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DIFFERENTIAL VARIATIONAL INEQUALITIES IN R^N

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In this paper, we study "Differential Variational Inequalities" defined in R^N. First we establish the existence of external trajectories. Then we show that those extremal trajectories are dense in the solution set of the original system. Afterwards we prove that the solution set is path connected. Also we view the solution set as multifunction of the initial datum and for this multifunction we show that we can construct a continuous selector. Using this selector we prove the existence of periodic trajectories. Finally we show that the solution set depends continuously (in both the Vietoris and Hausdorff hyperspace topologies) on the data of the problem.

Key Words : Differential Variational Inequality; Normal Cone; Tangent Cone; Vietoris Continuity; Hausdorff Continuity; Continuous Selector; Extremal Trajectory; Periodic Solution; Continuous Dependence

1. INTRODUCTION

In this paper we study "differential variational inequalities". So if $T = [0, b]$ (time horizon) and the state space is R^N, then the multivalued Cauchy problem under consideration is the following :

$$\left\{ \begin{array}{c} -\dot{x}(t) \in N_{K(t)}(x(t)) + F(t, x(t)) \; a.e. \\ x(0) = x_0 \end{array} \right\} \quad \ldots (1)$$

Here $N_{K(t)}(x(t))$ denotes the normal cone to the closed and convex set $K(t)$ at the point $x(t)$. Recall from convex analysis, that $N_{K(t)}(x(t)) = \partial\delta_{K(t)}(x(t))$, with $\delta_{K(t)}(\cdot)$ being the indicator function of the moving set $K(t)$; i.e. $\delta_{K(t)}(x) = 0$ if $x \in K(t)$ and $+\infty$ otherwise, while $\partial\delta_{K(t)}(x(t))$ is the convex subdifferential of the indicator function at $x(t)$.

Problems like (1), which are called "differential variational inequalities" (see Aubin-Cellina[1]), arise naturally in mathematical economics, in the study of planning procedures (resource allocation mechanisms; see Henry[2] and Cornet[3]), in mechanics in the study of the quasistatic evolution of elastoplastic systems (see Moreau[4]), in control theory in the analysis of feedback systems (see Aubin-Cellina[1]) and in obstacle problems (see Shuzhong[5]). When $K(t) = K$ (i.e. time-independent), closed and convex, Cornet[3] established that (1) is equivalent to the "projected differential inclusion"

$$\left\{ \begin{array}{c} \dot{x}(t) \in \text{proj}\,[F(t, x(t)); T_K(x(t))]\ a.e., \\ x(0) = x_0, \end{array} \right\}$$

with proj $[\cdot\, ; T_K(x(t))]$ being the metric projection onto $T_K(x(t))$. In many applications where there is a constraint to the system, in describing the effects of it to the dynamics, it can be assumed that the velocity vector $\dot{x}(t)$ is projected at each time instant on the set of allowed directions towards the constraint set K at the point $x(t)$. This is true for electrical networks with diode nonlinearities, as well as hysterisis loops of certain types (see Krasnoselski-Pokrovski[6]).

Working with the projection proj $[\cdot\, ; T_K(x(t))]$, if the usual Nagumo-type tangential condition $F(t, x) \cap T_K(x) \neq \phi$ for all $x \in K$, is not satisfied, is an effective device to enforce existence of at least one solution that stays in K (i.e. satisfies the constraint). Notice that if int $K \neq \phi$, then since for $x \in$ int K we have $T_K(x) = R^N$, the original equation is affected only at the boundary of K. So the "projected differential inclusion" and hence the equivalent "differential variational inequality", are seen to be good approximations of the original system.

A large class of optimal control problems of non-linear systems with dynamics governed by the "differential variational inequality" (1) were examined by Avgerinos[7], while problem (1) was studied (primarily in R^N) by Papageorgiou[8].

Our present work improves and extends the results of Ref. [8] and also produces new ones.

2. Preliminaries

Let (Ω, Σ) be a measurable space and X is a separable Banach space. Throughout this paper we will be using the following notations :

$$P_{f(c)}(X) = \{A \subseteq X \,:\, \text{nonempty, closed, (convex)}\}$$

and $$P_{wk(c)}(X) = \{A \subseteq X \,:\, \text{nonempty, } w\text{-compact, (convex)}\}.$$

A multifunction $F : \Omega \to P_f(X)$ is said to be measurable if for all $y \in X$, the R_+-valued function $\omega \to d(y, F(\omega)) = \inf\,\{\|x - y\| \,:\, x \in F(\omega)\}$ is measurable.

Also a multifunction $G : \Omega \to P_f(X)$ is said to be "graph measurable", if $GrG = \{(\omega, x) \in \Omega \times X : x \in G(\omega)\} \in \Sigma \times B(X)$, where $B(X)$ is the Borel σ-field of X. We mention that for closed valued multifunctions, measurability implies graph measurability. The converse is true if there exists a σ-finite measure $\mu(\cdot)$, with respect to which Σ is complete. For details we refer to Wagner[9] (see Theorem 5.1).

We also use S_F^1 to denote the set of measurable selectors of $F(\cdot)$ that belong in the space $L^1(\Omega, X)$. In general this set may be empty. However, a straightforward application of Aumann's selection theorem (see Wagner[9], theorem 5.10), shows that S_F^1 is nonempty if and only if $\omega \to \inf\{\|z\| : z \in F(\omega)\} \in L_+^1$.

On $P_f(X)$ we can define a generalized metric, known in the literature as the "Hausdorff metric", by setting for $A, B \in P_f(X)$

$$h(A, B) = \max\left[\sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A)\right]$$

with $d(a, B) = \inf\{\|a - b\| : b \in B\}$ and $d(b, A) = \inf\{\|b - a\| : a \in A\}$. It is well known that $(P_f(X), h)$ is a complete metric space and $(P_k(X), h)$ is a closed and separable subspace of it. For further details on these and related issues we refer to the book of Klein-Thompson[10]. A multifunction $F : X \to P_f(X)$ is said to be "Hausdorff continuous" (h-continuous), if it is continuous from X into the metric space $(P_f(X), h)$.

Let Y, Z be Hausdorff topological spaces and let $G : Y \to 2^Z \setminus \{\phi\}$ be a multifunction. We say that $G(\cdot)$ is upper semicontinuous (u.s.c.) if for all $V \subseteq Z$ open, the set $G^+(V) = \{y \in Y : G(y) \subseteq V\}$ is open. Also we say that $G(\cdot)$ is lower semicontinuous (l.s.c.) if for all $U \subseteq Z$ open the set $G^-(U) = \{y \in Y : G(y) \cap U \neq \phi\}$ is open. Observe that when $G(\cdot)$ is single valued, then the notions above coincide with the continuity of $G(\cdot)$. If a multifunction is both u.s.c. and l.s.c., we say that it is "continuous" or more precisely "Vietoris continuous", to emphasize that it is continuous into $2^Z \setminus \{\phi\}$, when the latter is equipped with the Vietoris hyperspace topology.

Suppose that Y, Z are metric spaces. On $P_k(Z)$ the Vietoris and Hausdorff hyperspace topology coincide (see Klein-Thompson[10], Corollary 4.2.3, p. 41). So a $P_k(Z)$ — valued multifunction $G(\cdot)$ (i.e. it has nonempty compact values) is Vietoris continuous if and only if is h-continuous. For a comprehensive introduction to these continuity concepts and additional results we refer to Klein-Thompson[10].

Let $K \in P_{fc}(R^N)$ and $x \in K$. Then the tangent cone to K at x, is defined by

$$T_K(x) = \left\{ v \in R^N : \lim_{\lambda \downarrow 0} \frac{d(x + \lambda v, K)}{\lambda} = 0 \right\}.$$

It is easy to see that this is a closed and convex cone. Furthermore if int K is nonempty, then so is int $T_K(x)$. The normal cone to K at x is defined by

$$N_K(x) = \left\{ x^* \in R^N : (x^*, x) = \sigma(x^*, K) = \sup_{z \in K} (x^*, z) \right\},$$

where by $(\cdot, \cdot)$ we denote the Euclidean inner product in R^N. As we already mentioned in the introduction, the normal cone to K at x is equal to the convex subdifferential of the indicator function of K at x (i.e. $N_K(x) = \partial\delta_K(x)$ for all $x \in K$). Also it is well-known that the normal cone is the negative polar cone of the tangent cone; i.e.

for all $x \in K$, $N_K(x) = T_K(x)^- = \{x^* \in R^N : (x^*, v) \leq 0$ for all $v \in T_K(x)\}$. Further properties of these cones can be found in Aubin-Cellina[1].

Finally, by a solution of (1) we mean an absolutely continuous function $x : T \to R^N$ such that

$$-\dot{x}(t) \in N_{K(t)}(x(t)) + f(t) \text{ a.e.,} \qquad x(0) = x_0$$

with $f(\cdot) \in L^1(T, R^N)$ and $f(t) \in F(t, x(t))$ a.e. (i.e. $f(\cdot) \in S^1_{F(\cdot, x(\cdot))}$).

Also by $L^1_w(T, R^N)$ we denote the space of equivalence classes of Lebesgue-integrable functions $x : T \to R^N$ equipped with the norm

$$\|f\|_w = \sup_{t \in T} \left\| \int_0^t f(s)\, ds \right\| .$$

The structure of the paper is the following :

In section 3, we establish the existence of "extremal solutions", i.e. trajectories which move through the extreme points of the orientor field $F(t, x)$.

In section 4, we show that these extremal trajectories are in fact dense in the set of trajectories of the original system. Of course in the context of control systems such a result is the well-known "bang-bang principle".

In section 5, we examine the topological structure of the solution set of (1) and show that under certain hypothesis on the orientor field, the solution set is path-connected.

In section 6, we view the solution set (which is in general non-convex), as a multifunction of the initial condition and for that multifunction we construct a continuous selector. Then we use that selector to prove the existence of periodic trajectories.

Finally in section 7, we consider a parametrized version of (1) with the parameter λ appearing in all the data of the problem and we study the dependence of the solution set on the parameter (sensitivity analysis).

3. Extremal Solutions

In conjuction with (1), we also consider the following multivalued Cauchy problem :

$$\left\{ \begin{array}{c} -\dot{x}(t) \in N_{K(t)}(x(t)) + \text{ext}\, F(t, x(t)) \; a.e. \\ x(0) = x_0 \end{array} \right\} . \qquad \ldots (2)$$

Here by ext $F(t, x)$ we denote the extreme points of the set $F(t, x)$. In this section we will prove the existence of a solution for (2).

First for future reference we recall the following existence theorem due to Moreau[4]. Assume that H is a separable Hilbert space.

Proposition 1 — If $K : T \to P_{fc}(H)$ is a multifunction such that $h(K(t'), K(t)) \le \int_t^{t'} v(s)\, ds$ for all $0 \le t \le t' \le b$ with $v \in L^1(T)$ and $x_0 \in K(0)$, then there exists a unique absolutely continuous functions $x : T \to H$ such that $-\dot{x}(t) \in N_{K(t)}(x(t))$ a.e. and $\| \dot{x}(t) \| \le v(t)$ a.e.

Remark : Recall that if X is a Banach space with the Radon-Nikodym Property (RNP) (see Diestel-Uhl)[11] and $x : T \to X$ is absolutely continuous, then $x(\cdot)$ is strongly differentiable almost everywhere. A Hilbert space and more generally a reflexive Banach or a separable dual Banach space has the RNP.

An immediate consequence of the Proposition 1 above is the following result.

Proposition 2 — If $K : T \to P_{fc}(H)$ is a multifunction such that $h(K(t'), K(t)) \le \int_t^{t'} v(s)\, ds$ for all $0 \le t \le t' \le b$ with $v \in L^1(T)$, $f \in L^1(T, H)$ and $x_0 \in K(0)$, then there exists a unique absolutely continuous function $x : T \to H$ such that $-\dot{x}(t) \in N_{K(t)}(x(t)) + f(t)$ a.e., $x(0) = x_0$ and $\| \dot{x}(t) \| \le v(t) + 2 \| f(t) \|$ a.e.

PROOF : Let $z(t) = \int_0^t f(s)\, ds$, $t \in T$. Then we have

$$\left\{ \begin{array}{c} -\dot{x}(t) - \dot{z}(t) \in N_{K(t)}(x(t)) \ \ a.e. \\ x(0) = x_0 \end{array} \right\}$$

Recall that $N_{K(t)}(x(t)) = \partial \delta_{K(t)}(x(t)) = \partial \delta_{K(t) + z(t)}(x(t) + z(t))$.

So we have

$$\left\{ \begin{array}{c} -(\dot{x} + \dot{z})(t) \in N_{K(t) + z(t)}(x(t) + z(t)) \ \ a.e. \\ (x + z)(0) = x_0 \end{array} \right\} \qquad \ldots (3)$$

Observe that

$$h(K(t') + z(t'), K(t) + z(t)) \le h(K(t'), K(t)) + \| z(t') - z(t) \|$$

$$\le \int_t^{t'} (v(s) + \| f(s) \|)\, ds \text{ for all } 0 \le t \le t' \le b.$$

So according to Proposition 1, there exists unique solution $y(\cdot) = x(\cdot) + z(\cdot)$ of (3) and we have $\| \dot{y}(t) \| = \| \dot{x}(t) + \dot{z}(t) \| \le v(t) + \| f(t) \|$ a.e.

$$\Rightarrow \| \dot{x}(t) \| \le v(t) + 2 \| f(t) \| \text{ a.e.}$$

Now we can state and prove our existence theorem concerning the Cauchy problem (2). We will need the following hypothesis on the data :

an state and prove our existence theorem concerning the Cauchy problem (2). We will need the following hypothesis on the data :

$H(K)$: $K : T \to P_{kc}(R^N)$ is a multifunction such that $h(K(t'), K(t)) \leq \int_t^{t'} v(s)\,ds$ for all $0 \leq t \leq t' \leq b$ with $v \in L^1(T)$.

$H(F)$: $F : T \times R^N \to P_{kc}(R^N)$ is a multifunction such that

(1) $t \to F(t,x)$ is measurable,

(2) $x \to F(t,x)$ is h-continuous,

(3) $|F(t,x)| = \sup \{\|v\| : v \in F(t,x)\} \leq a(t) + c(t)\|x\|$ a.e. with $a, c \in L^1(T)$.

Remark : The miltifunction $(t, x) \to \operatorname{ext} F(t, x)$ is not necessarily closed-valued and in general we cannot say anything about the continuity properties of $x \to \operatorname{ext} F(t,x)$. So our existence result cannot be deduced from the "nonconvex" existence theorem of Papageorgiou[8] (Theorem 3.1). Therefore, our results here can be viewed as an extension of Theorem 3.1 in [8].

Theorem 3.1 — *If hypotheses $H(K)$, $H(F)$ hold and $x_0 \in K(0)$, then problem (2) admits a solution.*

PROOF : First we will obtain an *a priori* bound for the solutions of (1). Let $x(\cdot) \in C(T, R^N)$ be such a solution. Then by definition there exists $f \in L^1(T, R^N)$ with $f(t) \in F(t, x(t))$ a.e. such that

$$\left\{ \begin{array}{c} -\dot{x}(t) \in N_{K(t)}(x(t)) + f(t) \ \ a.e. \\ x(0) = x_0 \end{array} \right\}.$$

Invoking Proposition 2, we get

$$\|\dot{x}(t)\| \leq v(t) + 2\|f(t)\| \text{ a.e.}$$

$$\Rightarrow \qquad \|x(t)\| \leq \|x_0\| + \int_0^t (v(s) + 2\|f(s)\|)\,ds$$

$$\Rightarrow \qquad \|x(t)\| \leq \|x_0\| + \int_0^t (v(s) + 2a(s) + 2c(s)\|x(s)\|)\,ds \text{ (hypothesis H(F)(3))}.$$

Invoking Gronwall's inequality, we deduce that there exists $M_1 > 0$ such that for every solution $x(\cdot) \in C(T, H)$ of (1) and every $t \in T$, we have $\|x(t)\| \leq M_1$.

Therefore, without any loss of generality we may assume that $|F(t,x)| \leq a(t) + M_1 c(t) = \phi(t)$ *a.e.* with $\phi(\cdot) \in L^1(T)$ (otherwise in what follows replace $F(t, x)$ by $F(t, p_{M_1}(x))$ with $p_{M_1}(\cdot)$ being the M_1-radial retraction in R^N).

Let $V = \{ h \in L^1(T, R^N) : \|h(t)\| \leq \phi(t) \text{ } a.e.\}$ and let $B = p(V) \subseteq C(T, R^N)$, where $p : L^1(T, R^N) \to C(T, R^N)$ is the map that assigns to each $h \in L^1(T, R^N)$ the unique solution of the Cauchy problem

$$\left\{ \begin{array}{c} -\dot{x}(t) \in N_{K(t)}(x(t)) + h(t) \ \ a.e. \\ x(0) = x_0 \end{array} \right\}$$

(see proposition 2). We claim that B is relatively compact in $C(T, R^N)$. To this end let

$$0 \le t \le t' \le b \text{ and } x \in B.$$

From Proposition 2, we have

$$\| \dot{x}(t) \| \le \nu(t) + 2\phi(t) \ a.e.$$

$$\Rightarrow \quad \| x(t') - x(t) \| \le \int_t^{t'} (\nu(s) + 2\phi(s))\, ds$$

$\Rightarrow$ B is equicontinuous.

Also we already established in the begining of the proof that it is bounded. Thus from the Arzela-Ascoli theorem, we deduce that indeed B is relatively compact in $C(T, R^N)$. Then from Mazur's theorem (see Diestel-Uhl[11], Theorem 12, p.51) we have that $B_0 = \overline{conv}\, B$ is a compact, convex subset of $C(T, R^N)$.

Let $R : B_0 \to P_{wkc}(L^1(T, R^N))$ be the multifunction defined by $R(x) = S^1_{F(\cdot, x(\cdot))}$. From Theorem 1 of Tolstonogov[12], we know that there exists a continuous map $r : B_0 \to L^1_w(T, R^N)$ such that $r(x) \in \text{ext } R(x) = \text{ext } S^1_{F(\cdot, x(\cdot))}$ for all $x \in B_0$. From Benamara[13], we know that ext $S^1_{F(\cdot, x(\cdot))} = S^1_{\text{ext } F(\cdot, x(\cdot))}$. So $r(x) \in S^1_{\text{ext } F(\cdot, x(\cdot))}$ for all $x \in B_0$. Let $\eta : B_0 \to B_0$ be defined by $\eta(x) = p(r(x))$. We claim that $\eta(\cdot)$ is in fact continuous. So let $x_n \to x$ in $B_0 \subseteq C(T, R^N)$ and set $z_n = \eta(x_n)$, $z = \eta(x)$. We have $r(x_n) \to r(x)$ in $L^1_w(T, R^N)$. From the Scorza-Dragoni theorem (see for example Kisielewicz[14], Theorem 3.7, p. 45), we know that given $\varepsilon > 0$, we can find $T_\varepsilon \subseteq T$ compact such that $\lambda(T \setminus T_\varepsilon) < \varepsilon$ (where $\lambda(\cdot)$ being the Lebesgue measure on T) and $F|_{T_\varepsilon \times R^N}$ is h-continuous (hence Vietoris continuous too). Therefore, $W_\varepsilon = \overline{\bigcup_{\substack{n \ge 1 \\ t \in T_\varepsilon}} F(t, x_n(t))} \in P_k(R^N)$ (see Klein-Thompson[10], Theorem 7.4.2, p. 90) and $r(x_n)(t), r(x)(t) \in W_\varepsilon$ a.e. on T_ε. Thus applying the result of Gutman[15], we get $r(x_n) \xrightarrow{w} r(x)$ in $L^1(T, R^N)$. Also from Proposition 2, we know that $\| \dot{\eta}(x_n)(t) \| = \| \dot{z}_n(t) \| \le \nu(t) + 2\phi(t)$ a.e. Since $B_0 \subseteq C(T, R^N)$ is compact, by passing to a subsequence if necessary, we may assume that $z_n \xrightarrow{s} u$ in $C(T, R^N)$ and also because of the Dunford-Pettis compactness criterion we can also have $\dot{z}_n \xrightarrow{w} \dot{u}$ in $L^1(T, R^N)$. Invoking theorem 3.1 of Papageorgiou[16], we get

$$-\dot{u}(t) - r(x)(t) \in \overline{conv}\, \overline{\lim}\, [-\dot{z}_n(t) - r(x_n)(t)]$$

$$\subseteq \overline{conv}\, \overline{\lim}\, N_{K(t)}(z_n(t)) \text{ a.e.}$$

From Theorem 1, p.220 of Aubin-Cellina[1], we have that

$$\overline{conv}\, \overline{\lim}\, N_{K(t)}(z_n(t)) \subseteq \overline{conv}\, N_{K(t)}(u(t)) = N_{K(t)}(u(t))$$

$\Rightarrow \quad -\dot{u}(t) \in N_{K(t)}(u(t)) + r(x)(t)$ a.e., $u(0) = x_0$

$\Rightarrow \quad u = p(r(x)) = z$

$\Rightarrow \quad \eta : B_0 \to B_0$ is continuous.

So we can apply Schauder's fixed point theorem and get $x = \eta(x)$. Clearly then $x(\cdot) \in C(T, R^N)$ solves (2).

4. A Strong Relaxation Theorem

In this section, by strengthening our hypothesis on $F(t, x)$, we can show that the solutions of (2) are dense in the solution set of (1), for the $C(T, R^N)$-topology. So every trajectory of (1) can be approximated arbitrarily close in the $C(T, R^N)$-topology by trajectories of (2). In what follows by $S(x_0)$ (resp. $S_e(x_0)$), we will denote the solution set of (1) (resp. (2)).

The new stronger hypothesis on $F(t, x)$ that we will need is the following :

H(F)$_1$: $F : T \times R^N \to P_{kc}(R^N)$ is a multifunction such that

(1) $t \to F(t, x)$ is measurable,

(2) $h(F(t, x), F(t, y)) \le k(t) \| x - y \|$ a.e. with $k(\cdot) \in L^1(T)$,

(3) $| F(t, x) | \le a(t) + c(t) \| x \|$ a.e. with $a, c, \in L^1(T)$.

Theorem 4.1 — *If hypotheses $H(K)$, $H(F)_1$ hold and $x_0 \in K(0)$, then* $S(x_0) = \overline{S_e(x_0)}$, *the closure taken in* $C(T, R^N)$.

Proof : Let $x \in S(x_0)$. Then by definition, $x = p(f)$ for some $f \in S^1_{F(\cdot, x(\cdot))}$ and with $p(\cdot)$ being the solution map introduced in the proof of Theorem 3.1. Also let $B_0 \in P_{kc}(C(T, R^N))$ be as in the proof of Thecrem 3.1. Given $z \in B_0$ and $\varepsilon > 0$, let $\Gamma : T \to 2^{R^N} \setminus \{\phi\}$ be defined by

$$\Gamma(t) = \left\{ u \in R^N : \| f(t) - u \| < \frac{\varepsilon}{2M_1 b} + d(f(t), F(t, z(t)), u \in F(t, z(t))) \right\}$$

with $M_1 > 0$ being the *a priori* bound for the elements in $S(x_0)$ obtained in the begining of the proof of the Theorem 3.1 Observe that

$$Gr\,\Gamma = \left\{ (t, u) \in GrF(\cdot, z(\cdot)) : \| f(t) - u \| - d(f(t), F(t, z(t))) < \frac{\varepsilon}{2M_1 b} \right\}$$

Because of hypotheses $H(F)_1$ (1) and (2) and theorem 3.3 of Papageorgiou[17], we have that $(t, x) \to F(t, x)$ is measurable $\Rightarrow t \to F(t, z(t))$ is measurable $\Rightarrow t \to d(f(t), F(t, z(t)))$ is measurable and also we have $GrF(\cdot, z(\cdot)) \in B(T) \times B(R^N)$ with $B(T)$ (resp. $B(R^N)$) being the Borel *σ-field* of T (resp. of R^N). Also since $(t, u) \to \| f(t) - u \| - d(f(t), F(t, z(t)))$ is a Caratheodory function (i.e. measurable in t, continuous in u), it is jointly measurable (see for example Kisielewicz[14], Proposition 3.2, p.40).

So we deduce that $Gr\,\Gamma \in B(T) \times B(R^N)$. Apply Aumann's selection theorem (see Wagner[9], Theorem 5.10), to get $u : T \to R^N$ measurable such that $u(t) \in \Gamma(t)$ a.e.

Therefore, if we define $L_\varepsilon : B_0 \to 2^{L^1(T, R^N)}$ by

$$L_\varepsilon(z)=\left\{u\in S^1_{F(\cdot,z(\cdot))}:\|f(t)-u(t)\|<\frac{\varepsilon}{4M_1b}+d(f(t),F(t,z(t))),\ a.e.\right\},$$

we see that $L_\varepsilon(\cdot)$ has nonempty values and in addition proposition 2.3 of Fryszkowski[18] tells us that $L_\varepsilon(\cdot)$ is l.s.c. and has decomposable values (recall that a subset $D\subseteq L^1(T,R^N)$ is said to be decomposable, if for all $u_1,u_2\in D$ and all $A\subseteq T$ measurable $\chi_A u_1+\chi_{A^c}u_2\in D$).

Therefore, $z\to\overline{L_\varepsilon(z)}$ is l.s.c. and has decomposable values. Apply Theorem 3.1 of Fryszkowski[18], to get $u_\varepsilon: B_0\to L^1(T,R^N)$ continuous such that $u_\varepsilon(z)\in\overline{L_\varepsilon(z)}$ for all $z\in B_0$.

Thus we have $\|f(t)-u_\varepsilon(z)(t)\|\le\frac{\varepsilon}{4M_1b}+d(f(t),F(t,z(t)))$

$$\le\frac{\varepsilon}{4M_1b}+k(t)\|x(t)-z(t)\|\ a.e.$$

In addition, from theorem 1 of Tolstonogov[12], we know that there exists a continuous map $w_\varepsilon: B_0\to L^1_w(T,R^N)$ such that $w_\varepsilon(z)\in \text{ext}\,S^1_{F(\cdot,z(\cdot))}=S^1_{\text{ext}\,F(\cdot,z(\cdot))}$ and $\|u_\varepsilon(z)-w_\varepsilon(z)\|_w<\varepsilon$ for all $z\in B_0$.

Next let $\varepsilon_n\downarrow 0$ and set $u_n=u_{\varepsilon_n}$, $w_n=w_{\varepsilon_n}$. Let $\eta_n: B_0\to B_0$ be defined by $\eta_n(x)=p(w_n(x))$ and let $x_n\in B_0$, $n\ge 1$ be such that $\eta_n(x_n)=x_n$ (their existence is guaranteed by Schauder's fixed point theorem, since B_0 is compact in $C(T,R^N)$ and $\eta_n(\cdot)$ is continuous; see the proof of Theorem 3.1). By passing to a subsequence if necessary, we may assume that $x_n\to z$ in $C(T,R^N)$. Also note that $x_n\in S_e(x_0)$. We will show now that $z=x$.

To this end, note that because of the monotonicity of $\partial\delta_{K(t)}(\cdot)=N_{K(t)}(\cdot)$, we have

$$(-\dot{x}(t)+\dot{x}_n(t),\ x_n(t)-x(t))\le(f(t)-w_n(x_n)(t),\ x_n(t)-x(t))\ a.e.$$

$$\Rightarrow\qquad \frac{1}{2}\frac{d}{dt}\|x_n(t)-x(t)\|^2\le(f(t)-w_n(x_n)(t),\ x_n(t)-x(t))\ a.e.$$

Integrating we get

$$\|x_n(t)-x(t)\|^2\le 2\int_0^t(f(s)-w_n(x_n)(s),\ x_n(s)-x(s))\,ds$$

$$\le 2\int_0^t(f(s)-u_n(x_n)(s),x_n(s)-x(s))\,ds$$

$$+\ 2\int_0^t(u_n(x_n)(s)-w_n(x_n)(s),x_n(s)-x(s))\,ds$$

$$\leq 2 \int_0^t \| f(s) - u_n(x_n)(s) \| \, \| x_n(s) - x(s) \| \, ds$$

$$+ \; 2 \int_0^t (u_n(x_n)(s) - w_n(x_n)(s), x_n(s) - x(s)) \, ds$$

$$\leq \varepsilon_n + 2 \int_0^t k(s) \| x_n(s) - x(s) \|^2 \, ds$$

$$+ \; 2 \int_0^t (u_n(x_n)(s) - w_n(x_n)(s), x_n(s) - x(s)) \, ds$$

Recall that $\| u_n(x_n) - w_n(x_n) \|_w \to 0$ as $n \to \infty$ and as in the proof of Theorem 3.1, using Gutman's theorem[15], we get that $(u_n(x_n) - w_n(x_n)) \xrightarrow{w} 0$ in $L^1(T, R^N)$. Since $x_n \to z$ in $C(T, R^N)$, we have

$$\int_0^t (u_n(x_n)(s) - w_n(x_n)(s), x_n(s) - x(s)) \, ds \to 0 \text{ as } n \to \infty.$$

Therefore, in the limit as $n \to \infty$, we get $\| z(t) - x(t) \|^2 \leq \int_0^t k(s) \| z(s) - x(s) \|^2 \, ds$

$\Rightarrow$ $x = z$ (by Gronwall's inequality); i.e. $x_n \to x$ in $C(T, R^N)$.

Recalling that $x_n \in S_e(x_0)$ we see that we have proved that $S(x_0) \subseteq \overline{S_e(x_0)}$, the closure taken in $C(T, R^N)$.

To conclude the proof of our theorem, we need to show that $S(x_0)$ is closed in $C(T, R^N)$. So let $x_n \in S(x_0)$, $n \geq 1$ and assume that $x_n \to x$ in $C(T, R^N)$. By definition $x_n = p(f_n)$ with $f_n \in S^1_{F(\cdot, x_n(\cdot))}$. As before we may assume that $\dot{x}_n \xrightarrow{w} \dot{x}$ in $L^1(T, R^N)$ and $f_n \xrightarrow{w} f$ in $L^1(T, R^N)$.

Denote by $\langle \cdot, \cdot \rangle$ the duality brackets for the pair $(L^1(T, R^N), L^\infty(T, R^N))$.

For every $g \in L^\infty(T, R^N)$ we have

$$\langle f_n, g \rangle \leq \sigma(g, S^1_{F(\cdot, x_n(\cdot))}) = \sup [\langle g, v \rangle : v \in S^1_{F(\cdot, x_n(\cdot))}]$$

$\Rightarrow$ $\langle f, g \rangle \leq \overline{\lim} \, \sigma(g, S^1_{F(\cdot, x_n(\cdot))}) = \sigma(g, S^1_{F(\cdot, x_n(\cdot))})$ (see theorem 4.5 of Papageorgiou[16]).

Since $g \in L^\infty(T, R^N)$ was arbitrary, we have proved that $f \in S^1_{F(\cdot, x(\cdot))}$. Then as in the proof of theorem 3.1, via Theorem 3.1 of Papageorgiou[16] and Theorem 1, p.220 of Aubin-Cellina[1], we get that

$$-\dot{x}(t) \in N_{K(t)}(x(t)) + f(t) \quad a.e., \quad x(0) = x_0$$

with $f \in S^1_{F(\cdot, x(\cdot))}$.

So $x \in S(x_0) \Rightarrow S(x_0)$ is closed in $C(T, R^N) \Rightarrow S(x_0) = \overline{S_e(x_0)}$.

5. The Topological Structure of the Solution Set

In this section we turn our attention to the topological structure of the solution set $S(x_0)$, of the "convexified" problem (1). We show that under the hypotheses of theorem 4.1, $S(x_0)$ is a nonempty, compact and path-connected subset $C(T, R^N)$.

***Theorem* 5.1** — *If hypotheses $H(K)$, $H(F)_1$ hold and $x_0 \in K(0)$, then $S(x_0)$ is a nonempty, compact and path-connected subset $C(T, R^N)$.*

PROOF : The nonemptiness and compactness of $S(x_0) \subseteq C(T, R^N)$, follow from theorems 3.1 and 3.2. So we have to prove the path connectedness property.

As in the proof of theorem 3.1, let $V = \{u \in L^1(T, R^N) : \| u(t) \| \leq \phi(t) \ a.e.\}$. Let $R : V \to 2^V \setminus \{\phi\}$ be defined by $R(u) = S^1_{F(\cdot, p(u)(\cdot))}$.

On $L^1(T, R^N)$, we consider the equivalent norm $| \cdot |$ defined by

$$| u | = \int_0^b \| u(t) \| \exp\left(-L \int_0^t k(s)\, ds \right) dt, \qquad L > 1.$$

In what follows, by h_1 and d_1 we will denote the Hausdorff metric and the distance function respectively, corresponding to this new norm $| \cdot |$. We will show that $R(\cdot)$ is h_1-Lipschitz, with constant $\frac{1}{L} < 1$ (since $L > 1$). To this end let $u_1, u_2 \in V$ and $f_1 \in R(u_1)$. Via a straightforward application of Aumann's selection theorem, we can find $f_2 \in R(u_2)$ such that $\| f_1(t) - f_2(t) \| = d(f_1(t), F(t, p(u_2)(t)))$ $a.e.$

Then we have :

$$\begin{aligned} d_1(f_1, R(u_2)) &\leq | f_1 - f_2 | \\ &= \int_0^b \| f_1(t) - f_2(t) \| \exp\left(-L \int_0^{t} k(s)\, ds \right) dt \\ &= \int_0^b d(f_1(t), F(t, p(u_2)(t))) \exp\left(-L \int_0^t k(s)\, ds \right) dt \\ &\leq \int_0^b h(F(t, p(u_1)(t)), F(t, p(u_2)(t))) \exp\left(-L \int_0^t k(s)\, ds \right) dt \end{aligned}$$

Due to the monotonicity of $N_{K(t)}(\cdot) = \partial \delta_{K(t)}(\cdot)$, we have

$$(-\dot{p}(u_1)(t) + \dot{p}(u_2)(t), p(u_2)(t) - p(u_1)(t))$$
$$\leq (u_1(t) - u_2(t), p(u_2)(t) - p(u_1)(t)) \ a.e.$$

$$\Rightarrow \qquad \frac{1}{2}\frac{d}{dt}\| p(u_1)(t)) - p(u_2)(t) \|^2 \le \| u_2(t) - u_1(t) \| \, \| p(u_1)(t)) - p(u_2)(t) \|$$

$$\Rightarrow \qquad \frac{1}{2}\| p(u_1)(t) - p(u_2)(t) \|^2 \le \int_0^t \| u_1(s) - u_2(s) \| \, \| p(u_1)(s) - p(u_2)(s) \| \, ds$$

Invoking lemma A.5, p.157 of Brezis[19], we get

$$\| p(u_1)(t) - p(u_2)(t) \| \le \int_0^t \| u_1(s) - u_2(s) \| \, ds$$

So using hypothesis $H(F)_1$ (2), we get

$$d_1(f_1, R(u_2) \le \int_0^b k(t) \int_0^t \| u_1(s) - u_2(s) \| \, ds \exp\left(-L \int_0^t k(s)\, ds \right) dt$$

$$= -\frac{1}{L} \int_0^b \left(\int_0^t \| u_1(s) - u_2(s) \| \, ds \right) d\left(\exp\left(-L \int_0^t k(s)\, ds \right)\right)$$

$$\le \frac{1}{L} \int_0^b \left(\exp\left(-L \int_0^t k(s)\, ds \right)\right) \| u_1(t) - u_2(t) \| \, dt$$

(integration by parts)

$$= \frac{1}{L} | u_1 - u_2 | .$$

Similarly by interchanging the roles of f_1 and f_2, we get $d_1(f_2, R(u_1)) \le \frac{1}{L} | u_1 - u_2 |$.

Hence, recalling the definition of the Hausdorff metric (see section 2), we get

$$h_1(R(u_1), R(u_2)) \le \frac{1}{L} | u_1 - u_2 | .$$

Let $\Theta = \{u \in V : u \in R(u)\}$. From Nadler[20], we know that $\Theta \neq \phi$, while from Ricceri[21] we have that Θ is an absolute retract, a fortiori then path-connected in $L^1(T, R^N)$ (see Kuratowski[22], p.339). Recalling that path-connectedness is a topological invariant, we get that $p(\Theta)$ is path-connected in $C(T, R^N)$. But observe that $p(\Theta) = S(x_0)$.

So $S(x_0)$ is path-connected in $C(T, R^N)$.

6. Solution Selector and Periodic Trajectories

In this section, we view the solution set $S(x_0)$ as a multifunction of the initial datum and we establish the existence of a continuous selector for that multifunction. Although as we will show in section 7, $x_0 \to S(x_0)$ may have nice continuity properties, the lack of convexity in the values of $S(\cdot)$ prohibits the use of Michael's selection theorem. Similarly the lack of decomposability in the values of $S(x_0)$, rules out the use of theorem 3.1 of Fryszkowski[18]. Nevertheless, the latter is an important tool in our proof, which is based on Filippov's construction of approximate solutions for a simple differential inclusion in R^N (see Filippov[23]).

***Theorem* 6.1** — *If hypotheses $H(K)$, $H(F)_1$ hold then there exists $r : K(0) \to C(T, R^N)$ a continuous map such that $r(x_0) \in S(x_0)$ for all $x_0 \in K(0)$.*

PROOF : First consider the following unperturbed differential variational inequality

$$\left\{ \begin{array}{c} -\dot{x}(t) \in N_{K(t)}(x(t)) \ a.e. \\ x_0 \in K(0) \end{array} \right\}$$

From proposition 1, we know that this has a unique solution $x_0(v)(\cdot) \in C(T, R^N)$.

Furthermore because of the monotonicity of $N_{K(t)}(\cdot) = \partial\delta_{K(t)}(\cdot)$, we can easily check that

$$\| x_0(v_1) - x_0(v_2) \|_{C(T, R^N)} \le \| v_1 - v_2 \|$$

Set $\gamma(v)(t) = a(t) + c(t)\, x_0(v)(t)$. Clearly $v \to \gamma(v)(\cdot)$ is continuous from $K(0)$ into $L^1(T, R^N)$.

Let $\varepsilon > 0$ and $\varepsilon_n = \frac{\varepsilon}{2^{n+1}}$, $n \ge 0$. Let $R_0 : K(0) \to 2^{L^1(T, R^N)} \setminus \{\phi\}$ be the multifunction defined by

$$R_0(v) = \left\{ u \in S^1_{F(\cdot, x_0(v)(\cdot))} : \| u(t) \| < \gamma(v)(t) + \varepsilon_0 \right\}$$

From proposition 2.3 of Fryszkowski[18], we know that $v \to R_0(v)$ is l.s.c. and has decomposable values. Hence $v \to \overline{R_0(v)}$ is l.s.c. with decomposable values. Apply Theorem 3.1 of Fryszkowski[18], to get $r_0 : K(0) \to L^1(T, R^N)$ a continuous map such that $r_0 \in \overline{R_0(v)}$ for all $v \in K(0)$. Let $\theta(t) = \int_0^t k(s)\, ds$ and

$$\eta_n(v)(t) = \int_0^t \gamma(v)(s) \frac{(\theta(t) - \theta(s))^{n-1}}{(n-1)!}\, ds + b \left(\sum_{k=0}^{n} \varepsilon_k \right) \frac{\theta(t)^{n-1}}{(n-1)!}, \quad n \ge 1$$

Clearly $v \to \eta_n(v)(\cdot)$ is continuous from $K(0)$ into $L^1(T, R^N)$. Consider the following Cauchy problem :

$$\left\{ \begin{array}{c} -\dot{x}(t) \in N_{K(t)}(x(t)) + r_0(v)(t) \ a.e. \\ x_0 = v \end{array} \right\}$$

From proposition 2, we know that this problem has a unique solution $x_1(v)(\cdot) \in C(T, R^N)$ and furthermore

$$\| x_1(v)(t) - x_0(v)(t) \| \leq \int_0^t \| r_0(v)(s) \| ds$$

$$\leq \int_0^t (\gamma(v)(s) + \varepsilon_0) ds$$

$$\leq \int_0^t \gamma(v)(s) ds + b\varepsilon_0 < \eta_1(v)(t).$$

By induction, we will show that we can obtain two sequences

$$\{x_n(v)(\cdot)\}_{n \geq 1} \subseteq C(T, R^N) \text{ and } \{r_n(v)(\cdot)\}_{n \geq 0} \subseteq L^1(T, R^N) \text{ such that}$$

(1) $x_n(v)(\cdot) = p(r_{n-1}(v))(\cdot)$ $n \geq 1$,

(2) $v \to r_n(v)(\cdot)$ is continuous from $K(0)$ into $L^1(T, R^N)$,

(3) $r_n(v)(t) \in F(t, x_n(v)(t))$ *a.e.*,

(4) $\| r_n(v)(t) - r_{n-1}(v)(t) \| \leq k(t)\, \eta_n(v)(t)$ *a.e.*

So suppose that we have already obtained $\left\{ x_n(v)(\cdot) \right\}_{n=1}^m$ and $\left\{ r_n(v)(\cdot) \right\}_{n=0}^m$ satisfying properties (1)-(4) above. Let $x_{m+1}(v)(\cdot) = p(r_m(v))(\cdot)$. Then as before, because of the monotonicity of $\partial\delta_{K(t)}(\cdot) = N_{K(t)}(\cdot)$, we have

$$\| x_{m+1}(v)(t) - x_m(v)(t) \| \leq \int_0^t \| r_m(v)(s) - r_{m-1}(v)(s) \| ds$$

$$\leq \int_0^t k(s)\, \eta_m(v)(s) ds$$

$$\leq \int_0^t k(s) \left[\int_0^s \gamma(v)(\tau) \frac{(\theta(s) - \theta(\tau))^{m-1}}{(m-1)!} d\tau + b \left(\sum_{k=0}^m \varepsilon_k \right) \frac{\theta(s)^{m-1}}{(m-1)!} \right] ds$$

$$\leq \int_0^t \gamma(v)(s) \int_s^t k(\tau) \frac{(\theta(\tau) - \theta(s))^{m-1}}{(m-1)!} d\tau\, ds$$

$$+ \int_0^t k(s)\, b \left(\sum_{k=0}^m \varepsilon_k \right) \frac{\theta(s)^{m-1}}{(m-1)!} ds$$

$$= \int_0^t \gamma(v)(s) \int_s^t \frac{1}{m!} \frac{d}{d\tau} (\theta(\tau) - \theta(s))^m \, ds + b \left(\sum_{k=0}^{m} \varepsilon_k \right) \frac{\theta(t)^m}{m!}$$

$$= \int_0^t \gamma(v)(s) \frac{(\theta(\tau) - \theta(s))^m}{m!} \, ds + b \left(\sum_{k=0}^{m} \varepsilon_k \right) \frac{\theta(t)^m}{m!} < \eta_{m+1}(v)(t)$$

Then let $R_{m+1} : K(0) \to 2^{L^1(T, R^N)} \setminus \{\phi\}$ be the multifunction defined by

$$R_{m+1}(v) = \{ u \in S^1_{F(\cdot, x_{m+1}(v)(\cdot))} : d(r_n(v)(t), \; F(t, x_{m+1}(v)(t))) < k(t)\, \eta_{m+1}(v)(t) \;\; a.e. \}$$

From proposition 2.3 of Fryszkowski[18], we know that $v \to R_{m+1}(v)$ is l.s.c. with decomposable values, and then so is $v \to \overline{R_{m+1}(v)}$. Thus Theorem 3.1 of Fryszkowski[18] gives us a continuous map $r_{m+1} : K(0) \to L^1(T, R^N)$ such that $r_{m+1}(v) \in R_{m+1}(v)$ for all $v \in K(0)$. Therefore, by induction we have the two sequences $\{x_n(v)(\cdot)\}_{n \geq 1} \subseteq C(T, R^N)$ and $\{r_n(v)(\cdot)\}_{n \geq 0} \subseteq L^1(T, R^N)$ satisfying (1)-(4) above.

Also from the above estimations, we have

$$\int_0^b \| r_n(v)(t) - r_{n-1}(v)(t) \| \, dt$$

$$\leq \int_0^b \gamma(v)(s) \frac{(\theta(b) - \theta(s))^n}{n!} \, ds + b \left(\sum_{k=0}^{n} \varepsilon_k \right) \frac{\theta(b)^n}{n!}$$

$$\leq \frac{\| k \|_1}{n!} (\| \gamma(v) \|_1 + b\varepsilon).$$

Since $v \to \gamma(v)$ is continuous, from the above inequality, we deduce that $\{r_n(v)(\cdot)\} n \geq 0$ is Cauchy in $L^1(T, R^N)$, uniformly in $v \in K(0)$ since the latter is compact in R^N. So $r_n(v)(\cdot) \xrightarrow{s} r(v)(\cdot)$ in $L^1(T, R^N)$, with $v \to r(v)(\cdot)$ continuous from $K(0)$ into $L^1(T, R^N)$. Also since for all $t \in T$

$$\| x_{n+1}(v)(t) - x_n(v)(t) \| \leq \| r_n(v) - r_{n-1}(v) \|_1$$

we have that $\{x_n(v)(\cdot)\}_{n \geq 1} \subseteq C(T, R^N)$ is Cauchy in $C(T, R^N)$, uniformly in $v \in K(0)$.

Hence $x_n(v)(\cdot) \to x(v)(\cdot)$ in $C(T, R^N)$ and $v \to x(v)(\cdot)$ continuous from $K(0)$ into $C(T, R^N)$.

Finally note that $r_n(v)(\cdot) \in S^1_{F(\cdot, x_n(v)(\cdot))}$ and $S^1_{F(\cdot, x_n(v)(\cdot))} \xrightarrow{h} S^1_{F(\cdot, x(v)(\cdot))}$ (see theorem 4.5 of Papageorgiou[16]). So $r(v) \in S^1_{F(\cdot, x(v)(\cdot))}$ and because of the continuity of the solution map $p(\cdot)$, it is easy to see that $x(v) = p(r(v))$. So $v \to x(v)$ is the desired continuous selector of $v \to S(v)$.

We can use theorem 6.1 to establish the existence of periodic solutions for (1). So our problem is now the following :

$$\left\{ \begin{array}{c} -\dot{x}(t) \in N_{K(t)}(x(t)) + F(t, x(t))\ a.e. \\ x_0 = x(b) \end{array} \right\} \qquad \dots (4)$$

Theorem 6.2 — *If hypotheses $H(K)$, $H(F)_1$ hold and $K(b) \subseteq K(0)$, then problem (4) admits a solution.*

PROOF : From theorem 6.1, we know that there exists $u : K(0) \to C(T, R^N)$ a continuous map such that $u(v) \in S(v)$ for all $v \in K(0)$. Let $\xi : K(0) \to K(b) \subseteq K(0)$ be defined by $\xi(v) = e_b(u(v)) = u(v)(b)$ (here $e_b : C(T, R^N) \to R^N$ is the evaluation at b map, which is continuous).

Clearly $\xi(\cdot)$ is continuous. So we can find $v \in K(0)$ such that $v = \xi(v)$.

Then $u(v)(\cdot) \in C(T, R^N)$ solves (4).

7. Continuous Dependence Results

In Papageorgiou[8] (section 5), we are considered parametrized differential variational inequalities and established that the solution set is an u.s.c. multifunction of the parameter. Only the orientor field F and the initial condition x_0 depended on the parameter. In our present work, we allow the parameter to appear also in the constraint multifunction K, we show that we actually have continuity of the solution multifunction $\lambda \to S(\lambda)$ (both in the Vietoris and Hausdorff topology), with λ being the parameter and in addition our overall hypotheses on the data are now weaker than those in [8].

So let Λ be a complete metric space (the parameter space). We consider the following family of differential variational inequalities, parametrized by $\lambda \in \Lambda$.

$$\left\{ \begin{array}{c} -\dot{x}(t) \in N_{K(t,\lambda)}(x(t)) + F(t, x(t), \lambda)\ a.e. \\ x_0 = x_0(\lambda) \end{array} \right\} \qquad \dots (5)$$

We denote the solution set of (5), by $S(\lambda)$. Our goal in this section is to establish the continuity properties of the multifunction $\lambda \to S(\lambda)$. For this we will need the following hypotheses on the data :

H(K)$_1$: $K : T \times \Lambda \to P_{kc}(R^N)$ is a multifunction such that

(1) $h(K(t', \lambda), K(t, \lambda)) \le \int_t^{t'} v_B(s)\, ds$ for all $0 \le t \le t' \le b$ and all $\lambda \in B \subseteq \Lambda$ compact,

(2) $\lambda \to K(t, \lambda)$ is continuous (note that here continuity is both in the Vietoris and Hausdorff hyperspace topologies since $K(\cdot, \cdot)$ is $P_{kc}(R^N)$-*valued*; see section 2).

H(F)$_2$: $F : T \times R^N \times \Lambda \to P_{kc}(R^N)$ is a multifunction such that

(1) $t \to F(t, x, \lambda)$ is measurable,

(2) $h(F(t,x,\lambda), F(t,y,\lambda)) \leq k_B(t) \| x-y \|$ *a.e.* for $x, y \in R^N$ and all $\lambda \in B \subseteq \Lambda$ compact,

(3) $| F(t,x,\lambda) | \leq a_B(t) + c_B(t) \| x \|$ for all $\lambda \in B \subseteq \Lambda$ compact and with $a_B, c_B \in L^1(T)$,

(4) $\lambda \to F(t,x,\lambda)$ is continuous

H_0 : $\lambda \to x_0(\lambda)$ is continuous from Λ into R^N.

Note that under these hypotheses, for every $\lambda \in \Lambda$, $S(\lambda) \in P_k(C(T, R^N))$; see theorem 4.1.

Theorem 7.1 — *If hypotheses $H(K)_1$, $H(F)_2$ and H_0 hold, then $\lambda \to S(\lambda)$ is continuous from Λ into $P_k(C(T, R^N))$.*

PROOF : First we will show that $S(\cdot)$ is l.s.c. To get this it suffices to show that if $\lambda_n \to \lambda$ then $S(\lambda) \subseteq \underline{\lim}\, S(\lambda_n) = \{x \in C(T, R^N) : \lim d(x, S(\lambda_n)) = 0\} = \{x \in C(T, R^N) : x = \lim x_n, x_n \in C(T, R^N), n \geq 1\}$ (see Klein-Tompson[10]). To this end $x(\cdot) \in S(\lambda)$. Then by definition $x = p(f)$ with $f \in S^1_{F(\cdot, x(\cdot), \lambda)}$.

Let $\lambda' \in \Lambda$ and define

$$m(t, \lambda') = proj\, [f(t); F(t, x(t), \lambda')]$$

$$u(t, z, \lambda') = proj\, [m(t, \lambda'); F(t, z, \lambda')]$$

where as before $proj(\cdot, \cdot)$ denotes the metric projection map. Note that $Grm\, (\cdot, \lambda') = \{(t, y) \in T \times R^N : \| f(t) - y \| = d(f(t), F(t, x(t), \lambda')),\ y \in F(t, x(t), \lambda')\}$.

From hypotheses $H(F)_2(1)$ and (2) and theorem 3.3 of Papageorgiou[17], we have that $t \to F(t, x(t), \lambda')$ is measurable. So $Grm(\cdot, \lambda') \in B(T) \times B(R^N) \Rightarrow t \to m(t, \lambda')$ is measurable. In a similar fashion, we also get that $t \to u(t, z, \lambda')$ is measurable. Also if $z_m \to z$, then $F(t, z_m, \lambda') \to F(t, z, \lambda')$ (both in the Hausdorff metric and Kuratowski senses, since by $H(F)_2$, F is $P_{kc}(R^N)$*-valued*) and so by Theorem 3.33, p.322 of Attouch[24], we have that $u(t, z_m, \lambda') \to u(t, z, \lambda')$ as $m \to \infty$. So $z \to u(t, z, \lambda')$ is continuous.

Now consider the following differential variational inequality :

$$\left\{ \begin{array}{c} -\dot{x}(t) \in N_{K(t, \lambda_n)}(x(t)) + u(t, x(t), \lambda_n)\ a.e. \\ x(0) = x_0(\lambda_n), \quad n \geq 1 \end{array} \right\}$$

From theorem 3.1, we know that this Cauchy problem has at least one solution $x_n(\cdot) \in C(T, R^N)$, $n \geq 1$. As in the begining of the proof of theorem 3.1, an easy *a priori* estimation, tells us that with $B = \{\lambda, \lambda_n\}_{n \geq 1} \subseteq \Lambda$, we can get $M_B > 0$ such that for all $n \geq 1$ and all $t \in T$, we have $\| x_n(t) \| \leq M_B$. Also from proposition 2, we have

$$\| \dot{x}_n(t) \| \leq a_B(t) + c_B(t) M_B + 2 \| u(t, x_n(t), \lambda_n) \| \ a.e.$$

$$\leq 3a_B(t) + 3c_B(t) M_B = \hat{\phi}_B(t) \ a.e.,$$

with $\hat{\phi}_B \in L^1(T)$ (see hypothesis $H(F)_2(3)$).

Thus via the Arzela-Ascoli and Dunford-Pettis theorems, we may assume by passing to a subsequence if necessary that $x_n \to z$ in $C(T, R^N)$ and $\dot{x}_n \xrightarrow{w} \dot{z}$ in $L^1(T, R^N)$. We will show that $x = y$.

Note that

$$\| u(t, x_n(t), \lambda_n) - u(t, z(t), \lambda) \|$$

$$= \| \text{proj} \ [m(t, \lambda_n); F(t, x_n(t), \lambda_n)] - \text{proj} \ [m(t, \lambda); F(t, z(t), \lambda)] \|$$

$$\leq \| \text{proj} \ [m(t, \lambda_n); F(t, x_n(t), \lambda_n)] - \text{proj} \ [m(t, \lambda); F(t, x_n(t), \lambda_n)] \|$$

$$+ \| \text{proj} \ [m(t, \lambda); F(t, x_n(t), \lambda_n)] - \text{proj} \ [m(t, \lambda); F(t, z(t), \lambda)] \|$$

$$\leq \| m(t, \lambda_n) - m(t, \lambda) \| + \| \text{proj} \ [m(t, \lambda); F(t, x_n(t), \lambda_n)]$$

$$- \text{proj} \ [m(t, \lambda); F(t, z(t), \lambda)] \|$$

From theorem 3.33, p.322 of Attouch[24], we get that $m(t, \lambda_n) \to m(t, \lambda)$, while $F(t, x_n(t), \lambda_n) \xrightarrow{h} F(t, z(t), \lambda)$ (hypothesis $H(F)_2(2)$ and (4)) from which follows that proj $[m(t, \lambda); F(t, x_n(t), \lambda_n)] \to$ proj $[m(t, \lambda); F(t, z(t), \lambda)]$ as $n \to \infty$ (see Attouch[24], p.322). So finally we have $\| u(t, x_n(t), \lambda_n) - u(t, z(t), \lambda) \| \to 0$ as $n \to \infty$.

From theorem 3.1 of Papageorgiou[16], we get that

$$-\dot{y}(t) - u(t, z(t), \lambda) \in \overline{conv} \ \overline{\lim} \ \{-\dot{x}_n(t) - u(t, x_n(t), \lambda_n)\}_{n \geq 1}$$

$$\subseteq \overline{conv} \ \overline{\lim} \ N_{K(t, \lambda_n)}(x_n(t)) \ a.e.$$

From Theorem 3.66, p.373 of Attouch[24], we know that

$$\overline{\lim} \ N_{K(t, \lambda_n)}(x_n(t)) \subseteq N_{K(t, \lambda)}(z(t))$$

$$\Rightarrow \qquad -\dot{z}(t) \in N_{K(t, \lambda)}(z(t)) + u(t, z(t), \lambda) \ a.e., \quad z(0) = x_0(\lambda).$$

As before, exploiting the monotonicity of $N_{K(t, \lambda)}(\cdot) = \partial \delta_{K(t, \lambda)}(\cdot)$, we get

$$\| x(t) - y(t) \| \leq \int_0^t \| f(s) - u(s, y(s), \lambda) \| ds$$

$$\leq \int_0^t (\| f(s) - m(s, \lambda_n) \| + \| m(s, \lambda_n) - u(s \ y(s), \lambda) \|) \, ds$$

$$\leq \int_0^t [d(f(s), F(s, x(s), \lambda_n)) + \| m(s, \lambda_n) - u(s, y(s), \lambda_n) \|$$

$$+ \| u(s, y(s), \lambda_n) - u(s, y(s), \lambda) \|] \, ds$$

$$\leq \int_0^t [d(f(s), F(s, x(s), \lambda_n)) + k_B(s) \| x(s) - y(s) \|$$

$$+ \| u(s, y(s), \lambda_n) - u(s, y(s), \lambda) \|] \, ds$$

Note that $d(f(s), F(s, x(s), \lambda_n)) \to 0$ as $n \to \infty$ and also

$$\| u(s, y(s), \lambda_n) - u(s, y(s), \lambda) \|$$

$$\leq \| \text{proj} [m(s, \lambda_n); F(s, y(s), \lambda_n)] - \text{proj} [m(s, \lambda); F(s, y(s), \lambda_n)] \|$$

$$+ \| \text{proj} [m(s, \lambda); F(s, y(s), \lambda_n)] - \text{proj} [m(s, \lambda); F(s, y(s), \lambda)] \|$$

$$\leq \| m(s, \lambda_n) - m(s, \lambda) \|$$

$$+ \| \text{proj} [m(s, \lambda); F(s, y(s), \lambda_n)] - \text{proj} [m(s, \lambda); F(s, y(s), \lambda)] \|$$

$$\to 0 \text{ as } n \to \infty.$$

So in the limit as $n \to \infty$, we get $\| x(t) - y(t) \| \leq \int_0^t k_B(s) \| x(s) - y(s) \| \, ds \Rightarrow x = y$ (Gronwall's inequality).

Therefore, every subsequence of $\{x_n\}_{n \geq 1}$ has a further subsequence that converges to x in $C(T, R^N)$. So $x_n \to x$ in $C(T, R^N)$ and since $x_n \in S(\lambda_n)$, $n \geq 1$, we have established that

$$S(\lambda) \subseteq \underline{\lim}\, S(\lambda_n)$$

$$\Rightarrow \qquad \lambda \to S(\lambda) \text{ is l.s.c.}$$

Next we will show that

$$\overline{\lim} S(\lambda_n) = \{x \in C(T, R^N) : \underline{\lim}\, d(x, S(\lambda_n)) = 0\}$$

$$= \{x \in C(T, R^N) : x = \lim x_{n_k}, \; n_1 < n_2 < \ldots < n_k < \ldots\}$$

So let $x \in \overline{\lim} S(\lambda_n)$. Then by definition and by denoting subsequences with the same index as the original sequences, we know that we can find $x_n \in S(\lambda_n)$ such that $x_n \to x$ in $C(T, R^N)$. We have $x_n = p(f_n)$, with $f_n \in S^1_{F(\cdot, x_n(\cdot), \lambda_n)}$. As before by passing to a subsequence if necessary, we may assume that $\dot{x}_n \xrightarrow{w} \dot{x}$ and $f_n \xrightarrow{w} f$ in $L^1(T, R^N)$. From theorem 4.5 of [16], we have that $f \in S^1_{F(\cdot, x(\cdot), \lambda)}$. Also from Theorem 3.1 of[16] and theorem 3.88 of Attouch[24], we get

$$-\dot{x}(t) - f(t) \in \overline{conv}\{-\dot{x}_n(t) - f_n(t)\}_{n \geq 1}$$

$$\subseteq \overline{conv}\ \overline{\lim}\ N_{K(t,\lambda_n)}(x_n(t)) \subseteq N_{K(t,\lambda)}(x(t)) \text{ a.e.}$$

$$\Rightarrow \quad -\dot{x}(t) \in N_{K(t,\lambda)}(x(t)) + f(t) \text{ a.e.}, \quad x(0) = x_0(\lambda),\ f \in S^1_{F(\cdot, x(\cdot), \lambda)}$$

$$\Rightarrow \quad x \in S(\lambda)$$

$$\Rightarrow \quad \overline{\lim} S(\lambda_n) \subseteq S(\lambda).$$

Since for every $B \subseteq \Lambda$ compact, $\overline{\bigcup_{\lambda \in B} S(\lambda)}$ is compact in $C(T, R^N)$ (easily we can see it in the proof of Theorem 3.1), from the above inclusion we get that $S|_B$ is u.s.c. for every $B \subseteq \Lambda$ compact (here $S|_B$ denotes the restriction of S on B). This then by lemma δ of Papageorgiou[25] implies that $S(\cdot)$ is u.s.c.. Therefore, $S(\cdot)$ is Vietoris continuous.

Recall (see section 2) that on $P_k(C(T, R^N))$ the Vietoris and Hausdorff metric hyperspace topologies coincide. So we also have :

Theorem 7.2 — *If hypotheses $H(K)_1$, $H(F)_2$ and H_0 hold, then $\lambda \to S(\lambda)$ is h-continuous from Λ into $P_k(C(T, R^N))$.*

An interesting consequence of Theorems 7.1 and 7.2 is the following corollary, which improves theorem 5.2 of [8].

Corollary — If hypotheses $H(K)$, and $H(F)_1$ hold, then $x_0 \to S(x_0)$ is Vietoris and h-continuous from $K(0)$ into $P_k(C(T, R^N))$.

References

1. J. P. Aubin and A. Cellina, *Differential Inclusions*, Springer, Berlin (1984).
2. C. Henry, *J. econ. Theory.* **4** (1972), 545-551.
3. B. Cornet, *JMAA* **96** (1983), 130-147.
4. J. J. Moreau, *J. Diff. Eq.* **26** (1977), 347-374.
5. S. Shuzhong, *Nonlinear Anal. TMA* **12** (1988), 951-967.
6. M. A. Krasnoselskii and A. V. Pokrovskii, *Systems with Hysteresis*, Nauka, Moscow, (1983), pp. 1-272, MR : 86e:93005.
7. E. P. Avgerinos, Optimal control of differential inclusions with state constraints, *Nonlinear Anal. TMA* (to appear).
8. N. S. Papageorgiou, *Proc. Edinb. math. Soc.* **32** (1989), 81-98.
9. D. Wagner, *SIAM J. Control Optim.* **15** (1977), 859-903.
10. E. Klein and A. Thomoson, *Theory of Correspondences*, Wiley, New York (1984).
11. J. Diestel and J. J. Uhl, *Math. Surveys* **15**, A.M.S. Providence, Rhode Island (1977).
12. A. A. Tolstonogov, *Sov. Math. Doklad.* **43** (1991), 481-85.
13. M. Benamara, Points Extremaux, Multiapplications et Fonctionelles Integrales, *These de 3eme Cycle*, Univ. Grenoble (1975).

14. M. Kisielewicz, *Differential Inclusions and Optimal Control*, Kluwer Academic Publ., Dodrecht, The Netherlands (1991).
15. S. Gutman, *Proc. AMS* **93** (1985), 40-42.
16. N. S. Papageorgiou, *Int. J. Math. math. Sci.* **10** (1987), 433-42.
17. N. S. Papageorgiou, *Math. Japonica* **32** (1987), 437-64.
18. A. Fryszkowski, *Studia Math.* **76** (1983), 163-74.
19. H. Brezis, *Math. Studies* **50**, North Holland, Amsterdam (1972).
20. S. B. Nadler, *Lect. Not. Math.* **171** (1971), Springer, Berlin, 64-69.
21. B. Ricceri, *Atti. Acad. Naz. Linci. U. Sci. Fiz. Math. Nat.* **81** (1987), 283-86.
22. K. Kuratowski, *Topology II*, Academic Press, London (1966).
23. A. F. Fillipov, *SIAM J. Control* **5** (1967), 609-21.
24. H. Attouch, *Variational Convergence for Functionals and Operators*, Pitman, London (1984).
25. N. S. Papageorgiou, *Proc. Indian Acad. Sci.* **100** (1990), 65-78.