_{n-1}\alpha + a_n\gamma)s = b. \qquad \ldots (7)$$

Hence, this equation contains $(n - 1)$ unknowns. Note that eq. (7) has the same structure as (3) and the coefficients of (7) are nonzero integers in $Q(\sqrt{5})$. If $n > 4$, then the similar process can be applied to eq. (7) to reduce to the linear diophantine equation with $(n - 2)$ unknowns.

By continuing this process, eq. (3) can be reduced to the linear diophantine equation with three unknowns and it can be solved with the method in [5]. By substituting these solutions backward to each repeated step in above, we get all the solutions of (3), $x_1, x_2, \ldots, x_{n-1}, x_n$ in terms of $(n - 1)$ new parameters.

Example — Consider the linear diophantine equation

$$(3 + \lambda)x + (3\lambda - 1)y + (2 + \lambda)z + (3\lambda - 2)w = 2\lambda - 1. \qquad \ldots (8)$$

From eq. (5) we have $\beta = -(3\lambda - 2)$, $\delta = (2 + \lambda)$. Substituting β and δ in (6), we obtain

$$\alpha = -(18\lambda + 11), \gamma = (23\lambda + 14).$$

Putting α, β, γ and δ in (4), one obtains

$$w = (23\lambda + 14)s + (2 + \lambda)t \qquad \ldots (9)$$

and $$z = -(18\lambda + 11)s - (3\lambda - 2)t.$$

By substituting values of w and z in (8), we get

$$(3+\lambda)x+(3\lambda-1)y+s=2\lambda-1, \qquad \ldots (10)$$

which is a linear diophantine equation with three unknowns. Since

$$\beta=-1, \quad \delta=3\lambda-1, \quad \alpha=\lambda, \quad \gamma=-2(\lambda+1)$$

and from (4) , we have

$$s=-2(\lambda+1)u+(3\lambda-1)r \qquad \ldots (11)$$

and $y=\lambda u-r.$

From (11) and (10), we get

$$(3+\lambda)x+u=2\lambda-1 \qquad \ldots (12)$$

with two unknowns.

It is easy to see that the coefficients of eq. (12) can be expanded to a λ_5 fraction, that is,

$$3+\lambda=\left[3\lambda, \frac{-1}{3\lambda}, \frac{-1}{\lambda}\right].$$

The right side when rolled up and using $\lambda^2=1+\lambda$, becomes $(14\lambda+9)/(3\lambda+2)$.

The penultimate convergent of this continued fraction is $[3\lambda, -1/3\lambda]=(9\lambda+8)/3\lambda$.

Thus

$$x'=-3(\lambda+2) \text{ and } u'=5(5\lambda+2)$$

is a solution of

$$(14\lambda+9)x'+(3\lambda+2)u'=2\lambda-1.$$

Hence, a solution of eq. (12) are in the form

$$x_0=-3(\lambda+2)(2+3\lambda)=-3(11\lambda+7)$$

and $u_0=5(5\lambda+2)(2+3\lambda)=5(31\lambda+19).$

All solutions of (12) are in the form

$$x=-3(11\lambda+7)-k \qquad \ldots (13)$$

and

$$u=5(31\lambda+19)+(3+\lambda)k, \quad k\in Q(\sqrt{5}).$$

From (13), (11) and (9) we have all solutions of (8) are given by

$$x=-3(11\lambda+7)-k,$$

$$y=(250\lambda+155)+(4\lambda+1)k-r,$$

$$z = (32490\lambda + 20080) + (434\lambda + 268)k - (69\lambda + 43)r - (3\lambda - 2)t$$

and $$w = -(41470\lambda + 25630) - (554\lambda + 342)k + (88\lambda + 55)r + (\lambda + 2)t$$

with three arbitrary integers k, r, t.

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