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EVALUATION OF INTEGRAL FORMING DIAGONAL ELEMENT OF MATRICES APPEARING IN PROBLEMS ON ELECTROMAGNETICS

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The paper presents the method of evaluation integral of which has been found to appear in all problems of evaluation of electrical capacitance of isolated metallic bodies of different geometrical shapes used in Satellite bodies and other electrical system. It is shown that the analysis based on consideration of (*i*) presence of an apparent singularity in and (*ii*) even functions for the integrand does not lead to a form suitable for general application. Method of derivation without consideration of these aspects is presented. The formulas derived for the general cases of rectangular plate and cylinder of finite length are applied to the cases of square plate and cylinder of infinite length and finite radius. The agreement of the formulas with those for limiting cases justifies the validity of the analysis.

Key Words : Integral Evaluation in Closed Form; Diagonal Element of Matrix; Moment Method in Electromagnetics; Integral Evaluation in the Presence of Apparent Singularity

1. INTRODUCTION

Moment method formulation based on pulse function and point matching has been widely used for evaluation of capacitance of metallic bodies isolated in free space and also metallic structures used in orbiting Satellites[1-5]. Method has been applied to a square plate on subdividing the surface into square subsections, and converting integral equation for the potential into a matrix equation which is inverted to obtain the desired result[1]. The integral appearing in the diagonal element of matrix based on square subsections has been evaluated in closed form and have been presented in eq. (2-31) of ref[1]. It is not always possible to subdivide the surface of the conductor into square or even almost square subsections. It is, therefore, essential to evaluate the integral, when the surface is divided into rectangular subsections. A rectangular plate is divided into planar rectangular subsections and curved surfaces are divided into curvillinear rectangular subsections[2-5]. In the case the number of subsections is very large, the curvlilinear subsections can be assumed to be planar for the purpose of analysis.

The expression for the integral in eq. (2.31) of ref.(1), appears to have singularity at the origin of two-dimensional co-ordinates and the integral appears to be an even function. When the subsections are rectangular, the limits of integration for the two variables of co-ordinates system are different.

It is shown in the present paper that the result of integration derived on the basis of above consideration leads to exactly the same result as eq. (2.31) of ref.[1] for a square plate. But when one of the dimensions of the rectangle tends to infinity as in the use of a cylinder of infinite length, the expression so obtained is not applicable.

It is also shown in the present paper that integrand in (2.31) of ref.[1] is not really an even function. The expression for the integral derived in closed form based on this concept can be applied to the special cases of (i) square plate and (ii) when one of the dimensions of the rectangular tends to infinity.

2. Analysis Based on Consideration of Even Function For The Integrand

Consider a planar rectangle of dimensions $L_1 \times L_2$ as shown in Fig. 1, and also a cylinder of finite length as shown in Fig. 2.

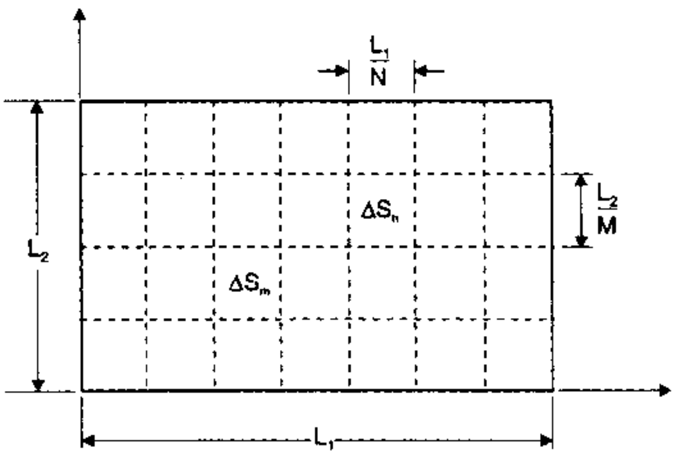

Fig. 1. A rectangular plate with subsections.

Considering the configuration of Fig. 1, the potential at any point is given by

$$\phi = \frac{1}{4\pi\varepsilon_0} \int_{-L_1/2}^{L_1/2} \int_{-L_2/2}^{L_2/2} \frac{\sigma(x', y')\, dx'\, dy'}{\sqrt{(x-x')^2 + (y-y')^2}}.$$

The surface of Fig. 1 is assumed to be divided into MN rectangular subsections each of sides $\frac{L_1}{N}, \frac{L_2}{M}$.

With $\phi = V$ for all points on the surface of the conductor, the integral equation for the unknown charge density $\sigma(x', y')$ assumes the form

$$V = \frac{1}{4\pi\varepsilon_0} \int_{-L_1/2}^{L_1/2} \int_{-L_2/2}^{L_2/2} \frac{\sigma(x', y')\, dx'\, dy'}{\sqrt{(x-x')^2 + (y-y')^2}} \quad \ldots (1)$$

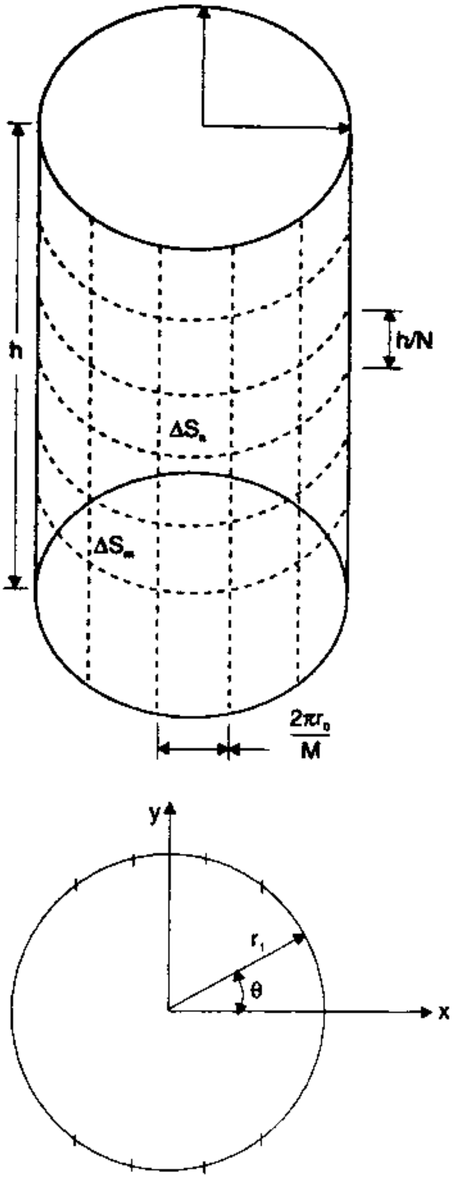

FIG. 2. An isolated metallic cylinder with subsections.

where ε_0 is the dielectric constant of free space.

The charge density $\sigma(x, y)$ is expanded in a series of pulse type basis functions as

$$\sigma(x, y) = \sum_{n=1}^{N} \alpha_n f_n \qquad \ldots (2)$$

where α_n are coefficients and the pulse function f_n is expressed as

$$f_n = \begin{cases} 1 & \text{on } \Delta S_n \text{ (}n\text{th subsection)} \\ 0 & \text{on any other } \Delta S_m \text{ (}m\text{th subsection)} \end{cases} \quad \text{... (3)}$$

f_n is the basis function in the application of the method of moments [1], and ΔS_n, ΔS_m are elementary rectangles having sides $\frac{L_1}{N}$ and $\frac{L_2}{M}$ as shown in Fig. 1. In Fig. 2 ΔS_n and ΔS_m represent curvilinear rectangular subsection having sides $\frac{2\pi r_0}{M}$ and $\frac{h}{N}$. The basis functions f_n are in the form of pulse functions over the subsection ΔS_n, where it is equal to unity and has zero value over any other subsection. The unknown function is then represented by a large number of discrete steps. In the limit ΔS_n is very very small, a close approximation to the desired function is obtained.

Substituting (2) and (3) in (1) and satisfying the resultant equation at the midpoint (x_m, y_m) of each subsection, which is point matching and taking the inner product of both sides of the resulting equation with Delta-function as the testing function[1], the following set of simultaneous equations are obtained

$$V = \sum_{n=1}^{MN} l_{mn}\,\alpha_n, \qquad m = 1, 2, \ldots MN$$

$$l_{mn} = \frac{1}{4\pi\varepsilon_0} \cdot \frac{\frac{L_1}{N} \cdot \frac{L_2}{M}}{\sqrt{(m-n)^2\left(\frac{L_1}{N}\right)^2 + (m-n)^2\left(\frac{L_2}{M}\right)^2}} \qquad m \neq n. \quad \text{... (4)}$$

Eq. (4) represents the nondiagonal elements of the matrix to be inverted. If $2a = \frac{L_1}{N}$, $2b = \frac{L_2}{M}$, the diagonal elements of the matrix assume the form[1]

$$l_{nn} = \frac{1}{4\pi\varepsilon_0} \int_{-a}^{+a} \int_{-b}^{+b} \frac{dx\,dy}{\sqrt{x^2 + y^2}} \quad \text{... (5)}$$

Considering the integrand as an even function, eq. (5) reduces to

$$l_{nn} = \frac{1}{\pi\varepsilon_0} \int_0^a \int_0^b \frac{dx\,dy}{\sqrt{x^2 + y^2}}.$$

Carrying out integration with respect to x,

$$l_{nn} = \frac{1}{\pi\varepsilon_0} \int_0^b \left[\ln \frac{a + \sqrt{a^2 + y^2}}{y} \right] dy.$$

Taking the lower limit of integration as δ, the expression for the diagonal element assumes the form

$$l_{nn} = \lim_{\delta \to 0} \frac{1}{\pi\varepsilon_0} \left[b \ln \left\{ \frac{a + \sqrt{a^2 + b^2}}{b} \right\} - \delta \ln \frac{2b}{\delta} + a \int_\delta^b \frac{dy}{\sqrt{a^2 + y^2}} \right]$$

$$= \frac{1}{\pi\varepsilon} \left[b \ln \left\{ \frac{a}{b} + \sqrt{1 + \frac{a^2}{b^2}} \right\} + a \ln \left\{ \frac{b}{a} + \sqrt{1 + \frac{b^2}{a^2}} \right\} \right].$$

Substituting $a = \frac{L_1}{2N}$; $b = \frac{L_2}{2M}$

$$l_{nn} = \frac{1}{\pi\varepsilon_0} \left[\frac{L_2}{2M} \ln \left\{ \frac{L_1}{L_2} \cdot \frac{M}{N} + \sqrt{1 + \left(\frac{L_1}{L_2} \frac{M}{N} \right)^2} \right\} + \frac{L_1}{2N} \ln \left\{ \frac{L_2}{L_1} \cdot \frac{N}{M} + \sqrt{1 + \left(\frac{L_2}{L_1} \frac{N}{M} \right)^2} \right\} \right] \qquad \dots (6)$$

For a square plate, $L_1 = L_2 = L, M = N$ and hence, the diagonal elements are obtained as

$$l_{nn} = \frac{L}{\pi\varepsilon_0 N} \ln (1 + \sqrt{2}) \qquad \dots (7)$$

which is the same as eq. (2-31) of reference [1].

For the structure of Fig. 2, $a = \frac{\pi r_0}{M}$, $b = \frac{h}{2N}$, the expression for the diagonal element assumes the form

$$l_{nn} = \frac{1}{2\pi\varepsilon_0} \left[\frac{h}{N} \ln \left\{ \frac{2\pi r_0}{h} \cdot \frac{N}{M} + \sqrt{1 + \left(\frac{2\pi r_0}{h} \frac{N}{M} \right)^2} \right\} + \frac{2\pi r_0}{M} \ln \left\{ \frac{h}{2\pi r_0} \cdot \frac{M}{N} + \sqrt{1 + \left(\frac{h}{2\pi r_0} \frac{M}{N} \right)^2} \right\} \right] \qquad \dots (8)$$

Eq. (8) is found to be valid for $h \approx 2\pi r_0$. But it is definitely not valid when h tends to infinity, since as $h \to \infty$, $l_{nn} \to \infty$. Thus the expression (8) and hence expression (6) are not correct expressions for the diagonal elements to be inverted.

The evaluation of integral (5) without considering the integrand as an even function should, therefore, be attempted.

3. Correct Analysis

Carrying out integration of (5) with respect to x first, without considering that the integrand is an even function, the following expression is obtained[6]

$$l_{nn} = \frac{1}{4\pi\varepsilon_0} \int_{-b}^{+b} \left[\left| \ln (x + \sqrt{x^2 + y^2}) \right|_{-a}^{0} + \left| \ln (x + \sqrt{x^2 + y^2}) \right|_{0}^{a} \right] dy$$

$$= \frac{1}{4\pi\varepsilon_0} \int_{-b}^{+b} \left[\ln y - \ln (-a + \sqrt{x^2 - y^2}) + \ln (a + \sqrt{x^2 + y^2}) - \ln y \right] dy$$

$$= \frac{1}{4\pi\varepsilon_0} \int_{-b}^{+b} \left[\ln (a + \sqrt{x^2 + y^2}) - \ln (-a + \sqrt{x^2 + y^2}) \right] dy$$

Carrying out integration by parts,

$$l_{nn} = \frac{1}{4\pi\varepsilon_0} \left[\left| y \ln (a + \sqrt{a^2 + y^2}) \right|_{-b}^{+b} - \int_{-b}^{+b} \frac{y^2}{a + \sqrt{a^2 + y^2}} \cdot \frac{dy}{\sqrt{a^2 + y^2}} - \left| y \ln (-a + \sqrt{a^2 + y^2}) \right|_{-b}^{+b} + \int_{-b}^{+b} \frac{y^2}{-a + \sqrt{a^2 + y^2}} \cdot \frac{dy}{\sqrt{a^2 + y^2}} \right]$$

$$= \frac{1}{4\pi\varepsilon_0} \left[2b \ln \frac{a + \sqrt{a^2 + b^2}}{-a + \sqrt{a^2 + b^2}} + \int_{-b}^{+b} \left(\frac{y^2}{-a + \sqrt{y^2 + a^2}} - \frac{y^2}{a + \sqrt{y^2 + a^2}} \right) \frac{dy}{\sqrt{a^2 + y^2}} \right]$$

$$= \frac{2}{4\pi\varepsilon_0} \left[b \ln \frac{a + \sqrt{a^2 + b^2}}{-a + \sqrt{a^2 + b^2}} + a \int_{-b}^{+b} \frac{dy}{\sqrt{a^2 + y^2}} \right]$$

$$= \frac{2}{4\pi\varepsilon_0} \left[b \ln \frac{a + \sqrt{a^2 + b^2}}{-a + \sqrt{a^2 + b^2}} + a \left| \ln y + \sqrt{a^2 + y^2} \right|_{-b}^{b} \right]$$

$$= \frac{1}{2\pi\varepsilon_0} \left[b \ln \frac{a + \sqrt{a^2 + b^2}}{-a + \sqrt{a^2 + b^2}} + a \ln \frac{b + \sqrt{a^2 + b^2}}{-b + \sqrt{a^2 + b^2}} \right]$$

Substituting $a = \frac{L_1}{2N}$; $b = \frac{L_2}{2M}$, the diagonal element of the matrix for a rectangular plate is of the form

$$l_{nn}=\frac{1}{4\pi\varepsilon_0}\left[\frac{L_2}{M}\ln\frac{\frac{L_1}{N}+\sqrt{\left(\frac{L_1}{N}\right)^2+\left(\frac{L_2}{M}\right)^2}}{-\frac{L_1}{N}+\sqrt{\left(\frac{L_1}{N}\right)^2+\left(\frac{L_2}{M}\right)^2}}+\frac{L_1}{N}\ln\frac{\frac{L_2}{M}+\sqrt{\left(\frac{L_1}{N}\right)^2+\left(\frac{L_2}{M}\right)^2}}{-\frac{L_2}{M}+\sqrt{\left(\frac{L_1}{N}\right)^2+\left(\frac{L_2}{M}\right)^2}}\right] \quad \ldots (9)$$

For a cylinder of finite axial length, $a=\frac{\pi r_0}{M}\,;\, b=\frac{h}{2N}$ the diagonal element assumes the form

$$l_{nn}=\frac{1}{2\pi\varepsilon_0}\left[\frac{h}{N}\ln\frac{\frac{2\pi r_0}{M}+\sqrt{\left(\frac{2\pi r_0}{M}\right)^2+\left(\frac{h}{N}\right)^2}}{-\frac{2\pi r_0}{M}+\sqrt{\left(\frac{2\pi r_0}{M}\right)^2+\left(\frac{H}{N}\right)^2}}\right.$$

$$\left.+\frac{2\pi r_0}{M}\ln\frac{\frac{h}{N}+\sqrt{\left(\frac{h}{N}\right)^2+\left(\frac{2\pi r_0}{M}\right)^2}}{-\frac{h}{N}+\sqrt{\left(\frac{h}{N}\right)^2+\left(\frac{2\pi r_0}{M}\right)^2}}\right] \quad \ldots (10)$$

4. DISCUSSION

The two forms for the expressions (6), (9) and (7), (10) for the diagonal elements of rectangular plate and cylinder of finite length respectively have been found to lead to almost identical values of numerical result on capacitance for $\frac{L_1}{L_2}\approx 4$ and $\frac{h}{r_0}\approx 25$. There is, however, one basic difference between the two forms. Considering the case of cylinder, for example one may try to apply the formulation for the capacitance of a single isolated cylinder or a pair of cylinders for the case the length of the cylinder tends to infinity. From the expression (8) it is found that as $h\to\infty, l_{nn}\to\infty$, even when $r_0\to 0$. In eq. (10), when h is taken outside the square bracket, 2nd term vanishes as $h\to\infty$. The limit of resulting first term multiplied by h tends to zero as $h\to\infty$. Thus the eq. (10) and hence eq. (9) are the correct expressions for the diagonal element, used widely for evaluation of electrical parameters of isolated conductors[1-5], pair of cylinders[7] and also multi-conductor lines using pulse function and point matching[8].

In the particular case $L_1=L_2=L, M$ and N are identical. Applyig these conditions in eq. (9) it is found that the expression of the diagonal element for square subsection of a square plate reduces to $l_{nn}=\frac{1}{4\pi\varepsilon}\cdot\frac{2L}{N}\ln\frac{\sqrt{2}+1}{\sqrt{2}-1}=\frac{1}{\pi\varepsilon}\cdot\frac{L}{N}\ln(\sqrt{2}+1)$, which is the same as eq. (7) and hence (2-31) of ref.

(1). It is worthwhile to mention that derivation of $l_{nn} = \frac{1}{\pi\varepsilon} \cdot \frac{L}{N} \ln(\sqrt{2} + 1)$ which is the result given in eq. (2-31) of ref. (1) has been obtained from the integral $\int \frac{dx}{\sqrt{a^2 + x^2}} = \ln(x + \sqrt{a^2 + x^2})$ given in 200.01 of ref. [8]. This form of integral is different from eq. (4) and hence eq. (5) of the present paper. The discrepancy for $h \to \infty$ can be attributed to this deviation in the method of evaluation of the integral

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REFERENCES

1. R. F. Harrington, *Field Computation by Moment Methods*, MacMillan, New York, 1968.
2. B. N. Das, S. B. Chakrabarty and A. K. Mallick, *Indian J. Radio Space Phys.* **24** (1995) pp. 151-57.
3. B. N. Das and S. B. Chakrabarty, *Indian J. Radio Space Phys.* **26** (1997) pp. 112-15.
4. B. N. Das and S. B. Chakrabarty, *IEEE Trans. on EMC*, **39** (1997) pp. 371-74.
5. B. N. Das and S. B. Chakrabarty, *IEEE Trans. EMC*, **39** (1997) No. 4, pp. 390-93.
6. Herbert Bristol Dwight, *Tables of Integrals and other Mathematical Data*, The Macmillan Company, New York 1961, formula 200.01, page 50.
7. B. N. Das and S. B. Chakrabarty, *Proc. IEE, Science, Measurement and Technology*, **44** (1997) No. 6, pp. 280-86.
8. J. C. Clements, C. R. Paul and A. T. Adams, *IEEE Trans. EMC*, **17** (1975) pp. 238-48.