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WEAKLY REGULAR NEAR-RINGS

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A near-ring N is said to be left weakly regular if $a \in (Na)^* (Na)$ for all $a \in N$. N is said to be left w-weakly regular (weak-weakly regular) if for any $x \in N$, $x = ux$ for some $u \in \langle x \rangle$. These two notions coincide in the case of rings with identity, but not in near-rings with identity. We have shown that a reduced near-ring N is left w-weakly regular if and only if N/P is a simple domain for every prime ideal P of N. We have also shown that if N is a reduced near-ring then N is weakly regular if and only if every ideal is completely semiprime and N/P is left weakly regular for all prime ideals P of N. If N is a reduced left w-weakly regular near-ring then for any proper ideal P, the following conditions are equivalent: (*i*) P is prime, (*ii*) P is completely prime and (*iii*) P is maximal.

Key words : Near-Rings-weakly Regular; Weak weakly Regular; Prime Ideal; Maximal Ideal

INTRODUCTION

The notion of weakly regular rings was introduced by Ramamurthi[1] and significant results have been obtained on weakly regular rings by Victor Camillo and Yufei Xiao[2]. In this paper, we introduce the notion of w-weakly regular near-ring and give several characterization of w-weakly regular near-ring. Victor Camillo and Yufei Xiao[2] have shown that for a reduced weakly regular ring R, R/P is a simple domain for every prime ideal P of R. We have shown that the converse is also true even in the case of near-rings. We have also shown that if N is a reduced near-ring then N is left weakly regular if and only if every ideal is completely semiprime and N/P is left weakly regular for all prime ideals P of N.

Throughout this paper N stands for a zerosymmetric right near-ring with identity. For any subsets A, B of N, $A * B$ denotes the set of all finite sums of the form $\Sigma\, a_k b_k$ with $a_k \in A, b_k \in B$. For any $x \in N$, $\langle x \rangle$ stands for the principal ideal of N generated by x. Following V.S. Ramamurthy[1], a ring R is left weakly regular if and only if $a \in (Ra) * (Ra)$ for all $a \in R$. If R contains an identity then R is left weakly regular if and only if for any a in R, $a = ua$ for some $u \in \langle a \rangle$. A near-ring N

is said to be left weakly regular if $x \in (Nx) * (Nx)$ for all $x \in N$. N is said to be left w-weakly regular (weak-weakly regular) if for any $x \in N, x = ux$ for some $u \in \langle x \rangle$. These two notions coincide in the case of rings with identity. But in the case of near-rings, left weakly regular always implies left w-weakly regular.

Example : On the Dihedral Group

$\{0, a, 2a, 3a, b, a+b, 2a+b, 3a+b\}$ define multiplication as follows :

$\cdot$	0	a	$2a$	$3a$	b	$a+b$	$2a+b$	$3a+b$
0	0	0	0	0	0	0	0	0
a	0	a	$2a$	$3a$	b	$a+b$	$2a+b$	$3a+b$
$2a$	0	$2a$	0	$2a$	0	0	0	0
$3a$	0	$3a$	$2a$	a	b	$a+b$	$2a+b$	$3a+b$
b	0	b	$2a$	b	b	$a+b$	b	$3a+b$
$a+b$	0	$a+b$	0	a	0	0	$2a$	0
$2a+b$	0	$2a+b$	$2a$	$2a+b$	b	$a+b$	b	$3a+b$
$3a+b$	0	$3a+b$	0	$a+b$	0	0	$2a$	0

This example is the scheme (6) is Pilz[3] (p. 345).

$$(a+b) \notin N(a+b)$$

but $(a+b) = (2a+b)(a+b)$ with $(2a+b) \in (a+b)$.

An ideal P of N is called completely prime (completely semiprime) if $ab \in P$ ($a^2 \in P$) implies $a \in P$ or $b \in P$ ($a \in P$). N is said to be a reduced near-ring if it is without nonzero nilpotent elements. If N is a zerosymmetric reduced near-ring, then by Pilz[3] (*p.* 290), N has IFP and for any x, y in N, $xy = 0$ implies $yx = 0$. N is called biregular if for every $x \in N$, $\langle x \rangle = eN$ for some central idempotent e. A near-ring N is said to be simple domain if it has no non-trivial ideals and if it has no non-zero divisors of zero. For the basic terminology and notation, we refer to Pilz[3].

Section I

Lemma 1 — If a near-ring N is left weakly regular then $I = I^2$ for every ideal I of N.

Lemma 2 — Every two-sided ideal and every quotient near-ring of a left weakly regular near-ring is left weakly regular. On the other hand if a near-ring N has a two-sided ideal I such that I and N/I are both left weakly regular then N is left weakly regular.

PROOF : It is clear from the definition that if N is left weakly regular then every quotient near-ring of N is also left weakly regular. Also if I is a two-sided ideal of N then I is also left weakly regular. On the other hand suppose that the near-ring N has a two sided-ideal I which is left weakly regular and that the quotient near-ring N/I is also left weakly regular. Let $x \in N$. Since N/I is left weakly regular,

$$\bar{x} \in (\bar{N}\bar{x}) * (\bar{N}\ \bar{x}).\ \text{Hence,}\ \bar{x} = \bar{x}_1\,\bar{n}_1\,\bar{x}\,\bar{n}_2\,\bar{x} + \bar{n}_3\,\bar{x}\,\bar{n}_4\,\bar{x} + \ldots + \bar{n}_k\,\bar{x}\,\bar{n}_{k+1}\,\bar{x}$$

for some $n_i \in N, i = 1, 2 \ldots. k + 1$. (i.e.) $x + I = (n_1 x n_2 x + I) + (n_3 x n_4 x + I) + \ldots + (n_k x n_{k+1} x + I) = (n_1 x n_2 x + \ldots + n_k x n_{k+1} x) + I$. Let $n_1 x n_2 x + \ldots + n_k x n_{k+1} x = x'$. Then $x + I = x' + I$, which implies $x - x' \in I$. Since I is left weakly regular, $x - x' \in [I(x - x^1)] * [I(x - x')]$. Now we claim that $[I(x - x')] * [I(x - x')] \subset (Nx) * (Nx)$. For, let $y \in [I(x - x')[* [I(x - x')]$. Then $y = z_1 (x - x') z_2 (x - x') + \ldots. + z_k (x - x') z_{k+1} (x - x')$ for some $z_j \in I, j = 1, 2 \ldots k + 1$. Now, $z_j (x - x') z_{j+1} (x - x') = z_j [x - (n_1 x n_2 x + \ldots + n_k x n_{k+1} x)] z_{j+1} [x - (n_1 x n_2 x + \ldots + n_k x n_{k+1} x)] = z_j (1 - (n_1 x n_2 + \ldots + n_k x n_{k+1}))] x z_{j+1} [(1 - (n_1 x n_2 + \ldots. + n_k x n_{k+1}))] x \in (Nx) * (Nx)$. Hence, $y \in (Nx) * (Nx)$. Thus $[I(x - x')] * [I(x - x')] \subseteq (Nx) * (Nx)$. Now, $x - x' \in (N(x) * (Nx)$. Clearly $x' \in (Nx) * (Nx)$. Hence, $x \in (Nx) * (Nx)$ so that N is left weakly regular.

Lemma 3 — Every ideal of a reduced left w-weakly regular near-ring N is completely semiprime,

PROOF : Suppose I is an ideal of N and $a^2 \in I$. Then $a^2 = ua^2$ for some $u \in \langle a^2 \rangle$. Hence, $(a - ua)\ a = 0$. Since N is reduced, $a(a - ua) = 0$. Hence, $(a - ua)^2 = 0$ which implies $a = ua \in I$.

Corollary 1 — Every ideal of a reduced left weakly regular near-ring is completely semiprime.

Lemma 4 — If I is a completely semiprime ideal of N then $ab \in I$ implies $ba \in I$.

PROOF : $(ba)^2 = b(ab)\,a \in I$ implies $ba \in I$.

Theorem 1 — *Let N be a reduced near-ring. N is left weakly regular if and only if*

(i) *Every ideal is completely semiprime.*

(ii) *N/P is left weakly regular for all prime ideals P of N.*

PROOF : Let us assume that N is left weakly regular. Part (i) alone follows from corollary 1 and (ii) follows from lemma 2.

Conversely, assuming that conditions (*i*) and (*ii*) are true, let us show that N is left weakly regular. If not, then there is an $x \in N$ such that $x \notin (Nx) * (Nx)$. Let $S = \{$completely semiprime ideals I of $N/x \notin (Nx) * (Nx) + I\}$. Since $(0) \in S$, $S \neq \varphi$. By Zorns lemma, S has a maximal element P with respect to the property that $x \notin (Nx) * (Nx) + P$. Thus P is not prime from (ii). So there exist ideals A, B of N such that $P \subset A, P \subset B$ but $AB \subseteq P$. Let $K = \{n \in N/nB \subseteq P\}$ and $L = \{n \in N/nK \subseteq P\}$. Clearly K is an ideal and hence L is also an ideal. Clearly, $A \subseteq K$. Since $KB \subseteq P$ and since P is completely semiprime $BK \subseteq P$. Hence, $B \subseteq L$. So $P \subset K$, $P \subset L$ but $K \cap L = P$. Because of the maximality of $P, x \in (Nx) * (Nx) + K$ and

$x \in (Nx) * (Nx) + L$. So $x - e_1 x \in K$ and $x - e_2 x \in L$ for some $e_1, e_2 \in (Nx) * N$. Let $e = e_1(1 - e_2) + e_2$. Then $x - ex = x - [e_1(1 - e_2) + e_2]x = x - [e_1(1 - e_2)x + e_2 x] = x - e_2 x - e_1(x - e_2 x) \in L$. Since $(1 - e_1)x \in K$, $x(1 - e_1) \in K$ as K is completely semiprime. Consider $[(1 - e_1)(x - e_2 x)]^2 = (1 - e_1)\ [(x - e_2 x)(1 - e_1)\ (x - e_2 x) = (1 - e_1)\ [x(1 - e_1) - e_2 x(1 - e_1)]\ (x - e_2 x) \in K$, since $x(1 - e_1) \in K$. Now, since K is completely semiprime $(1 - e_1)(x - e_2 x) \in K \Rightarrow x - [e_1(1 - e_2) + e_2]x \in K \Rightarrow x - ex \in K$. Thus $x - ex \in K \cap L = P$, a contradiction.

Lemma 5 — If I is a completely semiprime ideal of N then $(I : S) = \{n \in N/nS \subseteq I\}$ is an ideal for any subset S of N.

Lemma 6 — If I is a completely semiprime ideal of N then $x_1 x_2 .. x_n \in I$ implies $\langle x_1 \rangle \langle x_2 \rangle \ldots \langle x_n \rangle \subseteq I$.

PROOF : Suppose $x_1 x_2 \ldots x_n \in I$ for some $x_1, x_2, \ldots, x_n$ in N. Then by Lemma 5, $(I : x_1 x_2 \ldots x_n)$ is an ideal. Since $x_1 \in (I : x_2 x_3 \ldots x_n)$, we have $< x_1 > \subseteq (I : x_2 x_3 \ldots x_n)$ so that $\langle x_1 \rangle x_2 \ldots . x_n \subseteq I$. Hence, $x_2 x_3 \ldots . x_n \langle x_1 \rangle \subseteq I$. Since $x_2 \in (I : x_3 x_4 \ldots x_n < x_1 \rangle)$, we have $\langle x_2 \rangle \subseteq (I : x_3 x_4 \ldots x_n, \langle x_1 \rangle)$ so that $\langle x_2 \rangle x_3 \ldots x_n \langle x_1 \rangle \subseteq I$. Hence, $x_3 x_4 \ldots . x_n \langle x_1 \rangle \langle x_2 \rangle \subseteq I$. Continuing this process we get $\langle x_1 \rangle \langle x_2 \rangle \ldots \langle x_n > \subseteq I$.

Theorem 2 — *If N is a reduced left w-weakly regular near-ring then for any proper ideal P of N, the following are equivalent*:

(*i*) *P is prime,*

(*ii*) *P is completely prime and*

(*iii*) *P is maximal.*

PROOF : (i) $\Rightarrow$ (ii) Suppose $ab \in P$. By lemma 6, $\langle a \rangle \langle b \rangle \subseteq P$ which implies $\langle a \rangle \subseteq p$ or $\langle b \rangle \subseteq p$. This implies $a \in P$ or $b \in P$

(ii) $\Rightarrow$ (i) is obvious.

(i) $\Rightarrow$ (iii) — Let P be a proper prime ideal. Suppose P is properly contained in an ideal M. Let $x \in M/P$. Then $x = yx$ for some $y \in \langle x \rangle$. Now for any $n \in N$, $nx = nyx$ so that $(n\text{-}ny)\ x = 0$. By Lemma 6, $\langle n - n \rangle \langle x \rangle = 0 \subseteq P$. Since P is prime and $x \notin P$, we have $n\text{-}ny \in P \subset M$. Further $y \in \langle x \rangle \subseteq M$ whence $ny \in M$ so that $n \in M$. Hence, $M = N$. Thus P is maximal.

(iii) $\Rightarrow$ (i) is obvious.

SECTION 2

Victor Camillo and Yufei Xiao[2] have shown that if R is a reduced weakly regular ring then R/P is a simple domain for every prime ideal P. We have shown that this result is true in the case of near-rings and the converse is also true even in the case of near-rings.

Lemma 7 — In a reduced near-ring N, $A(x) = \text{ann}\ (x) = \{y \in N/yx = 0\}$ is an ideal.

PROOF : Clearly $A(x)$ is a left ideal. Let $n \in N$ and $y \in A(x)$. Then $y\ n\ x = 0$, since N satisfies IFP. Hence $y n \in A(x)$ so that $A(x) N \subseteq A(x)$. Thus $A(x)$ is a right ideal.

Theorem 3 — *A reduced near-ring N is left w-weakly regular if and only if N/P is a simple domain for every prime ideal P of N.*

PROOF : From Theorem 2, clearly N/P is a simple domain for every prime ideal P of N.

Conversely, let N/P be a simple domain for every prime ideal P of N. Let $0 \neq a \in N$. By P. Dheena[4] (Lemma 1) $\mathcal{N} = N/A\ (a)$ is reduced and $\bar{a}$ is not a zero divisor. Also every proper completely prime ideal of $\mathcal{N}$ is a maximal ideal of $\mathcal{N}$. Now let M be the multiplicative semigroup generated by all elements $\bar{a} - \bar{x}\,\bar{a}$ where $x \in \langle a \rangle$. We claim that $\bar{0} \in M$. If not, by Dheena[4] (Corollary 1) there exists a completely prime ideal $\mathcal{P}$ with $\mathcal{P} \cap M = \varphi$. Suppose $\langle \bar{a} \rangle \subseteq \mathcal{P}$. Then $\bar{a} \in \mathcal{P}$. Now for any $x \in \langle a \rangle$, $\bar{a} - \bar{x}\,\bar{a} \in \mathcal{P} \cap M$, a contradiction. Suppose $\langle \bar{a} \rangle \not\subset \mathcal{P}$. Since P is maximal $\mathcal{P} + \langle \bar{a} \rangle = \mathcal{N}$. Hence $\bar{1} = \bar{\alpha} + \bar{x}$ for some $\bar{\alpha} \in \mathcal{P}$ and for some $\bar{x} \in \langle \bar{a} \rangle$. So $\bar{1} - \bar{x} = \bar{\alpha}$ and hence $\bar{a} - \bar{x}\,\bar{a} = \bar{\alpha}\,\bar{a} \in \mathcal{P} \cap M$, a contradiction. Thus $\bar{0} \in M$. Now $\bar{0} = \overline{(a - \bar{x}_1\,\bar{a})}\,\overline{(a - \bar{x}_2\,\bar{a})} \ldots \overline{(a - \bar{x}_n\,\bar{a})}$ where $x_i \in \langle a \rangle$. Since $\mathcal{N}$ is reduced and $\bar{a}$ is not a zero divisor, we have $(\bar{1} - \bar{x}_1)(\bar{1} - \bar{x}_2) \ldots (\bar{1} - \bar{x}_n) = \bar{0}$. We claim that $\bar{1} = \bar{x}$ for some $x \in \langle a \rangle$. We prove it by taking n as 2. Now

$$(\bar{1} - \bar{x}_1)(\bar{1} - \bar{x}_2) = \bar{0} \Rightarrow \bar{1} - \bar{x}_2 - \bar{x}_1(\bar{1} - \bar{x}_2) = \bar{0}.$$

Thus $\bar{1} = \bar{x}_1(\bar{1} - \bar{x}_2) + x_2$.

Hence, $1 + P = [x_1(1 - x_2) + x_2] + P$.

Since $x_1, x_2 \in \langle a \rangle$, $x_1(1 - x_2) + x_2 \in \langle a \rangle$. Let $x = x_1(x_1 - x_2) + x_2$.

Then $\bar{1} = \bar{x}$ for some $x \in \langle a \rangle$.

Hence $1 - x \in A(a)$. Thus $a = xa$ for some $x \in \langle a \rangle$. Thus N is left w-weakly regular.

Theorem 4 — *A near-ring N is reduced and left w-weakly regular if and only if for every x in N, ann (x)* $\oplus \langle x \rangle = N$.

PROOF : Assume that ann $(x) \oplus \langle x \rangle = N$. Let $x \in N$ such that $x^2 = 0$. Then $x \in$ ann (x). Clearly $x \in \langle x \rangle$. Hence $x = 0$. Thus N is reduced. Also $x = 1 \cdot x = (y_1 + y_2)x$ (for some $y_1 \in$ ann (x) and $y_2 \in \langle x \rangle$) $= y_1 x + y_2 x = y_2 x$ which implies $x = y_2 x$ with $y_2 \in \langle x \rangle$. Thus N is left w-weakly regular.

Conversely, let us assume that N is reduced and left w-weakly regular. Let $x \in N$. Then there is an $a \in \langle x \rangle$ such that $x = ax$. So $1-a \in$ *ann*(x). This implies ann $(x) + \langle x \rangle = N$. Now let $y \in$ ann $(x) \cap \langle x \rangle$. Then $yx = 0$ and $y \in \langle x \rangle$. Since N is left w-weakly regular, $y = uy$ for some $u \in \langle y \rangle$. Since $yx = 0$, $\langle y \rangle \langle x \rangle = 0$ which implies $uy = 0 \Rightarrow y = 0$. Thus ann $(x) \cap \langle x \rangle = 0$.

Lemma 8 — If N is a reduced near-ring such that $N = A \oplus B$ where A and B are ideals of N then there exists a central idempotent e such that $A = eN$.

PROOF : Since $N = A \oplus B$, $a + b = 1$ for some $a \in A$ and $b \in B$. As A and B are ideals we have ab, $ba \in A \cap B$ and so $ab = 0 = ba$. Now $a = 1.\ a = (a + b)\ a = a^2$. Similarly, $b = b^2$. Hence, a and b are idempotents. We now show that $A = aN$. Obviously $aN \subseteq A$. Let $x \in A$. Then $x = (a + b)\ x = ax$. Therefore, $x \in aN$. This shows that $A = aN$. For any $n \in N$, $(an - ana)\ a = 0$. Hence, $a(an - ana) = 0$. Thus $(an- ana)^2 = 0$. Hence, $an = ana$. Also $1-a = a + b - a \in B$. Hence, $(1 - a)\ na \in A \cap B$ which implies na = ana. Thus a is central.

Lemma 9 — In a reduced near-ring, N is biregular if and only if N is left w-weakly regular.

PROOF : By Theorem 4 and lemma 8, N is left w-weakly regular implies N is biregular.

Conversely, for any $x \in N, \langle x \rangle = eN$. Hence, $x = en$. Thus $x = ex$ where $e \in \langle x \rangle$. Hence, N is left w-weakly regular.

Corollary 2 — If the ring R is reduced then it is biregular if and only if R is left weakly regular.

Theorem 5 — *A reduced near-ring N is left w-weakly regular if and only if for each prime ideal P of N,* $P = \bigcup_{x \in N/P} ann(x)$ *and P is completely prime.*

PROOF : Let N be a reduced near-ring. Suppose N is left w-weakly regular. Let P be a prime ideal and let $Q = \bigcup_{x \in N/P}$ ann(x). Then Q is an ideal. Let $a \in Q$. Then $ax = 0$ for some $x \in N \setminus P$. Hence, $ax \in P$. Since p is completely prime $a \in p$. Thus $Q \subseteq p$. Now let $x \in p$. Then $x = ux$ for some $u \in \langle x \rangle$ as N is left w-weakly regular. Hence, $(1 - u)x = 0$ which implies $x \in$ ann $(1 - u)$ and $1 - u \in N \setminus P$. Therefore, $x \in Q$. Thus $P \subseteq Q$.

Conversely, let $P = \bigcup_{x \in N \setminus P} ann(x)$ for each prime ideal P. Let us show that N is left w-weakly regular. In view of Theorems 2 and 3, it is enough to prove that P is maximal. Let $M = \{1\}$. Let $S = \{I\ \Delta N/I \cap M = \varphi$ and $P \subseteq I\}$. By Zorn's lemma there exists a maximal element say M' such that $P \subseteq M'$. Since M' is maximal it is prime by Pilz[3] (Corollary 2.72). Hence, $M' = \bigcup_{x \in N \setminus M} ann(x) \subseteq \bigcup_{x \in N \setminus P} ann(x) = P$. Therefore $M' \subseteq P$. Hence P is maximal.

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