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EFFECTS OF NONLINEAR AND LINEARIZED COUPLING ON THE DYNAMIC THERMOELASTIC RESPONSE OF A THIN ROD

M. J. RODGERS[1] AND L. M. BROCK[2]

[1]*McCormick School of Engineering and Applied Science, Northwestern University, Evanston, Illinois* 60208 *U.S.A.*

[2]*Engineering Mechanics, University of Kentucky, Lexington, Kentucky* 40506 *U.S.A.*

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The dynamic steady-state thermoelastic response of a thin, infinitely-long insulated rod to a thermo-mechanical load moving along it at a constant speed is analyzed. The governing set of differential equations follow from the momentum balance, thermoelastic constitutive law and heat equation for the rod, and the thermoelastic coupling between them is often ignored or weakened. Here, however, three different cases are chosen to illustrate the effects of coupling — partial coupling, linearized full coupling, and nonlinear full (exact) coupling. A fourth linearized coupling case based on a heat equation of the non-Fourier type is also treated.

Exact closed-form solutions for the temperature change and displacement are presented for each case, and show that thermoelastic coupling — and its type — clearly influence rod behaviour. In particular, coupling introduces a transonic speed range for the moving thermo-mechanical load which produces rod responses that differ markedly from those occurring for loads moving in the sub- and supersonic ranges. For the non-Fourier heat law case, for example, loads moving in the transonic range can propagate sinusoidal temperature change and strain fields, which suggests the possibility of a dynamic fatigue mechanism for the rod.

Key Words : Thermoelastic Coupling; Non-Fourier Heat Equation; Nonlinear Coupling; Sub/trans/Supersonic Speeds; Temperature and Displacement

INTRODUCTION

The role of fibers in modern communication systems, superconductors and composites has placed new emphasis on the modelling of thin rods in environments where rapidly-applied thermal and mechanical loading may both occur. Therefore, accurate thermoelastic responses are required. Considerations of thermodynamic principles and small-strain elasticity have produced[1-4] a set of differential equations for the displacements and (absolute) temperature. These equations are thermoelastically coupled in the sense that the temperature change gradient appears in the momentum balance equation while, simultaneously, the absolute temperature and time derivatives of the strains form a product in the equation for temperature, thereby introducing a non-linear coupling of otherwise linear equations. The temperature equation itself is generally based on the classical[5,6] Fourier law of heat conduction.

Because the changes in absolute temperature in mechanical systems are often small, its (given) initial value is generally[1-4] used in the aforementioned product term. In application, moreover, the equations are often partly uncoupled by neglecting the product term altogether[1-4]. In a static (equilibrium) analysis, this simplification is strictly valid because the strain time derivatives disappear.

This article, however, considers the effects of full nonlinear and linearized coupling in the thermoelastic response of a thin rod to rapidly-applied thermal and mechanical loads. In addition, for the linearized case, a temperature equation of the Jeffreys[7] type is treated. This equation allows the possibility that heat propagates in a wave-like manner, whereas the Fourier-based equation by itself predicts that a thermal source will instantaneously establish a temperature field everywhere[5,6], albeit often with exponential decay with distance from the source. It is chosen because it is a generalization of the well-known Cattaneo non-Fourier equation[8] and can be easily reduced to either that equation or even a Fourier-based equation. In what follows, therefore, it is referred to as the J-C equation, and the associated rod analysis as the J-C model.

To focus more readily upon key differences in the various cases, the thin rod is insulated and infinite in length, and the thermo-mechanical loads move with a constant speed along the rod. This will create an essentially 1-D steady-state for the temperature change and (axial) displacement. Moreover, the rod properties are themselves independent of temperature. The speed of the load is *a priori* arbitrary i.e., it need not be subsonic. Insulating the rod means that no heat is lost or gained except perhaps at the moving loads site. While it is argued[2] that heat exchange by convection/radiation is more realistic, its adoption requires knowledge of additional parameters that are generally specific to a situation. The case of a non-insulated rod is to be treated, therefore, in a sequel.

The study begins in the next section with the adaptation of the Fourier model-based nonlinearly coupled differential equations for an isotropic homogeneous thermoelastic body[1-4] to the dynamic steady-state 1-D rod problem. The partly-coupled and linearly-coupled versions of these equations are then treated as cases 1 and 2, before addressing the exact (nonlinearly coupled) equations themselves as case 3. The linearly-coupled equations for the J-C model are then introduced and solved as case 4.

Nonlinearly Coupled Equations Based on the Fourier Model

Consider an insulated straight rod of uniform circular cross-section. The rod is initially at rest at a uniform absolute temperature $T_0(K)$, and is assumed long enough to be considered for analytical purposes as infinite in length. As indicated schematically in Figure 1, a ring is then placed about the rod, and moved along it with a constant speed v. The sliding of the ring generates a net axial force through shear, and perhaps an associated heat gain/loss. We assume that the ring and the rod are thin enough in comparison to the actual rod length that these thermo-mechanical loads can be treated as point sources, and that the rod behaves in an essentially 1-D manner. Thus, we locate positions on the rod with respect to the initial ring location

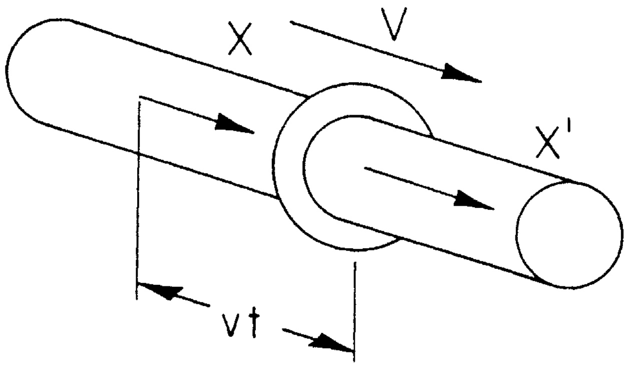

Fig. 1. Schematic of ring sliding on insulated rod.

by the variable x, measured positive in the direction of ring travel, and figure time t from the instant that the ring is affixed. The governing differential equations for all x and $t > 0$ can then be obtained[1-4] as

$$\frac{\partial\sigma}{\partial x} + 2\frac{\tau_0}{r}\delta(x - vt) = \rho\frac{\partial^2 u}{\partial t^2},\ \tau_0 > 0, \qquad \text{... (1a)}$$

$$\frac{\partial u}{\partial x} = \frac{\sigma}{E} + \alpha(T - T_0) \qquad \text{... (1b)}$$

and

$$k\frac{\partial^2 T}{\partial x^2} = E\alpha T\frac{\partial^2 u}{\partial x \partial t} + \rho_c h\frac{\partial T}{\partial t} + Q_0\delta(x - vt). \qquad \text{... (1c)}$$

Eqs. 1*a*, *b* are, respectively, the 1-D momentum balance equation and thermoelastic constitutive law for the rod, while (1*c*) is the temperature equation based on the Fourier model. In (1*a*-*c*), $u(x, t)$, $\sigma(x, t)$ and $T(x, t)$ are, respectively, the axial displacement and stress and absolute temperature in the rod, while (τ_0, Q_0, r) are, respectively, the shear force per unit of ring circumference exerted by the ring, the associated heat loss due to τ_0-induced extension of the rod, and the radius of rod and ring, while δ is the Dirac function. That a heat loss is assumed $(Q_0 > 0)$ is not critical to the analysis. The parameters $(E, \rho, \alpha, k, c_h)$ for the rod are, respectively, Young's modulus, mass density, coefficient to linear expansion, conductivity and specific heat. These are assumed to remain constant in this first-step study.

In what follows it is convenient to employ the thermoelastic constants

$$v_0 = \sqrt{\frac{E}{\rho}}, \ \varepsilon = \frac{T_0}{c_h}(\alpha v_0)^2, \ h = \frac{k}{\rho c_h v_0}, \qquad \text{... (2)}$$

where v_0 is the isothermal speed of sound in the rod[9], the dimensionless constant ε is the 1-D version of the thermoelastic coupling constant[2] and h is a thermoelastic characteristic length. Generally (ε, h, v_0) are of $O(10^{-2})$, $O(10^{-4})\mu m$ and $O(10^3)m/s$, respectively[2,9,10]. In addition, we now assume that the sliding ring/rod system reaches a steady-state in which (τ_0, Q_0) are essentially constants. Then, one can replace the independent variables (x, t) by the set (x', t), where $x' = x - vt$, that moves with the ring, and suppress variation with t. Finally, it is convenient to work with the quantities

$$\theta = T - T_0, \ \xi = \frac{x'}{h}, \ L_0 = 2\frac{\tau_0 h}{Er} > 0, \ \theta_0 = \frac{Q_0}{\rho c_h v_0}, \ a = 1 - c^2, \ c = \frac{v}{v_0}, \qquad \text{... (3)}$$

where θ is the change in (absolute) temperature, (c, ξ) are the dimensionless ring speed and position with respect to the ring, while the loading parameters (L_0, θ_0) have the dimensions, respectively, of length and temperature change. In this study, the case $c > 1$, and thus, $a < 0$, is not *a priori* disallowed. In view of all this, (1*a*-*c*) combine to give the two differential equations

$$au'' - h\alpha\theta' = -L_0 h\delta(h\xi) \qquad \text{... (4a)}$$

and

$$\theta' + c\left(\theta + \frac{\varepsilon}{h\alpha}u'\right)' + \frac{c\varepsilon}{h\alpha T_0}\theta u'' = \theta_0 h\delta(h\xi), \qquad \text{... (4b)}$$

that govern (u, θ) for all ξ, where $(\ )'$ denotes ξ-differentiation. The partly-coupled case is now considered.

CASE 1 : PARTLY-COUPLED EQUATIONS

Because ε is generally of $O(10^{-2})$, it is argued[1-4] that the ε-terms in (4*b*) can be dropped, leading to, upon integration, the two partly-coupled equations

$$\theta' + c\theta = \theta_1 \ (\xi < 0), \ \theta_0 + \theta_1 \ (\xi > 0) \qquad \text{... (5a)}$$

and

$$ah' - h\alpha\theta = L_1(\xi < 0), \ L_0 + L_1(\xi > 0) \qquad \text{... (5b)}$$

where (θ_1, L_1) are constants of integration. Eqs. 5*a*, *b* have piecewise constant inhomogeneous terms, but the solutions (u, θ) should be continuous at $\xi = 0$ because the rod does not break and there is no heat source/sink within the rod to cause a jump in temperature. The solutions should also be bounded as $|\xi| \to \infty$ and, because the rod is at rest at constant temperature T_0 prior to action of the ring, they should vanish an infinite distance ahead ($\xi \to \infty$) of the moving ring.

Solutions to (5) that satisfy these physically reasonable additional requirements are easily obtained as

$$\theta = -\frac{\theta_0}{c} e^{-c\xi},\ u = \frac{h\alpha}{ca}\frac{\theta_0}{c} e^{-c\xi}\ (\xi > 0) \qquad \text{... (6a)}$$

and

$$\theta = -\frac{\theta_0}{c},\ u = \frac{h\alpha}{ca}\frac{\theta_0}{c}\ (\xi < 0) \qquad \text{... (6b)}$$

where the constraint

$$\frac{\theta_0}{cL_0} = \frac{1}{h\alpha} \qquad \text{... (7)}$$

on two positive quantities that involve the loading parameters (θ_0, L_0) applies. This constraint arises in the course of the solution procedure specifically because of the aforementioned continuity requirements; that is, (7) does not arise if the rod is allowed to break or the temperature is allowed to jump at the (instantaneous) location of the ring ($\xi = 0$). Eqs. 6 indicate that (u, θ) are uniform in the wake of the ring. The constraint (7) demonstrates that the ring-induced mechanical loading $\tau_0(L_0)$ and the associated thermal effect $Q_0(\theta_0)$ are not independent. Indeed, (7) corresponds to a self-equilibrating load requirement for a non-transient non-thermal elasticity problem[11]. Its sign indicates that, as assumed, the ring acts as a heat sink on the rod. Thus, (6) predicts drops in rod temperature.

The presence of a in (6) indicates that the displacement u is singular when the ring reaches the isothermal speed of sound ($c = 1$, $v = v_0$). The general forms of both (u, θ) are, however, insensitive as to whether the ring motion is subsonic ($0 < c < 1$, $0 < v < v_0$) or not ($c > 1, v > v_0$). It should also be noted in view of (3) that, becaue the thermoelastic characteristic length h is of $O(10^{-4})\mu m$, the exponential deay with distance x' from the ring is especially pronounced. Moreover, this decay is enhanced by ring speed (c).

CASE 2 : LINEARIZED FULL COUPLING

To preserve a full coupling in the governing eqs. (4), it is argued[1-4] that $|\theta/T_0| \ll 1$ in many circumstances, i.e., the change in absolute temperature from its initial value is small compared to the value itself. Then the last term on the left-hand side of (4b), which introduces a nonlinearity into the equation set, can be dropped; the remaining coupling (ε) term remains, however. Then while (5b) remains valid, (5a) is replaced by

$$\theta' + c\left(1+\frac{\varepsilon}{a}\right)\theta = \theta_1 + \frac{c\varepsilon}{ah\alpha} L_1\ (\xi < 0),\ \theta_0 + \theta_1 + \frac{c\varepsilon}{ah\alpha}(L_0 + L_1)\ (\xi > 0). \quad \text{... (8)}$$

The same conditions imposed upon (5*a*, *b*) are again used, with the result that (5*b*) and (8) yield

$$\theta = -\frac{\theta_0}{c} e^{-ca\xi},\ u = \frac{h\alpha}{ca}\frac{\theta_0}{c\omega} e^{-ca\xi}\ (\xi > 0) \quad \text{... (9a)}$$

and

$$\theta = -\frac{\theta_0}{c},\ u = \frac{h\alpha}{ca}\frac{\theta_0}{c\omega}\ (\xi < 0) \quad \text{... (9b)}$$

when either $0 < c < 1$ or $c > \sqrt{1+\varepsilon}$, while the relations

$$\theta = 0,\ u = 0\ (\xi > 0) \quad \text{... (10 a)}$$

$$\theta = \frac{\theta_0}{c}(e^{-ca\xi} - 1),\ u = \frac{h\alpha}{ca}\frac{\theta_0}{c\omega}(1 - e^{-ca\xi})\ (\xi < 0) \quad \text{... (10 b)}$$

hold when $1 < c < \sqrt{1+\varepsilon}$. In (9) and (10) the dimensionless quantity

$$\omega = 1 + \frac{\varepsilon}{a} \quad \text{... (11)}$$

is employed and the constraint (7) still holds.

Comparison of (9) and (10) with (6) shows that the linearized coupling solution is more noticeably affected by ring speed. Moreover, there are now three speed ranges:

$$0 < v < v_0\ (0 < c < 1),\ v_0 < v < v_0\sqrt{1+\varepsilon}(1 < c < \sqrt{1+\varepsilon}),\ v > v_0\sqrt{1+\varepsilon}(c > \sqrt{1+\varepsilon}).$$

These can be denoted, respectively, as the sub-, trans- and supersonic ranges. Then (9) and (10) show that the sub- and supersonic solutions are similar in form, and differ from those for the transonic range. In particular, the behaviour of (u, θ) ahead $(\xi > 0)$ and in the wake $(\xi < 0)$ of the ring moving in the transonic range correpsonds roughly with the behaviour for, respectively, $\xi < 0$ and $\xi > 0$ when the ring moves in the sub- and supersonic ranges. Consequently (9) and (10) show that for sub- and supersonic speeds, the ring leaves constant (u, θ) in its wake, while the body is undisturbed ahead of the ring for the transonic range. That a transonic range exists at all is clearly a direct consequence of the full coupling, i.e., the appearance of ε gives rise to the isentropic (adiabatic) wave speed $v_0\sqrt{1+\varepsilon}$.

Case 3 : Nonlinear Full Coupling

The exact equation set (4) is now considered: Combining and integrating these relations gives (5*b*) and the replacement

$$\theta' + c\left(1+\frac{\varepsilon}{a}\right)\theta + \frac{c\varepsilon}{2aT_0}\theta^2 = \theta_1 + \frac{c\varepsilon}{ah\alpha}L_1\ (\xi<0),$$

$$\theta_0 + \theta_1 + \frac{c\varepsilon}{ah\alpha}\left[L_0 + L_1 + \frac{\theta(0)}{T_0}L_0\right](\xi>0) \qquad \dots (12)$$

for (5*a*). Eq. 12 is a nonlinear equation of the Riccati type with a piecewise constant inhomogeneous term. Moreover, it is somewhat implicit because the term $\theta(0)$ appears for $\xi>0$. Nevertheless, under the physically reasonable requirements that (u, θ) be bounded as $|\xi|\to\infty$ and continuous at $\xi = 0$, and that (u, θ) vanish as $\xi\to\infty$, the expressions

$$\frac{(\alpha v_0)^2}{c_h}\theta = \frac{-2a\omega}{1+\dfrac{\lambda}{2(\lambda-\omega)\chi+2\omega-\lambda}e^{c\omega\xi}} \qquad \dots (13a)$$

and

$$u = \frac{2c_h h}{c\alpha v_0^2}\ln\left[1+\frac{2(\lambda-\omega)\chi+2\omega-\lambda}{\lambda}e^{-c\omega\xi}\right] \qquad \dots (13b)$$

for $\xi>0$ can be obtained, while for $\xi<0$ the expressions

$$\frac{(\alpha v_0)^2}{c_h}\theta = a\left[\lambda-2\omega+\frac{2(\lambda-\omega)}{1+\dfrac{2\omega+(\lambda-\omega)\chi}{\chi^{\lambda}}e^{c(\lambda-\omega)\xi}}\right]. \qquad \dots (14a)$$

and $$u = \frac{2c_h h}{c\alpha v_0^2}\ln\left[\chi+\frac{2\omega+(\lambda-\omega)\chi}{\lambda}e^{c(\lambda-\omega)\xi}\right] \qquad \dots (14b)$$

hold. In (13) and (14) ω is again defined by (11), the dimensionless constants (λ, χ) are

$$\lambda = \frac{\alpha v_0^2}{ac_h}\frac{L_0}{h},\ \ln\chi = \frac{2c_h h}{c\alpha v_0^2}u_0 \qquad \dots (15a,\ b)$$

and the constraints

$$\frac{\theta_0}{cL_0} = \frac{1}{h\alpha} + \frac{\omega}{h\alpha}\left[2\omega - \frac{\lambda}{2} - \frac{\lambda}{\omega+(\lambda-\omega)\chi}\right], \qquad \dots (16a)$$

$$2\omega<\lambda<\lambda^*(0<\chi<0.5),\ \lambda>\omega(\chi>0.5), \qquad \dots (16b)$$

$$\frac{L_0}{a}>0,\ \omega>0 \qquad \dots (16c,d)$$

and

$$\chi>0 \qquad \dots (16e)$$

hold. In (15b) the constant u_0 is the displacement an infinite distance behind the ring, and

$$\lambda^* = \frac{2(1-\chi)}{1-2\chi}\,\omega > 2. \qquad \text{... (17)}$$

Eq. 16a shows that, like the previous cases, the loading parameters (θ_0, L_0) are not independent.

Comparison of (16a) and (7) indicates, however, that the relation here is not linear. Moreover, study of (16b) for $0 < \chi < 0.5$ gives the limiting cases

$$\frac{\theta_0}{cL_0} = \frac{1}{h\alpha}\left[1 - \left(\frac{1+\chi}{1-\chi}\right)\omega\right](\lambda = 2\omega) \qquad \text{... (18}a\text{)}$$

and

$$\frac{\theta_0}{cL_0} = \frac{1}{h\alpha}\left[1 - \frac{(1-\chi-\chi^2)}{(1-\chi)(1-2\chi)}\,\omega\right](\lambda = \lambda^*) \qquad \text{... (18}b\text{)}$$

which show that the temperature change θ_0 can vanish and change sign. That is, the ring acts as either a heat source or sink, depending on ring loading τ_0 and speed v through the parameters L_0 and c.

The constraints (16b-d) have no analogy in the partly-coupled and linearly-coupled cases, and introduce two restrictions hitherto not seen: First, (16b) indicates that solutions do not exist for all finite values of the mechanical loading (τ_0). Then (16c) cannot be satisfied for $c > 1$ unless τ_0 is allowed to take on negative values. That is, under the loading $\tau_0 > 0$ (13) and (14) hold only for subsonic ring speeds ($v < v_0, c < 1$). Clearly (16d) is automatically satisfied when $c < 1$.

Finally, (16e) arises in light of (15b) order to guarantee real-valued solutions. The quantity (15b) has no analogy in the cases considered previously, as well: In those cases (u, θ) were completely determined. In this case they are determined only to within the constant displacement u_0 an infinite distance behind the ring. Clearly any positive value of u_0 will give $\chi > 0$, so that an additional requirement must be imposed in order to completely determine (u, θ). While such a requirement need not be on u_0 itself, one possible choice would be to fix, in effect, the rod by setting $u_0 = 0$ ($\chi = 1$).

The additional constraints in this non-linear case arise because, in the course of solving (12), the physically-reasonable continuity requirements at $\xi = 0$ actually combine with the other requirements to influence the very form of the solution, not just, as in the previous linear studies, to influence the choice of loading parameters (θ_0, L_0). Another distinctive feature of the non-linear coupling—seen in (13) and (14)-is that, while the same speed-enhanced exponential decay found in the previous cases influences the behaviour of (u, θ), they themselves are no longer simple exponential functions.

CASE 4 : LINEARIZED COUPLING BASED ON A JEFFREYS MODEL

The same situation outlined for the Fourier models is considered again, except that a temperature equation of the J-C type[7,8] is modified to reflect linearized coupling and used in place of (1*c*):

$$\left(k+k_1\frac{\partial}{\partial T}\right)\frac{\partial^2 T}{\partial x^2}=E\alpha T_0\frac{\partial^2 u}{\partial x\partial t}+\rho c_h\left(1+t_1\frac{\partial}{\partial t}\right)\frac{\partial T}{\partial t}+Q_0\delta(x-vt). \quad \text{... (19)}$$

In (19) the new material constants ($k_1>0, t_1>0$) are, respectively, a conductivity rate and a relaxation time. In addition to (2), therefore, we introduce for convenience the new thermoelastic characteristic lengths

$$h_1=v_0t_1,\ h_2=\sqrt{\frac{k_1}{\rho c_h}}. \quad \text{... (20)}$$

In contrast to the quantities in (2), however, generally accepted measurements of (k_1, t_1) are few[7]. This lack of information is not critical to the analysis, but it should be noted that experiments[7] in metals do suggest a heat propagation speed v_h of $O(10^6)m/s$. If k_1 is small, then an approximation $v_h=\sqrt{k/\rho c_h t_1}$ may be valid, whereupon (2) and (20) give the result

$$\frac{h_1}{h}=\left(\frac{v_0}{v_h}\right)^2. \quad \text{... (21)}$$

The orders of magnitude for (v_0, h) then suggest that h_1/h is $O(10^{-5})$ and, consequently, that h_1 is $O(10^{-9})\mu m$.

In any event, the steady-state situation discussed earlier is again assumed, so that the variables (3) prove convenient. The result is that, while (4*a*) still holds, (4*b*) must be replaced by

$$\theta'+c(\theta-cl_1\theta'-l_2^2\theta'')+\frac{c\varepsilon}{h\alpha}u''=\theta_0 h\delta(h\xi), \quad \text{... (22)}$$

where (l_1, l_2) are the dimensionless ratios

$$l_1=\frac{h_1}{h},\quad l_2=\frac{h_2}{h}. \quad \text{... (23)}$$

Combining (4*a*) with (22) and integrating the result then gives the equation set

$$-cl_2^2\theta''+(1-c^2l_1)\theta'+c\left(1+\frac{\varepsilon}{a}\right)\theta=\theta_1+\frac{c\varepsilon}{ah\alpha}L_1(\xi<0),\ \theta_0+\theta_1+\frac{c\varepsilon}{ah\alpha}(L_0+L_1)(\xi>0) \quad \text{... (24a)}$$

$$au'-h\alpha\theta=L_1(\xi<0),\ L_0+L_1(\xi>0). \quad \text{... (24b)}$$

In contrast to the Fourier model cases, a 2nd-order differential equation is involved. Indeed the coefficient of the second derivative in (24a) suggests that the length h_2 be used instead of h for non-dimensionalization of x'. However, h_2 depends on k_1 for which, as noted above, accepted values are few. Moreover, use of h will facilitate comparisons with the Fourier model results.

Upon yet again imposing the conditions of continuity at $\xi = 0$, boundedness as $|\xi| \to \infty$ and the vanishing of the solution as $\xi \to \infty$, it can be shown for subsonic ($0 < c < 1$) and supersonic ($c > \sqrt{1+\varepsilon}$) ring speeds that

$$\theta = \frac{\Omega_+}{2\Omega}\left(\frac{a\Omega_- u_0}{h\alpha} - \frac{\theta_0}{c}\right)e^{-\Omega_+\xi},\ u = \frac{1}{2\Omega}\left(\frac{h\alpha\theta_0}{ac} - \Omega_- u_0\right)e^{-\Omega_+\xi}\ (\xi > 0) \quad \ldots (25a)$$

and

$$\theta = \frac{\theta_0}{c} + \frac{\Omega_-}{2\Omega}\left(\frac{\theta_0}{c} - \frac{a\Omega_+ u_0}{h\alpha}\right)e^{-\Omega_-\xi},\ u = u_0 + \frac{1}{2\Omega}\left(\frac{h\alpha\theta_0}{ac} - \Omega_+ u_0\right)e^{-\Omega_-\xi}\ (\xi < 0). \quad \ldots (25b)$$

Here the constant u_0 is again the displacement an infinite distance behind the ring, the dimensionless parameters $(\Omega_\pm, \Omega)$ are

$$\Omega_\pm = \omega_1 \pm \Omega,\ \Omega = \sqrt{\omega_1^2 + \frac{\omega}{l_2^2}},\quad \omega_1 = \frac{1 - c^2 l_1}{2c l_2^2}, \quad \ldots (26)$$

we have $\Omega_- < 0 < \Omega_+$ for all positive real (l_1, l_2), and the constraint (7) again applies. For the transonic ($1 < c < \sqrt{1+\varepsilon}$) case, however, the values of (l_1, l_2) affect this solution form. In view of the already-noted lack of information necessary to determine these parameters, we present here the results for all positive values of (l_1, l_2): When these parameters for $1 < c < \sqrt{1+\varepsilon}$ are such that $\omega + l_2^2\omega_1^2 > 0$, we have

$$\theta = 0,\ u = 0\ (\xi > 0), \quad \ldots (27a)$$

$$\theta = -\frac{\theta_0}{c} + \frac{\Omega_+}{2\Omega}\left(\frac{\theta_0}{c} + \frac{a\Omega_- u_0}{h\alpha}\right)e^{\Omega_+\xi} - \frac{\Omega_-}{2\Omega}\left(\frac{\theta_0}{c} + \frac{a\Omega_+ u_0}{h\alpha}\right)e^{\Omega_-\xi},$$

and

$$u = u_0 + \frac{1}{2\Omega}\left(\frac{h\alpha\theta_0}{ac} + \Omega_- u_0\right)e^{\Omega_+\xi} - \frac{1}{2\Omega}\left(\frac{h\alpha\theta_0}{ac} + \Omega_+ u_0\right)e^{\Omega_-\xi}\ (\xi < 0) \quad \ldots (27b)$$

when $\omega_1 > 0 (\Omega_\pm > 0)$, while for $\omega_1 < 0\ (\Omega_\pm < 0)$ the results

$$\theta = -\frac{\Omega_+}{2\Omega}\left(\frac{\theta_0}{c} + \frac{a\Omega_- u_0}{h\alpha}\right)e^{\Omega_+\xi} + \frac{\Omega_-}{2\Omega}\left(\frac{\theta_0}{c} + \frac{a\Omega_+ u_0}{h\alpha}\right)e^{\Omega_-\xi},$$

and

$$u = -\frac{1}{2\Omega}\left(\frac{h\alpha\theta_0}{ac} + \Omega_- u_0\right)e^{\Omega_+\xi} + \frac{1}{2\Omega}\left(\frac{h\alpha\theta_0}{ac} + \Omega_+ u_0\right)e^{\Omega_-\xi}(\xi > 0) \quad \ldots (28a)$$

and

$$\theta = -\frac{\theta_0}{c},\ u = u_0\ (\xi < 0) \quad \ldots (28b)$$

hold. Again the constraint (7) applies. When (l_1, l_2) for $1 < c < \sqrt{1+\varepsilon}$ are such that $\omega + l_2^2\omega_1^2 < 0$, however, we have

$$\theta = 0,\ u = 0(\xi > 0), \quad \ldots (29a)$$

$$\theta = -\frac{\theta_0}{c} + \left[\frac{\theta_0}{c}\cos\Omega^*\xi + \left(\frac{\omega_1\theta_0}{c} - \frac{a\omega u_0}{h\alpha l_2^2}\right)\frac{\sin\Omega^*\xi}{\Omega^*}\right]e^{\omega_1\xi},$$

and

$$u = u_0 + \left[u_0\left(\frac{\omega_1}{\Omega^*}\sin\Omega^*\xi - \cos\Omega^*\xi\right) + \frac{h\alpha\theta_0}{ac}\frac{\sin\Omega^*\xi}{\Omega^*}\right]e^{\omega_1\xi}\ (\xi < 0), \quad \ldots (29b)$$

when $\omega_1 > 0$, while when $\omega_1 < 0$,

$$\theta = \left[-\frac{\theta_0}{c}\cos\Omega^*\xi + \left(\frac{a\omega u_0}{h\alpha l_2^2} - \frac{\omega_1\theta_0}{c}\right)\frac{\sin\Omega^*\xi}{\Omega^*}\right]e^{\omega_1\xi},$$

$$u = \left[u_0\left(\cos\Omega^*\xi - \frac{\omega_1}{\Omega^*}\sin\Omega^*\xi\right) - \frac{h\alpha\theta_0}{ac}\frac{\sin\Omega^*\xi}{\Omega^*}\right]e^{\omega_1\xi}\ (\xi > 0) \quad \ldots (30a)$$

and

$$\theta = -\frac{\theta_0}{c},\ u = u_0\ (\xi < 0). \quad \ldots (30b)$$

In (29) and (30) the constraint (7) applies and the dimensionless parameter Ω^* is

$$\Omega^* = \sqrt{-\omega_1^2 - \frac{\omega}{l_2^2}} \quad \ldots (31)$$

Examination of (25)-(30) shows that the solutions (u, θ) are known only to within the constant displacement u_0. An analogous generality arose for the exact Fourier model, but then the cause was problem nonlinearity. Here, the situation arises because of the 2nd-order nature of (24a). As before, an additional restriction must be imposed, and one choice would be to fix the rod by setting $u_0 = 0$.

Comparison of (25)-(30) with the corresponding solution fields for the Fourier model-based cases also indicate that the former are more sensitive to ring speed. In particular, the sinusoidal behaviour of (29) and (30) demonstrates that the rod can have a speed-controlled periodic length

$$L^* = \frac{\pi h}{\Omega^*} \qquad \text{... (32)}$$

in the transonic $(1 < c < \sqrt{1+\varepsilon})$ range. The possibility of multiple sign reversals in u(and, thus, in the axial strain $\partial u/\partial x$) that travel through the rod suggest the possibility of fatigue-like material behaviour for sufficiently large τ_0. As already mentioned, however, not enough is known about (l_1, l_2) to specify which of the various conditions on $(\omega_1, \omega + l_2^2\,\omega_1)$ associated with (25)-(30) actually arise. It is also of interest to note that, despite the possible variety of the Jeffreys model-based response, the constraint (7) holds, i.e. the ring always acts as a heat sink for the rod.

Finally, in contrast to the Fourier model cases, the exponential decay of (u, θ) seen in (25) and (27)-(30) is not necessarily enhanced by ring speed, because the exponential arguments are now complicated functions of c. The decay imposed by the arguments is controlled more by the three thermoelastic characteristic lengths (h, h_1, h_2). For example, use of (3) and (23) give for the exponential arguments in (25) the more explicit forms

$$\left[-\left(\frac{h - c^2 h_1}{2h_2}\right) \pm \sqrt{\left(\frac{h - c^2 h_1}{2h_2}\right)^2 + \omega}\,\right]\frac{x'}{h_2}. \qquad \text{... (33)}$$

This confirms the earlier observation that h_2 would be the more natural choice for non-dimensionalization of x', were it not for the relative lack of knowledge about h_2 in comparison to h.

Some Observations

This article considered the dynamic steady-state thermoelastic response of a thin, infinitely-long insulated rod to a ring moving along it at a constant speed. It was assumed that the ring induced a net axial force and associated heat gain/loss on the rod, and that these thermo-mechanical loads could be treated as point sources, and the rod response 1-D.

Four different cases were chosen to illustrate the effects of thermoelastic coupling in the governing differential equations — coupling that is generally[1-4] neglected. In case 1, the equations were partly-coupled by the appearance of the temperature change gradient in the momentum balance equation. In case 2, a linearized full coupling was preserved by also including the product of initial temperature and strain rate in the temperature equation. This is, however, an approximation because the exact theory[1-4] features the instantaneous value of the temperature in the product. Thus, case 3 treated the exact (nonlinearly) coupled set of differential equations.

These three cases were based on the standard Fourier law of heat conduction; in case 4 the linearized coupling equations were used, but the temperature equation itself was based on a heat conductor of the Jeffreys-Cattaneo (J-C) type[7,8], which allows the possibility of heat waves.

The coupled differential equations for all four cases — including the nonlinear case 3 — were solved exactly for the axial displacement u and temperature change θ as functions of the instantaneous non-dimensionalized distance from the moving ring. The solutions (u, θ) for the four cases exhibited some similarities in behaviour: In particular, the thermo-mechanical loads induced by the ring in cases (1, 2, 4) were subject to the same linear relation that showed the ring to be a heat sink for the rod, and cases (3, 4) required the imposition of an additional condition in order to completely determine (u, θ).

The differences (u, θ) among the four cases were, however, even more prominent, especially the variation with ring speed. In particular, case 1 exhibited only the standard[9] sub- and supersonic speed ranges, defined by the isothermal speed of sound v_0 in the rod, and the forms of (u, θ) were the same in either range. The fully-coupled cases 2-4, however, exhibited sub-, trans- and supersonic speed ranges, defined by v_0 and the thermoelastic coupling constant ε, and the behaviour of (u, θ) depended greatly on whether the speed was in the sub/super-sonic range or in the transonic range.

The coupling constant, therefore largely defined the four cases, but was itself dependent on the thermal expansion coefficient α which always appeared explicitly in the momentum balance equation. Thus, once constructed, the solutions (u, θ) for the various cases could not all be readily recovered from each other by limiting procedures. Similarly, the additional parameters that characterized the J-C model case 4 changed the very order of the governing differential equations from that for the three Fourier model cases. It was, therefore, more useful simply to present the complete solutions for each case and then discuss aspects of their general behaviour for purposes of comparison.

It should be noted that various aspects of these types of equations have been addressed: For example, summaries of several treatments of non-Fourier heat equations with limited consideration of coupling can be found in[7,8,12] and efforts to study partly-coupled Fourier model systems when changes in absolute temperature are not small exist[13]. However, the 1-D steady-state cases treated here allowed a variety of situations to be formulated in terms of coupled ordinary differential equations that could be solved exactly. Moreover, the cases covered a range of thermoelastic coupling possibilities — including nonlinear and non-Fourier.

As noted at the outset, a perhaps more realistic rod would not be insulated. Also, material properties that define equation parameters, e.g. the conductivity k, may themselves change with temperature[14]. A study now underway includes these effects. Nevertheless, despite the idealizations of the present models, the results show clearly in terms of closed-form solutions that thermoelastic coupling — and the particular type of coupling—can be important in determining the dynamic thermoelastic response of thin rods.

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