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PACKINGS AND COVERINGS

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I am very grateful to the Academy for the honour being given to me. I am overwhelmed by your kindness.

In this talk, I shall try to give a general overview of the work that my colleagues and I have done for some time. There is bound to be a considerable overlap between this talk and other similar talks given by me at other occasions.

PACKINGS

Lagrange proved in 1773 the following :

Let $f(x, y) = ax^2 + bxy + cy^2$ be a real positive definite quadratic form with discriminate $d = d^2 - 4ac < 0$. Then there exist integers x, y, not both 0, such that

$$f(x, y) \leq \sqrt{\frac{-d}{3}},$$

and the result is best possible.

In 1831, Seeber looking for an analogous result for ternary forms observed that the more natural instead of the discriminant α would be the determinant.

$$D = \begin{vmatrix} a & b/2 \\ b/2 & c \end{vmatrix} = -\frac{1}{4}d.$$

Generalising the ideas of Lagrange he developed a "reduction" theory for ternary definite forms and obtained some inequalities which, he expected, would lead to a Lagrange type of theorem for ternary forms Gauss (1831, 1840)[26], while reviewing Seeber's work, gave a geometric interpretation to the problem and in a natural way obtained the Seeber inequalities, from which he deduced the best Lagrange type result for positive definite ternary quadratic forms.

In 1850, Hermite[28] proved :

Let $f(x_1, ..., x_n) = \sum_{1 \leq i, j \leq n} a_{ij} x_i x_j$, $a_{ij} = a_{ji}$ real, be a positive definitite quadratic form in n variables with determinant D = det (a_{ij}). Then there exist integers $x_1, ..., x_n$, not all 0, such that

$$f(x_1, ..., x_n) \leq \left(\frac{4}{3}\right)^{\frac{n-1}{2}} D^{1/n}$$

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Thus, Hermite showed the existence of numbers α_n, such that for forms f like the above, $f(x_1, ..., x_n) \le \alpha_n D^{1/n}$ has a non-trivial solution in integers x_i.

Once the existence of numbers α_n is established, the natural thing is to ask for smallest such number. Let us denote it by γ_n. Then γ_n is called the Hermite constant, and

$$\gamma_2 = \sqrt{\frac{4}{3}},\ \gamma_3 = \sqrt[3]{2},\ \gamma_n \le \left(\frac{4}{3}\right)^{\frac{n-1}{2}},$$

the first is the result of Lagrange (1773)[33], the second of Gauss[26] and the last of Hermite[28].

One would now naturally like to know the value of γ_n for all n. γ_4, γ_5 were determined by Korkine and Zolatareff[32] in 1872 and 1877. The values of γ_6, γ_7 were announced by Blichfeldt in 1925, 1926 and the proof published with that of γ_3 also in 1934. γ_8 is not known for $n \ge 9$.

You may be wondering what this has to do with the title "packings" given by me to this part of the talk. It took the mathematicians a long time to see the connection between these arithmetic results and packings. Hence, I also have has to wait for some time to justify the title I have given.

Gauss associated with positive definite quadratic forms some geometrical entities, which we now call Lattices. Let us illustrate this in two dimensions.

Let $A_1 = (\alpha, \beta)$, $A_2 = (\gamma, \delta)$ be two points in the plane, not collinear with the origin O. Extend the lines OA_1, OA_2, indefinitely in both directions. On these lines mark the points $\pm A_1, \pm 2A_1, \pm 3A_1, ..., \pm A_2, \pm 2A_2 \pm 3A_2, ...$, in the obvious way. Through the points $0, \pm A_1, \pm 2A_1, ...$ draw lines parallel to OA_2 and through $\pm A_2, \pm 2A_2, ...$ draw lines parallel to OA_1. Then the whole plane is divided into parallelograms whose corners are of the type $uA_1 + vA_2$, where u, v are integers. The set Λ of these corners $uA_1 + vA_2$, uv integers, is called the lattice generated by the basis A_1, A_2. The area of the parallelogram $OA_1A_3A_2$, where $A_3 = A_1 + A_2$, is called the determinant $d(\Lambda)$ of the lattice Λ. The same lattive Λ can arise from many bases, but the value of its determinant $d(\Lambda)$, luckily, is independent of the choice of the basis and comes out to be the same for all its generating bases.

If $f(x, y) = ax^2 + bxy + cy^2$ is a positive definite binary quadratic forms with det D we can write it as

$$(*)\ f(x, y) = (\alpha x + \beta y)^2 + (\gamma x + \delta y)^2$$

for suitable $\alpha, \beta, \gamma, \delta$ and the determinant $|\alpha\delta - \beta\gamma|^2$ = determinant D of f. With f we can associate the lattice Λ with basis (α, γ), (β, δ) whose determinent $d(\Lambda) = \sqrt{D}$.

Various representations (*) of f give rise to various lattices, but there are natural connections between them, that I shall not describe here.

Conversely, let Λ be a lattice generated by $(\alpha, \gamma), (\beta, \delta)$. Then we can associate with it the form

$$f(\alpha x + \beta y)^2 + (\gamma x + \delta y)^2.$$

Corresponding to different bases of Λ, we get different f, but they are also related to each other (in fact they are forms called equivalent to f by Lagrange in his theory).

Gauss observed this correspondence between positive forms and lattices and saw at once that if $f(x, y)$ is a positive definite quadratic form and Λ a corresponding lattice, then the values of $f(x, y)$ for integers x, y are the squared distances of points of Λ from O, or, what is the same, squared distances of points of Λ from each other. Then Gauss observed that if we choose a suitable basis for Λ, we can show that if $\rho(\Lambda)$ in the minimum distance of points of Λ from O, then $\rho^2(\Lambda) \leq \sqrt{\frac{4}{3}}\, d(\Lambda)$ which is Lagrange theorem.

Now let us generalize these ideas to n dimensions. Let $A_1, ..., A_n$ be n linearly independent points of the n dimensional Euclidean space R_n. The set of points $u_1A_1 + ... + u_nA_n$, u_i integers, is called a lattice, more precisely the lattice with basis $A_1, ..., A_n$. If A_i are considered column vectors then

$$d(\Lambda) = |\det(A_1, ..., A_n)|$$

is called the determinant $d(\Lambda)$ of the lattice Λ. The same lattice Λ may arise from different bases, but the value of $d(\Lambda)$ is independent of the choice of the basès.

There is a correspondence between the positive definite quadratic forms in n variables and n dimensional lattices similar to the one described in two dimensions. The value of the Hermite constant γ_n is related in a very natural way to the maximum length that the shortest vector of a lattice of determinant 1 can have (more precise relation will appear latter). Gauss used this formulation to prove the Gauss-Seeber theorem for ternary forms.

Minkowski[36], analysing the Gauss approach, observed that the Gauss approach can be rephrased in terms of Lattice Packings that we now describe.

Let Λ be a lattice in R_n and K a set in R_n, with volume $v(K)$. We say (K, Λ) is a lattice packing of K, or Λ is a packing lattice for K, if no two translates $K + A$, $A \in \Lambda$ of K overlap (i.e., have common interior points). We define its density $\delta(K, \Lambda)$ to be $v(K)/d(\Lambda)$, i.e., the "proportion of space R_n occupied by the packed sets $K + A$, $A \in \Lambda$". We define $\delta(K) = \sup \delta(K, \Lambda)$, taken over all lattice packings (K, Λ) of K. Clearly, $\delta(K) \leq 1$, and is called the density of the best lattice packings of K.

Minkowski observed that the Gauss approach really amounted to the following : Let K be the n dimensional sphere (ball) of radius 1 and volume J_n. Let $\delta_n = \delta(K)$ be the density of the best lattice packing of K and γ_n the Hermite constant, then

$$\delta_n = J_n\left(\frac{\gamma_n}{4}\right)^{n/2} = J_n\left(\frac{\sqrt{\gamma_n}}{2}\right)^n.$$

Thus the problem of determination of Hermite constant γ_n is that of finding the density of the best lattice packings of n dimensional spheres.

Since $\delta_n \leq 1$, it follows that

$$J_n\left(\frac{\gamma_n}{4}\right)^{n/2} \leq 1, \text{ or } \gamma_n \leq 4\left(\frac{1}{j_n}\right)^{n/2} \sim \frac{2n}{\Pi e}, \text{ as } n \to \infty.$$

This observation of Minkowski in 1891, gives a qualitative improvement to the estimate $\gamma_n \leq \left(\frac{4}{3}\right)^{\frac{n-1}{2}}$ of Hermite.

Minkowski realised that his observations about packings of spheres and their relations with arithmetical questions could be carried on to very general class of objects and formulated his fundamental theorem, which not only gave a host of Lagrange-Gauss-Hermite type results, but laid the foundation of a new branch of Number Theory which he called Geometry of Numbers. This branch has developed into a very rich independent one and has been studied extensively for its own sake as also for many applications elsewhere.

Minkowski not only obtained the upper bound $\gamma_n \leq \alpha_n \simeq \frac{2n}{\Pi e}$ as $n \to \infty$ mentioned earlier, but also showed in 1903 that

$$\gamma_n \geq \beta_n \sim \frac{2n}{\Pi e} \text{ as } n \to \infty.$$

The upper bound of Minkowski was improved by Blichfeldt[13], Sidlenikov[42], Levenstein[34], Kabatjanski and Levenstein[30] and others. The best known present estimates appear to be

$$\frac{n}{2\Pi e} \sim \beta_n \leq \gamma_n \leq c_n \sim \frac{n}{\Pi e} \cdot \frac{1.743...}{2} \text{ as } n \to .$$

To get back to the concept of packings, while the lattice packings have been motivated by number theory, physically or geometrically, a packing can be more general, viz., let K be a subset of R_n and (K, Σ) a set of translates of K (of the type $K + A$, $A \in \Sigma$), such that no two overlap. Then we can say (K, Σ) is a packing of K. We can define its density $\delta(K, \Sigma)$ suitably to reflect the "proportion" of space "occupied" by the translates of K and define $\delta^*(K) = \sup \delta(K, \Sigma)$ over all packings (K, Σ) to be the density of the best (general) packings of K. We then have

$$\delta(K) \leq \delta^*(K) \leq 1$$

and one can ask many natural questions, which form the subject of Packings in general.

If K is an n dimensional sphere and we write $\delta(K) = \delta_n$, $\delta^*(K) = \delta_n^*$. Then

$$\delta_n \leq \delta_n^* \leq 1.$$

A natural question is : is $\delta_n = \delta_n^*$? If $n = 2$. Thue (1892, 1910)[45], answered the question in the affirmative; but completely rigorous proofs were given by L. Fejes Toth[24] and Segre and Marhler independently only in the 1940's. For $n = 3$, every one believes $\delta_3 = \delta_3^*$, a belief which can be traced back to Kepler in 1637, but no proof has been established. (The recent claim of Hales is being examined). Partial results of various types are available.

COVERINGS

Motivated by the concept of packings we now introduce the concept of coverings.

Let K be a subset of R_n (usually closed and with volume $v(K)$). Let Σ be another (discrete) set in K. If the translates $K + A$, $A \in \Sigma$, of K cover R_n, we call (K, Σ) a covering by K. We can suitably define its density $\theta(K, \Sigma)$ to reflect the "ratio" of the amount of "covering material" to the covered space. Clearly $\theta(K, \Sigma) \geq 1$. If Σ is a lattice we say (K, Σ) is a lattice covering by K and Σ is a covering lattice for K. The density $\theta(K, \Sigma)$ comes out to be $v(K)/d(\Sigma)$. We define

$$\theta(K) = \inf\ \theta(K, \Sigma) : K(\Sigma) \text{ is a lattice covering by } K, \text{ and}$$

$$\theta^*(K) = \inf\ \theta(K, \Sigma) \text{ over all coverings } (K, \Sigma) \text{ by } K.$$

We call $\theta(K), \theta^*(K)$. The densities of the best (thinnest) lattice coverings and general coverings by K. Clearly

$$\theta(K) \geq \theta^*(K) \geq 1.$$

The concept of coverings might be lying hidden in earlier work and at least in the case of circles it was probably known to Dirichlet. But the earliest explicit mention seems to be due to Kershner[31] in 1939, who proved

$$\theta(K) = \theta^*(K) = \frac{2\Pi}{3\sqrt{3}} \text{ for circles } K.$$

Hlawka[29] explicitly introduced the concept of coverings and lattice coverings by general sets K in 1949 and proved some upper bounds for $\theta(K)$ for convex K.

Kershner's[31] results $\theta(K) = \theta^*(K)$ for circles was extended to symmetric plane convex domains K by L. Fejes Toth[24] and R. P. Bambah and C. A. Rogers[4] in early 1950's. Other proofs and some extensions have been given, among others, by R. P. Bambah, C. A. Rogers and H. Zassenhaus[5], as also R. P. Bambah and A. C. Woods[6]. These include various refinements which seem to have generated further work by many others.

That the result $\theta(K) = \theta^*(K)$ is not true for the larger class of plane star sets is due to R. P. Bambah, V. C. Dumir and R. J. Hans-Gill (1977)[3].

Let us denote by θ_n, θ_n^*, the values of $\theta(K), \theta^*(K)$, when K is an n-dimensional sphere (ball). Then as stated earlier θ_2, θ_2^* were determined by Kershner[31]. The value of θ_3 was determined by the author in 1954[2]. The geometric result has the following arithmetical interpretation :

Let $f(x, y, z) = ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy$ be a real positive definite quadratic form with det. D. Then there exists real α, β, γ, such that for all integers x, y, z,

$$f(x + \alpha, y + \beta, z + \gamma) \geq \left(\frac{125}{1024} D\right)^{1/3},$$

and the result is best possible.

Upper and lower bounds for θ_4, were obtained by the author[2] in 1954, who also conjectured the exact value for θ_4. This conjecture was proved in 1963 independently by Barnes and Dickson[9] in Australia and Delone and Ryskov in Russia. The value of θ_5 was determined by Ryskov and Baranovsky in 1975. The value of θ_4 for larger n is not known. Good upper and lower bounds of

θ_n, θ_n^* are available after the work of many workers, including H. Davenport[15], author, G. L. Watson[46], C. A. Rogers[40] and W. Schmidt.

Every one believes $\theta_3^* = \theta_3$. But as in the corresponding problem for packings, very little headway has been made. Some partial results are available, e.g., the author and A. C. Woods[6] have shown that the density of best covering by spheres does not improve if we allow "double" lattices as translation sets, and Coxeter, Few Rogers (1959) have shown

$$1.431 \ldots \leq \theta_3^* \leq \theta_3 = \frac{5\sqrt{5} - \pi}{24} = 1.464 \ldots$$

the R.H.S. being the author's 1954 result.

Bounds for $\theta(K)$, $\theta^*(K)$ for larger classes of sets K have been determined by many authors, including C. A. Rogers[40], W. Schmidt and the author.

By now a great deal of work has been generated on the covering problems and related areas.

MINKOWSKI'S CONJECTURE

We next talk about another covering theorem. I will first give its number theoretic formulation :

Minkowski's Theorem (31.12.1899) —

Let $L_1 = \alpha x + \beta y$, $L_2 = \gamma x + \delta y$, $\alpha, \beta, \gamma, \delta$ *real* $\Delta = \alpha\delta - \beta\gamma \neq 0$. *Then given any real* c_1, c_2, *there exist integers* x, y, *such that*

$$|(L_1 + c_1)(L_2 + c_2)| \leq \frac{1}{4}|\Delta|.$$

The result is obviously best possible.

In the geometric language, Minkowski's theorem states that every plane lattice of determinant 1 is a covering lattice for the region $|xy| \leq \frac{1}{4}$.

Although there is no written record of it, there is a general agreement that Minkowski believed in an n-dimensional analogue also, and the following conjecture is known as Minkowski's Conjecture.

Let

$$L_1 = a_{11} x_1 + \ldots + a_{1n} x_n$$

$$L_n = a_{n1} x_1 + \ldots + a_{nn} x_n$$

be n real linear forms with determinant $\Delta = \det(a_{ij}) \neq 0$. Then given real $c_1, \ldots, c_n$, there exist integers $x_1, \ldots, x_n$, with

$$|(L_1 + c_1) \ldots (L_n + c_n)| \leq \frac{1}{2^n}|\Delta|.$$

The case $n = 3$ was proved by Remark[39] in 1923 in two papers covering 62 printed pages. The proof was simplified by Davenport[16] in 1939 who completed the proof in 5 pages, including the introduction.

For $n = 4$, the first proof was given by Dyson[23] in 1948. He split the proof into two parts, say A & B. While proof of A was elementary, though complicated, the proof of part B used tools very sophisticated for the period and prople generally shied away from the problem. In 1956, Birch and Swinnerton-Dyer[11] showed that if Minkowski's conjecture is true for n, then it is true for $n + 1$ for at least a certain type of forms. Alan Woods and his students took up part A and in the 60's and 70's gave simple proof for $n = 4$ and also proved it for $n = 5$ and 6[47]. In 1974, Alan Woods and the author could give an elementary proof for part B for those forms not of the type covered by Birch and Swinnerton-Dyer[11], so that one now had an elementary, though not easy, proof also for the Minkowski conjecture for $n = 4$. About the same time, Skubenko[43] announced a proof of part B for $n = 4$ and 5 also. When the proof was published later, there was agreement that it was on the whole correct, especially when applied to the forms not covered by Birch Swinnerton-Dyer[11] theorem, but was far from clear or even complete. A. C. Woods and R. P. Bambah[6] in 1980 gave a version of the Skubenko proof for $n = 5$ for the relevant forms which is accepted as complete and clear. Thus Minkowski's conjecture has been proved for $n \leq 5$. For larger n the problem is still open.

A weaker theorem that shows the existence of a Minkowski type result for all n with $\left(\frac{1}{\sqrt{2}}\right)^n$ instead of $\frac{1}{2^n}$ was proved by Tchebotoroff[44] in 1940. The result has been improved by various authors, including Mordell, Davenport[15], Woods[47], Gruber, Bombieri, and more recently by Skubenko and his school. But the result is still far from the Minkowski conjecture.

The conjecture has naturally given rise to many variants and related problems to which many people, including the speaker and his collaborator A. C. Woods have made contributions.

Non-Homogeneous Indefinite Quadratic Form : Conjectures of Watson, Bambah, Dumir and HANS-GILL

Minkowski's theorem on the product of two inhomogeneous linear forms can be stated as a theorem on indefinite quadratic forms as follows :

Let $f(x, y) = ax^2 + 2bxy + cy^2$

be a real indefinite quadratic forms with $\det D = ac - b^2 < 0$. Then given real α, β, there exist integers x, y with

$$|f(x + \alpha, y + \beta)| \leq \sqrt{\frac{|D|}{4}}$$

(the result is best possible)

Davenport succeeded in 1948 in proving a similar best possible result from forms in 3 variables.

In his Ph.D. thesis, written under the direction of Davenport, Blaney[12] (1948) answered in the affirmative the question whether such results occur in higher number of variables. Before stating his result, we note that any non-degenerate quadratic form in n variables can be written as $\pm L_1^2 \pm L_2^2 \ldots \pm L_n^2$, where $L_1, \ldots, L_n$ are linearly independent linear forms. If the signs are all +ve, f is positive definite, if all negative, then it is negative definite. In case of indefinite forms

$f = L_1^2 + L_2^2 + \ldots + L_r^2 - L_{r+1}^2 - \ldots - L_{r+s}^2$ $r > 0$, $s > 0$, $r + s = n$. The form f can have many such representations, but for all such representations the values of r, s are the same. We call (r, s) the type of f and $\sigma = r - s$ its signature. Blaney proved :

For all $r \geq 1$, $s \geq 1$ there exist constants C depending on r, s with the following property :

Let f be a real indefinite quadratic form of the type (r, s). Given real $\alpha_1, \ldots, \alpha_n$, there exist integers $x_1, \ldots, x_n$ with

$$|f(x_1 + \alpha_1, \ldots, x_n + \alpha_n| \leq (C|D|)^{1/n}$$

If we write $C_{r,s} = \inf \{C : C \text{ has this property}\}$, the obvious question is to determine the value of $C_{r,s}$ for different r, s.

Blaney's theorem shows the existence of $C_{r,s}$. He also has an upper bound for $C_{r,s}$, which was improved by C. A. Rogers in 1952. The exact value of $C_{r,s}$ for various r, s was determined by various authors over the years :

$C_{1,1}$ (Minkowski 1899)[36]

$C_{1,2} = C_{2,1}$ (Davenport 1948)[15]

$C_{r,s}$ for all r (Birch 1958)[10]

$C_{r,s}$ for $r + s = 4$ (Dumir 1967)[17]

for $r + s = 5$ (R. J. Hans-Gill and Madhu Raka, 1979-80)[27].

Watson[46] showed that if $n \geq 21$,

$$C_{r,s} = \begin{cases} \frac{1}{4} \text{ if } \sigma = r - s = 0, \pm 1 \\ \frac{1}{3} \text{ if } \sigma = \pm 2 \\ \frac{1}{2} \text{ if } \sigma = \pm 3 \\ 1 \text{ if } \sigma = \pm 4 \end{cases}$$

and if $r + s = r' + s' = n$, $\sigma = r - s \equiv \sigma = r' - s'$ (mod $-$ 8) then $C_{r,s} = C_{r's'}$ the conjectured that the above holds for all $n = 4$.

Madhu Raka[38] (1982-83) proved the Watson conjecture for $\sigma = \pm 1, \pm 2, \pm 3, \pm 4$. To complete the proof of Watson's conjecture we needed only to prove that if $r + s = r' + s' = n$, $r - s \equiv r' - s'$ (mod 8), then $C_{r,s} = C_{r',s'}$. Using the result of Margulis in the homogeneous case and other devices, finally V. C. Dumir[21], R. J. Hand-Gill and A. C. Woods proved this result in 1994, thus completing the evaluation of $C_{r,s}$ for all r, s.

Since the indefinite quadratic forms take both positive and negative values, one can ask : Do there exist number Γ, such that if $f(x_1, \ldots, x_n)$ is a real indefinite quadratic form of type (r, s), then for all given real $\alpha_1, \ldots, \alpha_n$, there exist integers $x_1, \ldots, x_n$ with

$$0 < f(x_1 + \alpha_1, \ldots, x_n + \alpha_n) \leq (\Gamma |D|)^{1/n}$$

The existence of there number Γ was proved by Blaney in 1948. Define $\Gamma_{r,s} = \inf \{\Gamma : \Gamma$ with this property$\}$. Then the question is : what is the value $\Gamma_{r,s}$? $\Gamma_{1,1}$ was determined by Davenport and Heilbronn in 1947, $\Gamma_{2,1}$ by Blaney (1950), and Barnes (1961), $\Gamma_{1,2}$, $\Gamma_{3,1}$, $\Gamma_{2,2}$ by Dumir[17], $\Gamma_{1,3}$ by Dumir and Hans-Gill (1981), $\Gamma_{3,2}$, $\Gamma_{4,1}$ by Hans-Gill and Madhu Raka[18] (1981), $\Gamma_{2,3}$ by Bambah, Dumir and Hans-Gill[3] (1981), $\Gamma_{r,s}$ for $n \geq 6, -1 \leq \sigma \leq 3$ by Bambah, Dumir and Hans-Gill[3] (1984). Bambah, Dumir and Hans-Gill conjectured that if $r + s \equiv r' + s'$ and $r - s \equiv r' - s'$ (mod 8), then $\Gamma_{r,s} = \Gamma_{r',s'}$. They also conjectured the values of $\Gamma_{r,s}$ for all r, s. Flahive, Aggarwal and Gupta[1] confirmed the value of $\Gamma_{r,s}$ for various r, s. Finally, Dumir, Hans-Gill and Woods[21] (1994) proved the first part of the conjecture. As a result the value of $\Gamma_{r,3}$ was known for all r, s except $(r, s) = (2, 5)$, $(1, 4)$ and $(2, 4)$. The cases $(2, 5)$ and $(2, 4)$ have been settled by Dumir and Sehmi. The only case left is $(r, s) = (1, 4)$; where one knows $8 \leq \Gamma \leq 12$, it is expected that $\Gamma_{1,4} = 8$.

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