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TEMPERATURE DISTRIBUTION IN A GENERALIZED THERMOELASTIC SOLID DUE TO HEAT SOURCES

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The distribution of temperature in an infinite homogeneous transversely isotropic thermoelastic spherically symmetric space due to heat sources has been studied by using the method of potentials. The fundamental equations of generalized thermoelasticity for the potential are solved using the operational method along with the Laplace transform techniques to obtain the solution in the form of an infinite series of functions which are valid for all times. It is found that the first three (four) terms of the series dominate the rest of the terms in both of the L-S (G-L) theories of generalized thermoelasticity. The term(s) representing generalized thermoelasticity dominate the term representing coupled thermoelasticity when the ratio of time to the radial distance is small and for other values of distance and time, both the terms have same order of magnitude. The corrections due to the L-S and G-L theories of generalized thermoelasticity on the magnitude of limiting temperature function are found to be in the ratio of 3 : 5 in both the cases of considered heat sources.

Key Words : Temperature Distribution; Generalised Thermoelasticity; Laplace Transform; Coupled Thermoelasticity

1. INTRODUCTION

The coupled theory of thermoelasticity has been extended by including the thermal relaxation times in the constitutive equations by Lord and Shulman[1] and Green and Lindsay[2]. These theories eliminate the paradox of infinite velocity of heat propagation and here in after are named as L-S and G-L generalized theories of thermoelasticity, respectively. The theory[1] was extended by Dhaliwal and Sherief[3] to general anisotropic materials in the presence of heat sources. Hetnarski used the method of asymptotic expansions valid for small values of time to obtain the solution for a thermal shock half-space problem[4] and to solve a spherically symmetric problem

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with a point source of heat[5], both in the context of coupled thermoelasticity. Sherief and Dhaliwal[6] and Sherief[7] used the same method to solve these problems in generalized thermoelasticity, respectively. Sherief and Anwar[8] treated the cylindrically symmetric case. Hetnarski[9] used the operational method successfully to obtain the solution to a problem similar to that of Ref. 5 in the form of a series of functions. Sherief and Ezzat[10] solved the fundamental problem of thermoelasticity for an infinite spherically symmetric space by using the method of potentials and employing the operational method together with the Laplace transform techniques to obtain the solution in the form of an infinite series of functions. Noda *et al.*[11] studied the problem of generalized thermoelasticity in an infinite solid with a hole.

In this article, the operational method is used to find the temperature distribution in generalized thermoelasticity for a spherically symmetric homogeneous transversely isotropic space with instantaneous and continuous heat sources, in the form of a series of known functions. It is found that the first three terms of the series dominate the rest of the terms for L-S theory and the first four terms dominate the rest for G-L theory.

2. Formulation of the Problem

We consider an infinitely extended homogeneous transversely isotropic thermoelastic solid, initially at temperature T_0, occupying the whole space. A spherically symmetric point heat source of intensity $Q(r, t)$ is assumed to be situated at the origin of the spherical polar coordinate system (r, θ, ϕ). In the case of spherical symmetry, the non-zero displacement $u(r, t)$ and temperature change $T(r, t) = T_1 - T_0$ satisfy the differential equations[3, 11],

$$c_{11}\left[u_{,rr}+\frac{2}{r}u_{,r}-\frac{2}{r^2}u\right]-\beta_1\,(T+t_1\,\dot{T})_{,r}=\rho\ddot{u}, \qquad \ldots (1)$$

$$K\left(T_{,rr}+\frac{2}{r}T_{,r}\right)-\rho C_e\,(\dot{T}+t_0\,\ddot{T})$$

$$=\beta_1\,T_0\left(\frac{\partial}{\partial r}+\frac{2}{r}\right)(\dot{u}+\delta_{1K}\,t_0\,\ddot{u})-\rho(Q+\delta_{1K}\,t_0\,\dot{Q}), \qquad \ldots (2)$$

where $\beta_1=(c_{11}+c_{12})\,\alpha_1+c_{13}\,\alpha_3$, c_{ij} are isothermal elastic parameters; K, ρ, c_e and t_0, t_1 are respectively the thermal conductivity, density, specific heat at constant strain and thermal relaxation times; α_1, α_3 are linear thermal expansion coefficients. The comma notation is used for spatial derivatives and superposed dot represents time derivatives. δ_{1K} is the Kronecker's delta. For L-S theory $t_1 = 0$, $\delta_{1K} = 1$ and for G-L theory $t_1 > 0$, $\delta_{1k} = 0$ (i.e. $k = 1$ for L-S and $k = 2$ for G-L theory). The thermal relaxation times t_0 and t_1 satisfy the inequalities,

$$t_1 \geq t_0 \geq 0, \quad \text{for G-L theory only.} \qquad \ldots (3)$$

We define the quantitites,

$$\left.\begin{array}{llll} R = w^* r/v\,, & \tau = w^* t\,, & U = \rho w^* v u/\beta_1 T_0\,, & Z = T/T_0, \\ \tau_0 = w^* t_0\,, & \tau_1 = w^* t_1\,, & \varepsilon = \beta_1^2 T_0/\rho C_e c_{11}, & w^* = c_{11} C_e/K, \\ v^2 = c_{11}/\rho\,, & Q^* = Q/Q_0\,, & K_0 = K Q_0/T_0 C_e v. & \end{array}\right] \quad \ldots (4)$$

Here ε is the thermoelastic coupling constant, w^* is the characteristic frequency of the medium and v is the velocity of the QL-wave, Q_0 is the intensity of the heat source at undisturbed state. Introducing the quantities (4) in eqs. (1) and (2), we obtain

$$U_{,RR} + 2R^{-1}U_{,R} - 2R^{-2}\,U = (Z + \tau_1 \dot{Z})_{,R} + \ddot{U}, \quad \ldots (5)$$

$$Z_{,RR} + 2R^{-1}Z_{,R} - \dot{Z} - \tau_0 \ddot{Z} = \varepsilon\left(\frac{\partial}{\partial R} + \frac{2}{R}\right)(\dot{U} + \delta_{1k}\,\tau_0\,\ddot{U}) - K_0\left(1 + \tau_0\,\delta_{1K}\frac{\partial}{\partial \tau}\right)Q^* \quad \ldots (6)$$

We assume that the medius is at rest and undistributed initially. The initial and regularity conditions can be written as,

$$\left.\begin{array}{l} U(R, \tau) = \dot{U}(R, \tau) = 0, \quad Z(R, \tau) = \dot{Z}(R, \tau) = 0, \\ \sigma_{RR}(R, \tau) = \dot{\sigma}_{RR}(R, \tau) = \sigma_{\phi\phi}(R, \tau) = \dot{\sigma}_{\phi\phi}(R, \tau) = 0 \end{array}\right] \quad \ldots (7)$$

at $\tau = 0$ and the boundary conditions are

$$U,\ Z,\ \sigma_{RR},\ \sigma_{\phi\phi} \to 0, \quad \text{for} \quad R \to \infty. \quad \ldots (8)$$

We now introduce the potential function ψ defined by

$$U = \psi_{,R}. \quad \ldots (9)$$

Using relation (9) in equations (5) and (6), we obtain

$$\nabla^2\psi = (1 + \tau_1\,\partial/\partial\tau)Z + \partial^2\,\psi/\partial\tau^2, \quad \ldots (10)$$

and

$$(\nabla^2 - \partial/\partial\tau - \tau_0\,\partial^2/\partial\tau^2)Z = \varepsilon(\partial/\partial\tau + \delta_{1K}\,\tau_0\,\partial^2/\partial\tau^2)\,\nabla^2\psi - (1 + \tau_0\,\delta_{1k}\,\partial/\partial\tau)\,Q^* K_0 \quad \ldots (11)$$

where

$$\nabla^2 = \partial^2/\partial R^2 + 2R^{-1}\,\partial/\partial R.$$

The initial conditions (7) expressed in terms of ψ become

$$\psi(R, 0) = \partial\psi(R, 0)/\partial\tau = \partial^2\,\psi(R, 0)/\partial\tau^2 = \partial^3\,\psi(R, 0)/\partial\tau^3 = 0 \quad \ldots (12)$$

While the boundary conditions (8) can be expressed as

$$\psi, \partial\psi/\partial R, \quad \partial^2\psi/\partial R^2 = 0, \text{ as } R \to \infty. \qquad \ldots (13)$$

3. Solution of the Problem

We take $Q^* = Q_0^* \delta(x) f(\tau) = Q_0^* \delta(R) f(\tau)/4\pi R^2$. Taking the Laplace transform with parameter p (denoted by a bar) of both sides of equations (10) and (11), we get

$$(\nabla^2 - p^2)\,\bar{\psi} = (1 + \tau_1 p)\,\bar{Z} \qquad \ldots (14)$$

and

$$(\nabla^2 - p - \tau_0 p^2)\,\bar{Z} = \varepsilon\,(p + \delta_{1K}\,\tau_0 p^2)\,\nabla^2\,\bar{\psi} - K_0(1 + \delta_{1k}\,\tau_0 p)\,Q^*. \qquad \ldots (15)$$

Eliminating $\bar{Z}$ between equations (14) and (15), we obtain

$$\{(\nabla^2 - p)(\nabla^2 - p^2) - [\varepsilon p(1 + \tau_1 p)(1 + \delta_{1k}\,\tau_0 p)\nabla^2 + \tau_0 p^2(\nabla^2 - p^2)]\}\bar{\psi} = -K_0(1 + \tau_1 p)(1 + \delta_{1K}\,\tau_0 p)\,\bar{Q}^*, \qquad \ldots (16)$$

which leads to

$$\bar{\psi} = \frac{-K_0(1 + \tau_1 p)(1 + \delta_{1k}\,\tau_0 p)}{(\nabla^2 - p)(\nabla^2 - p^2)} \left[1 - \frac{\varepsilon p(1 + \tau_1 p)(1 + \delta_{1k}\tau_0 p)\nabla^2 + \tau_0 p^2(\nabla^2 - p^2)}{(\nabla^2 - p)(\nabla^2 - p^2)}\right]^{-1} \bar{Q}^* \qquad \ldots (17)$$

Now we consider two cases of heat sources in the following sections of this article.

Case - I (Instantaneous Source of Heat)

We take $f(\tau) = \delta(\tau)$, then $\bar{Q}^* = Q_0^*\,\delta(x)$

and we have

$$\bar{\psi} = -K_0(1 + \tau_1 p)(1 + \delta_{1k}\,\tau_0 p)Q_0^* \sum_{n=0}^{\infty} \frac{[\varepsilon p(1 + \tau_1 p)(1 + \delta_{1k}\,\tau_0 p)\nabla^2 + \tau_0 p^2(\nabla^2 - p^2)]^n}{(\nabla^2 - p)^{n+1}(\nabla^2 - p^2)^{n+1}}\,\delta(x) \qquad \ldots (18)$$

On noting that ε, τ_0 and τ_1 are very small parameters and applying the binomial theorem to the expression in square brackets of (17). Applying the binomial theorem a second time, we obtain,

$$\overline{\psi} = -K_0 Q_0^* \sum_{n=0}^{\infty} \sum_{m=0}^{n} \binom{n}{m}$$

$$\frac{\varepsilon^m p^{2n-m} (1+\delta_{1k}\tau_0 p)^{m+1} (1+\tau_1 p)^{m+1} (\nabla^2) \tau_0^{n-m}}{(\nabla^2 - p)^{n+1} (\nabla^2 - p^2)^{m+1}} \delta(x). \quad \ldots (19)$$

Also

$$(1+\delta_{1k}\tau_0 p)^{m+1} (1+\tau_1 p)^{m+1}$$

$$= \begin{cases} \sum_{s=0}^{m+1} \binom{m+1}{s} \tau_0^s p^s, & \text{for L–S Theory} \\ \sum_{s=0}^{m+1} \binom{m+1}{s} \tau_1^s p^s, & \text{for G–L Theory.} \end{cases} \quad \ldots (20)$$

Therefore, eq. (19) takes the form

$$\overline{\psi} = -K_0 Q_0^*$$

$$\sum_{n=0}^{\infty} \sum_{m=0}^{n} \sum_{s=0}^{m+1} \frac{\binom{n}{m}\binom{m+1}{s} \varepsilon^m \tau_0^{n-m} (\delta_{1k}\tau_0^s + \delta_{2k}\tau_1^s) p^{2n-m+s} (\nabla^2)^m}{(\nabla^2 - h_1^2)^{m+1} (\nabla^2 - h_2^2)^{n+1}} \delta(x)$$

$$\ldots (21)$$

where

$h_1 = p$, $h_2 = \sqrt{p}$, $k = 1$ for L-S theory and $k = 2$ for G-L theory.

Case - II (Continuous Source of heat)

Let $f(\tau) = H(\tau)$, then $Q^* = Q^* \delta(x)/p$ and we get

$$\overline{\psi} = -K_0 Q_0^*$$

$$\sum_{n=0}^{\infty} \sum_{m=0}^{n} \sum_{s=0}^{m+1} \frac{\binom{n}{m}\binom{m+1}{s} \varepsilon^m \tau_0^{n-m} (\delta_{1k}\tau_0^s + \delta_{2k}\tau_1^s) p^{2n-m+s-1} (\nabla^2)^m}{(\nabla^2 - h_1^2)^{m+1} (\nabla^2 - h_2^2)^{n+1}} \delta(x)$$

$$\ldots (22)$$

Replacing h_i by $\sqrt{w_i}\, h_i$, $i = 1, 2$ and differentiating successively $(\nabla^2 - w_i h_i^2)^{-1}$ with respect to w_i, we get

$$(\nabla^2 - w_1 h_1^2)^{-(m+1)} = \frac{1}{m!\, h_1^{2m}} \frac{\partial^m}{\partial w_1^m} [\nabla^2 - w_1 h_1^2]^{-1}, \quad \ldots (23.1)$$

and

$$(\nabla^2 - w_2 h_2^2)^{-(n+1)} = \frac{1}{n!\, h_2^{2n}} \frac{\partial^n}{\partial w_2^n} [\nabla^2 - w_2 h_2^2]^{-1} \quad \ldots (23.2)$$

Substituting from (23) in equations (21) and (22), we get

$$\overline{\psi} = -K_0 Q_0^* \sum_{n,m,S} \frac{B_{nms}\, \varepsilon^m\, \tau_0^{n-m}\, (\delta_{1k}\, \tau_0^2 + \delta_{2k}\, \tau_1^s)\, (\nabla^2)^m\, p^{2n-m+s}}{h_1^{2m}\, h_2^{2n}}$$

$$\frac{\partial^{m+n}}{\partial w_1^m\, \partial w_2^n} [(\nabla^2 - w_1 h_1^2)\, (\nabla^2 - w_2 h_2^2)]^{-1}_{w_1 = w_2 = 1}\, \delta(x) \qquad \ldots (24)$$

in case - I and

$$\overline{\psi} = -K_0 Q_0^* \sum_{n,m,S} \frac{B_{nms}\, \varepsilon^m\, \tau_0^{n-m}\, (\delta_{1k}\, \tau_0^s + \delta_{2k}\, \tau_1^s)\, p^{2n-m+s-1}\, (\nabla^2)^m}{h_1^{2m}\, h_2^{2n}}$$

$$\frac{\partial^{m+n}}{\partial w_1^m\, \partial w_2^n} [(\nabla^2 - w_1 h_1^2)\, (\nabla^2 - w_2 h_2^2)]^{-1}_{w_1 = w_2 = 1}\, \delta(x) \qquad \ldots (25)$$

in case - II, where

$$B_{nms} = \frac{1}{n!\, m!} \begin{pmatrix} n \\ m \end{pmatrix} \begin{pmatrix} m+1 \\ s \end{pmatrix} = \frac{m+1}{(n-m)!\, m!\, s!\, (m+1-s)!} \qquad \ldots (26)$$

Using Helmholtz equations in space [13], namely

$$(\nabla^2 - k^2)^{-1}\, \delta(x) = -\exp(-kR)/4\pi R \qquad \ldots (27)$$

We can write

$$\frac{1}{(\nabla^2 - w_1 h_1^2)\, (\nabla^2 - w_2 h_2^2)}\, \delta(x) = -\frac{\exp(-h_1 R\sqrt{w_1}) - \exp(-h_2 R\sqrt{w_2})}{4\pi R (w_1 h_1^2 - w_2 h_2^2)} \qquad \ldots (28)$$

Thus the expressions (24) and (25) become

$$\overline{\psi} = Q_0' \sum_{n,m,s} \frac{B_{nms}\, \varepsilon^m\, \tau_0^{n-m}}{h_1^{2m}\, h_2^{2n}}\, (\delta_{1k}\, \tau_0^s + \delta_{2k}\, \tau_1^s)\, p^{2n-m+s}\, (\nabla^2)^m\, \frac{\partial^{m+n}}{\partial w_1^m\, \partial w_2^n}$$

$$\left[\frac{\exp(-h_1 R\sqrt{w_1}) - \exp(-h_2 R\sqrt{w_2})}{R(w_1 h_1^2 - w_2 h_2^2)} \right]_{w_1 = w_2 = 1} \qquad \ldots (29)$$

$$\overline{\psi} = Q_0' \sum_{n,m,s} \frac{B_{nms}\, \varepsilon^m\, \tau_0^{n-m}\, (\delta_{1k}\, \tau_0^s + \delta_{2k}\, \tau_1^s)\, p^{2n-m+s-1}}{h_1^{2m}\, h_2^{2n}}\, (\nabla^2)^m\, \frac{\partial^{m+n}}{\partial w_1^m\, \partial w_2^n}$$

$$\left[\frac{\exp(-h_1 R\sqrt{w_1}) - \exp(-h_2 R\sqrt{w_2})}{R(w_1 h_1^2 - w_2 h_2^2)} \right]_{w_1 = w_2 = 1} \qquad \ldots (30)$$

in case - II

$$Q_0' = -K_0 Q_0^*/4\pi. \qquad \ldots (31)$$

Making use of the relations

$$(\nabla^2)^m \left[\frac{1}{R} \exp(-h_i R\sqrt{w_i}) \right] = \frac{1}{R} w_i^m h_i^{2m} \exp(-h_i R\sqrt{w_i}), \quad i = 1, 2$$

eq. (29) and (30) become

$$\bar{\psi} = \frac{Q_0'}{R} \sum_{n=0}^{\infty} \sum_{m=0}^{n} \sum_{s=0}^{m+1} B_{nms}\, \varepsilon^m\, \tau_0^{n-m} (\delta_{1k}\, \tau_0^s + \delta_{2k}\, \tau_1^s) \left[\frac{\partial^{m+n}\, \bar{\psi}_{nms}}{\partial w_1^m\, \partial w_2^n} \right]_{w_1 = w_2 = 1}$$

$$\ldots (32)$$

where

$$\bar{\psi}_{nms} = w_1^{m-1} \left[\frac{p^{n+s}}{p - w} \left\{ \frac{e^{-up}}{p^{m+1}} - \frac{w^m\, e^{-v_1\sqrt{p}}}{p^{2m+1}} \right\} \right],$$

and

$$u = R\sqrt{w_1}, \quad v = R\sqrt{w_2}, \quad w = w_2/w_1,$$

in case - I, and

$$\bar{\bar{\psi}} = \frac{Q_0'}{R} \sum_{n=0}^{\infty} \sum_{m=0}^{n} \sum_{s=0}^{m+1} B_{nms}\, \varepsilon^m\, \tau_0^{n-m} (\delta_{1k}\, \tau_0^s + \delta_{2k}\, \tau_1^s) \left[\frac{\partial^{m+n}\, \bar{\bar{\psi}}_{nms}}{\partial w_1^m\, \partial w_2^n} \right]_{w_1 = w_2 = 1}$$

$$\ldots (33)$$

in case - II, where

$$\bar{\bar{\psi}}_{nms} = w_1^{m-1} \left[\frac{p^{n+s}}{p - w} \left\{ \frac{e^{-up}}{p^{m+2}} - \frac{w^m\, e^{-v_1\sqrt{p}}}{p^{2m+2}} \right\} \right]$$

The linear terms of the series (32) and (33) can be written as

$$\bar{\psi} = \frac{Q_0'}{R} (\bar{\psi}_0 + \varepsilon\, \bar{\psi}_1 + \tau_0\, \bar{\psi}_2 + \ldots), \text{ for L-S theory} \qquad \ldots (34)$$

and

$$\bar{\psi} = \frac{Q_0'}{R} (\bar{\psi}_0 + \varepsilon\, \bar{\psi}_1 + \tau_0\, \bar{\psi}_2 + \tau_1\, \vec{\psi}_2 + \ldots), \text{ for G-L theory} \qquad \ldots (35)$$

where

$$\left.\begin{aligned}
&\bar{\psi}_0 = \frac{e^{-Rp}}{p(p-1)} - \frac{e^{-R\sqrt{p}}}{p(p-1)}, \quad \bar{\psi}_2^* = \exp(-Rp)/(p-1) - \exp(-R\sqrt{p})/(p-1),\\
&\bar{\psi}_1 = -[(p+1)/p(p-1)^3 + R/2(p-1)^2]\, e^{-Rp}\\
&\qquad\qquad + [(p+1)/p(p-1)^3 - R\sqrt{p}/2p(p-1)^2]\, e^{-R\sqrt{p}},\\
&\bar{\psi}_2 = p\, e^{-Rp}/(p-1)^2 + [R\sqrt{p}/2(p-1) - p/(p-1)^2] \exp(-R\sqrt{p})
\end{aligned}\right\} \quad \ldots (36)$$

for case - I, and

$$\left.\begin{aligned}
&\bar{\psi}_0 = \exp(-Rp)/p^2(p-1) - \exp(-R\sqrt{p})/p^2(p-1),\\
&\bar{\psi}_1 = -\left[\frac{p+1}{p^2(p-1)^3} + \frac{R}{2p(p-1)^2}\right]\exp(-R\sqrt{p})\\
&\qquad\qquad + \left[\frac{p+1}{p^2(p-1)^3} - \frac{R\sqrt{p}}{2p^2(p-1)^2}\right]\exp(-R\sqrt{p})\\
&\bar{\psi}_2 = \exp(-Rp)/p(p-1) - \left[\frac{1}{(p-1)^2} - \frac{R}{2\sqrt{p}\,(p-1)}\right]\exp(-R\sqrt{p});\\
&\qquad\qquad \bar{\psi}_2^* = \frac{e^{-Rp}}{p(p-1)} - \frac{e^{-R\sqrt{p}}}{p(p-1)},
\end{aligned}\right\} \quad \ldots (37)$$

for case - II

4. Temperature Distribution

The transformed temperature $\bar{Z}$ can be obtained from equation (14) with equations (34) to (37) as

$$\bar{Z} = Q_0' \left\{ \frac{e^{-R\sqrt{p}}}{R} + \varepsilon \left[\frac{p e^{-Rp}}{R(p-1)^2} + \left(\frac{\sqrt{p}}{2(p-1)} - \frac{p}{R(p-1)^2} \right) e^{-R\sqrt{p}} \right] \right.$$
$$\left. + \tau_0 \left[\frac{p}{R} - \frac{p\sqrt{p}}{2} \right] \exp(-R\sqrt{p}) \right\}, \quad \ldots (38.1)$$

for L-S theory, and

$$\bar{Z} = Q_0' \left\{ \frac{e^{-R\sqrt{p}}}{R} + \varepsilon \left[\frac{p e^{-Rp}}{R(p-1)^2} + \left(\frac{\sqrt{p}}{2(p-1)} - \frac{p}{R(p-1)^2} \right) e^{-R\sqrt{p}} \right] \right.$$
$$\left. + \tau_0 \left(\frac{p}{R} - \frac{p\sqrt{p}}{2} \right) e^{-R\sqrt{p}} + \tau_1\, p\, e^{-R\sqrt{p}}/R \right\}, \quad \ldots (38.2)$$

for G-L theory in case-I, and

$$\bar{Z} = Q_0' \left\{ \frac{e^{-R\sqrt{p}}}{Rp} + \varepsilon \left[\frac{e^{-Rp}}{R(p-1)^2} + \left(\frac{1}{2\sqrt{p}(p-1)} - \frac{1}{R(p-1)^2} \right) e^{-R\sqrt{p}} \right] + \tau_0 (1/R - \sqrt{p}/2)\, e^{-R/\sqrt{p}} \right\}, \quad \ldots (39.1)$$

for L-S theory, and

$$\bar{Z} = Q_0' \left\{ \frac{e^{-R\sqrt{p}}}{Rp} + \varepsilon \left[\frac{e^{-Rp}}{R(p-1)^2} + \left(\frac{1}{2\sqrt{p}(p-1)} - \frac{1}{R(p-1)^2} \right) e^{-R\sqrt{p}} \right] + \tau_0 (1/R - \sqrt{p}/2)\, e^{-R/\sqrt{p}} + \tau_1 e^{-R\sqrt{p}}/R \right\}, \quad \ldots (39.2)$$

for G-S theory in case-II

Inverting the above Laplace transforms, we get

$$Z = Q_0' [Z_0 + \varepsilon Z_1 + \tau_0 Z_2 + \ldots], \quad \text{for L-S theory} \quad \ldots (40)$$

and

$$Z = Q_0' [Z_0 + \varepsilon Z_1 + \tau_0 Z_2 + \tau_1 Z_2^* + \ldots], \text{ for G-L theory} \quad \ldots (41)$$

where

$$\left.\begin{aligned} Z_0 &= \exp(-R^2/4\tau)/2\tau\sqrt{\pi\tau}, \\ Z_1 &= \exp(-R^2/4\tau)/2\sqrt{\pi\tau} + (\tau + 1 - R)\, W/R - (\tau + 1)\, M/R + V, \\ Z_2 &= [\exp(-R^2/4\tau)/8\tau^2\sqrt{\pi\tau}]\,[4R^2/\tau - 9 - R^4/4\tau^2], \\ Z_2^* &= (R^2/\tau - 6)\exp(-R^2/4\tau)/8\tau^2\sqrt{\pi\tau}, \end{aligned}\right\} \quad \ldots (42)$$

for case - I, and

$$\left.\begin{aligned} Z_0 &= erfc\,(R/2\tau)/R, \\ Z_1 &= -\exp(-R^2/4\tau)/2\sqrt{\pi\tau} + (\tau - R)\, W/R + (R - 2\tau)\, M/2R, \\ Z_2 &= (6\tau - R^2)\exp(-R^2/4\tau)/8\tau^2\sqrt{\pi\tau}, \\ Z_2^* &= \exp(-R^2/4\tau)/2\tau\sqrt{\pi\tau}. \end{aligned}\right\} \quad \ldots (43)$$

for case - II, where

$$W = W(R, \tau) = \exp(\tau - R)\, H(\tau - R),$$

$$M = M(R, \tau) = \left[e^{\tau - R}\, erfc\left(\frac{R}{2\sqrt{\tau}} - \sqrt{\tau} \right) + e^{\tau + R}\, erfc\left(\frac{R}{2\sqrt{\tau}} + \sqrt{\tau} \right) \right]/2, \quad \ldots (44)$$

$$V = V(R, \tau) = \left[e^{\tau - R}\, erfc\left(\frac{R}{2\sqrt{\tau}} - \sqrt{\tau} \right) - e^{\tau + R}\, erfc\left(\frac{R}{2\sqrt{\tau}} + \sqrt{\tau} \right) \right] / 2,$$

$$erfc\,(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty \exp(-y^2)\, dy, \text{ is the complementary error function.}$$

5. Special Cases

We shall now consider some limiting values of Z. Clearly as $R \to 0$, we have

$$M \to \exp(\tau), \quad V \to \exp(\tau)\,[1 - erfc\,(\sqrt{\tau})], \quad W \to \exp(\tau).$$

Then

$$\lim_{R \to 0} \frac{Z}{Q_0'} = \frac{1}{2\tau\sqrt{\pi\tau}} + \varepsilon \left[\frac{1}{2\sqrt{\pi\tau}} - \exp(\tau)\, erfc\,(\sqrt{\tau}) \right] - 9\tau_0 / 8\tau^2 \sqrt{\pi\tau}, \qquad \text{... (45.1)}$$

for L-S theory and

$$\lim_{R \to 0} \frac{Z}{Q_0'} = \frac{1}{2(\sqrt{\pi\tau})\,\tau} + \varepsilon \left[\frac{1}{2\sqrt{\pi\tau}} + erfc\,(\sqrt{\tau}) \exp(\tau) \right] - 3\tau_1 / 4\tau^2 \sqrt{\pi\tau} - 9\tau_0 / 8\tau^2 \sqrt{\pi\tau}, \qquad \text{... (45.2)}$$

for G-L theory in case - I and

$$\lim_{R \to 0} \frac{Z}{Q_0'} = \frac{1}{\sqrt{\pi\tau}} - \frac{\varepsilon}{2} \left[\frac{1}{\sqrt{\pi\tau}} + \exp(\tau) \right] + 3\tau_0 / 4\tau \sqrt{\pi\tau}, \qquad \text{... (45.3)}$$

for L-S theory and

$$\lim_{R \to 0} \frac{Z}{Q_0'} = \frac{1}{\sqrt{\pi\tau}} - \frac{\varepsilon}{2} \left[\frac{1}{\sqrt{\pi\tau}} + \exp(\tau) \right] + 3\tau_0 / 4\tau \sqrt{\pi\tau} + \frac{\tau_1}{2\tau\sqrt{\pi\tau}}, \qquad \text{... (45.4)}$$

for G-L theory in case - II.

(i) In case of uncoupled theory, we take $\tau_1 = 0 = \tau_0$ and $\varepsilon = 0$. We obtain

$$\lim_{R \to 0} \frac{Z}{Q_0'} = \frac{1}{2\tau\sqrt{\pi\tau}} \text{ in case - I and } \lim_{R \to 0} \frac{Z}{Q_0'} = \frac{1}{\sqrt{\pi\tau}} \text{ in case - II.}$$

(ii) In case of coupled theory we take $\varepsilon \neq 0$, $\tau_1 = 0 = \tau_0$. We get

$$\lim_{R \to 0} \frac{Z}{Q_0'} = \frac{1}{2\tau\sqrt{\pi\tau}} + \varepsilon \left[\frac{1}{2\sqrt{\pi\tau}} - e^{\tau}\, erfc\,(\sqrt{\tau}) \right],$$

for case - I and

$$\lim_{R \to 0} \frac{Z}{Q_0'} = \frac{1}{\sqrt{\pi\tau}} - \frac{\varepsilon}{2} \left[\frac{1}{\sqrt{\pi\tau}} + \exp(\tau) \right], \text{ for case - II.}$$

(iii) If we take $\varepsilon \neq 0$, $\tau_1 = \tau_0 \neq 0$ then in case of G-L theory we get

$$\lim_{R \to 0} \frac{Z}{Q_0'} = \frac{1}{2\tau\sqrt{\pi\tau}} + \varepsilon\left[\frac{1}{2\sqrt{\pi\tau}} - \exp(\tau)\, erfc(\sqrt{\tau})\right] - 15\tau_0/8\tau^2\sqrt{\pi\tau}, \quad \ldots (46.1)$$

in case - I, and

$$\lim_{R \to 0} \frac{Z}{Q_0'} = \frac{1}{\sqrt{\pi\tau}} - \frac{\varepsilon}{2}\left[\frac{1}{\sqrt{\pi\tau}} + \exp(\tau)\right] + 5\tau_0/4\tau\sqrt{\pi\tau}, \quad \ldots (46.2)$$

in case - II.

The comparison of these results in the light of inequalities (3) with whose of (45.1) and (45.3) in case of L-S theory leads to the conclusion that the corrections due to the L-S and G-L theories of generalized thermoelasticity on the magnitude of limiting value of temperature function are found to be in the ratio of 3 : 5 in both the cases - I and II.

As $R \to \infty$, then M, V and $W \to 0$ and thus we get $\lim_{R \to \infty} Z = 0$, which is in agreement with eq. (8).

Also by using the initial and final value theorems of the Laplace transform[12] namely,

$$\operatorname*{Lim}_{\tau \to 0} Z(\tau) = \operatorname*{Lim}_{p \to \infty} p\,\bar{Z}(p) \text{ and}$$

$$\operatorname*{Lim}_{p \to 0} p\,\bar{Z}(p) = \operatorname*{Lim}_{\tau \to \infty} Z(\tau),$$

we obtain $\operatorname*{Lim}_{\tau \to 0} Z(\tau) = 0$ and $\operatorname*{Lim}_{\tau \to \infty} Z(\tau) = 0$ in both the cases and theories of thermoelasticity. The behaviour of the discontinuity of Z at the position of the wave front $R = \tau$ has been investigated below.

The jumps in temperature are given by

$$[Z] = Z(R^+, R) - Z(R^-, R) = Q_0'\,\varepsilon/R, \text{ in case - I}$$

and

$$[Z] = 0, \text{ in case - II}$$

where R^+ and R^- denote limits as $\tau \to R$ from the right and left respectively. The above relation shows that the jump in temperature decreases as R increases and tends to zero as $R \to \infty$ in case - I and the temperature is continuous in case - II. Also the uncoupled theory predicts a continuous temperature function and the corrections due to generalized theories of thermoelasticity has no effect on the magnitude of the jump.

The distributions of displacement and stresses can be similarly obtained by using the expressions in the relevant relations. The result for isotropic materials can be obtained by setting.

$$c_{11} = \lambda + 2\mu, \quad \beta_1 = \beta = (3\lambda + 2\mu)\,\alpha_t, \quad c_{12} = c_{13} = \lambda,$$

in the above analysis. In this situation the results so obtained in case - I agree with those of Sherief and Ezzat[10] for Lord and Shulman theory of thermoelasticity.

6. Numerical Results

We take the zinc material as a numerical example, the material constants for which are given as Singh and Sharma[13]

$$\varepsilon = 0.0221,\ \tau_0 = 0.0,\ 0.02;\ \tau_1 = 0.0,\ 0.02,\ 0.05,\ K = 1.24 \times 10^2 \ \text{Wm}^{-3}$$

$$c_{11} = 1.628 \times 10^{11}\ \text{Nm}^{-2},\ \beta_1 = 5.75 \times 10^6\ \text{Nm}^{-2}\ \text{deg}^{-1},$$

$$T_0 = 296.0^{\circ}\text{K},\ \rho = 7.14 \times 10^3\ \text{Kgm}^{-3},$$

The temperature values obtained in equations (40) and (41) have been evaluated for different values of R for both $\tau = 0.5$ and $\tau = 2.0$ and the results are illustrated graphically in Figures 1 and 2 for cases - I and II respectively. The graph shows the four curves predicted by the different theories of thermoelasticity. Solid lines represent the uncoupled theory (only one term is kept) while dotted lines represent the coupled theory (two terms of the series are kept). The dashed lines represent the generalized theory of Lord and Shulman corresponding to three terms of the series. The values predicted by the generalized theory of Green and Lindsay, shown by the encircled lines correspond to taking into account four terms of the series in equation (41). The temperature in case - II (Fig. 2) is observed to decrease because of the dependence of the dominant terms in series (40) and (41) on inverse powers of R. The curves in case of $\tau_0 = \tau_1$ in G-L theory are almost observed to coincide with those of L-S theory for the respective values of thermal relaxation.

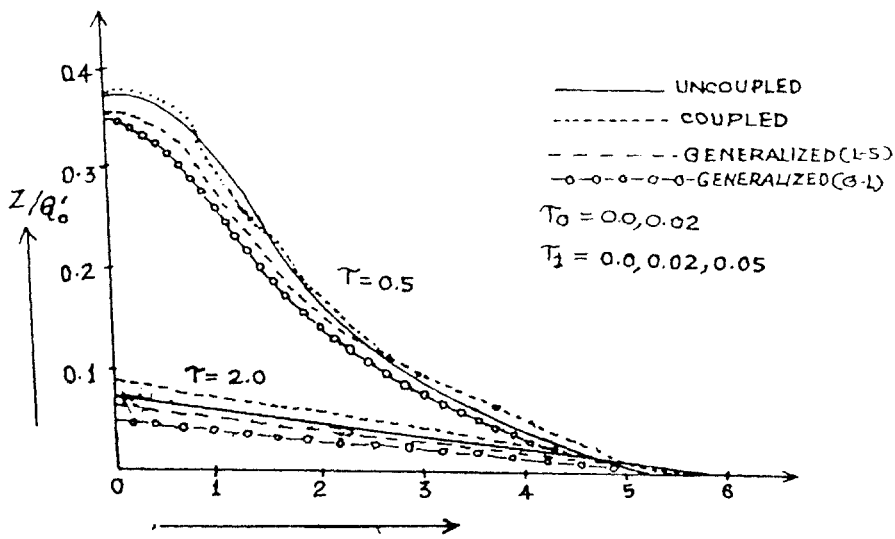

FIG. 1. Temperature distribution (Case - I)

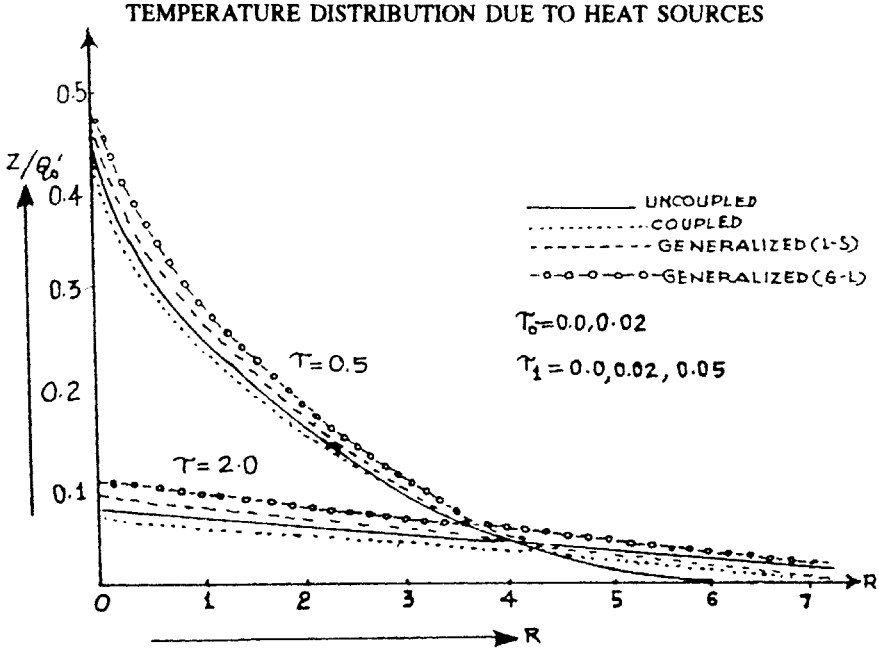

FIG. 2. Temperature distribution (Case - II)

7. CONCLUSIONS

The operational method have been used to find the solution of the fundamental problems of generalized theories of thermoelasticity developed by the Lord and Shulman and Green and Lindsay for an infinite spherically symmetric transversely isotropic space in the form of a series of known functions. It is found that the first three terms of the series dominate the rest of the terms. The third term representing generalized thermoelasticity dominate the second term representing coupled thermoelasticity, when the ratio $\tau : R$ is small and for all other values of R and τ, both the terms have the same order of magnitude. The behaviour of the discontinuity of temperature Z at the wave front $R = \tau$ has also been investigated, which shows that the jump in temperature decreases as R increases and tends to zero as $R \to \infty$ in case - I and the temperature is found to be continuous in case II. The uncoupled theory predicts a continuous temperature function and the correction due to generalized theories has no effect on the magnitude of the jump. It is observed that the corrections due to the (L-S) and (G-L) theories of generalized thermoelasticity on the magnitude of limiting value of temperature function are in the ratio of 3 : 5 in both the cases. The temperature in case of continuous heat source (Fig. 2) observed to decrease because of the dependence of dominant terms on inverse power of R.

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