Indian J. pure appl. Math., **28**(10) : 1367-1389, October 1997

A DISCRETE ANALOG OF KAC'S FORMULA AND OPTIMAL APPROXIMATION OF THE SOLUTION OF THE HEAT EQUATION

GEORGE A ANASTASSIOU AND ALEXANDER D BENDIKOV

Department of Mathematical Sciences, The University of Memphis, Memphis, TN 38152, *U.S.A.*

(*Received* 25 *September* 1996; *accepted* 21 *May* 1997)

In this article we obtain a closed form solution of the Dirichlet problem of the discretized heat equation with potential. Sharp quantitative estimates of the difference between actual and approximate solutions are given in terms of the first and second moduli of continuity of some first and second order partial derivatives of the exact solution. This is achieved probabilistically by using the appropriate random walk.

Key Words : Dirichlet Problem with Potential-Continuous and Discrete; Heat Equation; Space-time Wiener Process; Space-time Random Walk; Kac's Formula; Convergence with Rates; First and Second Moduli of Continuity; Sharp Inequality; Approximate Solution; Average Operators; First Exit Time; Uniform Grid; Lipschitz Class; Cylinder; Minimum Principle

INTRODUCTION

Let Ω be the open unit cube in $I\!R^l$, $l \geq 1$ and $\dot{\Omega} := \Omega \times I$ be an "interval" in space-time $\dot{I\!R}^l = I\!R^l \times I\!R$, where $I := (0, T)$, $T > 0$. Let us denote by $\dot{\Delta} = \frac{1}{2}\Delta - \partial_t$ the heat operator, where Δ stands for the Laplacian operator in $I\!R^l$. Given a continuous function $\lambda \geq 0$ on $\dot{\Omega}$ consider the Dirichlet problem in $\dot{\Omega}$:

$$\dot{\Delta}\, u(\dot{x}) - \lambda(\dot{x})\, u(\dot{x}) = -f(\dot{x}), \quad \dot{x} \in \dot{\Omega},$$

$$\lim_{\dot{x} \to \dot{y}} u(\dot{x}) = \varphi(\dot{y}), \qquad \forall \quad \dot{y} \in \partial\dot{\Omega} - \{\dot{x} = (x, t) : t = T\}.$$

It is a well-known fact (due to Kac[1]) that the solution u of this equation can be expressed by a very elegant formula

$$u(\dot{x}) = E_{\dot{x}} \left\{ \int_0^{\tau_{\dot{\Omega}}} M_s \cdot f(\dot{x}(s))\, ds + M_{\tau_{\dot{\Omega}}} \cdot \varphi(\dot{x}(\tau_{\dot{\Omega}})) \right\},$$

where

$$M_t = \exp\left\{ - \int_0^t \lambda(\dot{x}(s))\, ds \right\}, \quad t \geq 0$$

and $\{\dot{x}(t, w),\ t < \tau_{\dot{\Omega}}\}$ is a space-time Brownian motion in domain $\dot{\Omega}$. We also consider the discrete Dirichlet problem of obtaining u_h such that

$$\dot{\Delta}_h u_h(\dot{x}) - \lambda(\dot{x})\, u_h(\dot{x}) = -f(\dot{x}), \quad \forall \quad \dot{x} \in \dot{\Omega}_{h,T},$$

$$u_h(\dot{x}) = \varphi(\dot{x}), \quad \forall \quad \dot{x} \in \dot{\Omega}_{h,T} - \{\dot{x} = (x, t) : t = T\},$$

where $\dot{\Delta}_h$ is the discrete heat operator

$$\dot{\Delta}_h u_h(\dot{x}) = \frac{1}{2} h^{-2} \left(\sum_{k=1}^{l} u_h(x \pm h e_k, t - k(h)) - 2l u_h(x, t) \right),$$

$$\forall \ \dot{x} = (x, t) \in \dot{\Omega}_{h,T}$$

Here $k(h) = \frac{h^2}{l}$ and $\{e_k\}$ is the natural basis in $\mathbb{R}^l$. Using the space-time random walk $\{\dot{x}(k, w),\ k < \tau_{\dot{\Omega}_{h,T}}\}$ in $\dot{\Omega}_{h,T}$ we give a discrete analog of Kac's formula for the solution u_h

$$u_h(\dot{x}) = E_{\dot{x}} \left\{ k \cdot \sum_{n=0}^{\tau_{\dot{\Omega}_{h,T}}-1} M_n \cdot f(\dot{x}(n)) + M_{\tau_{\dot{\Omega}_{h,T}}-1} \cdot \varphi(\dot{x}(\tau_{\dot{\Omega}_{h,T}})) \right\},$$

where

$$M_n = \prod_{s=0}^{n} (1 + k \cdot \lambda(\dot{x}(s)))^{-1}, \quad n = 0, 1, 2, \ldots$$

and $k = h^2/l$.

Putting together all the above we are able to produce the following sharp inequality

$$\text{(i)} \qquad \| u - u_h \|_{\overline{\dot{\Omega}}_{h,T}} \leq \min\left\{ T, \frac{1}{4} \right\} \cdot \left\{ \frac{1}{4} \cdot \sum_{i=1}^{l} \omega_{2,i}(h, \partial^2_{x_i} u; \overline{\dot{\Omega}}_{h,T}) \right.$$

$$\left. + \frac{1}{2} \cdot \sum_{i=1}^{l} \omega_1(k, \partial^2_{x_i} u; \overline{\dot{\Omega}}_{h,T}) + \omega_1(k, \partial_t u; \overline{\dot{\Omega}}_{h,T}) \right\},$$

where $\| \cdot \|_{\overline{\dot{\Omega}}_{h,T}}$ is the supremum norm in $\overline{\dot{\Omega}}_{h,T}$.

Here $\omega_{2,i}$ is the second modulus of continuity of the second single partial of u with respect to x_i, $i = 1, \ldots, l$; while ω_1 stands for the first modulus of continuity of the indicated function with respect to the variable t. See Theorem 3.1.

The sharpness of inequality (i) regarding to general Lipschitz classes is established in a similar manner as in the papers[2, 3, 4].

1. BACKGROUND

Dirichlet problem for the "heat" operator. Let R^l be the Euclidean space, and $\Delta = \partial^2_{x_1} + \ldots + \partial^2_{x_l}$ be the Laplacian. We denote $\dot{R}^l = \{\dot{x} = (x, t) : x \in R^l,\ t \in R^1\}$ and let $\dot{\Delta} = \frac{1}{2}\Delta - \partial_t$ be the "heat" operator (i.e., the parabolic Laplacian). Let $\dot{\Omega} \subset \dot{R}^l$ be an open subset with nonempty boundary $\partial\dot{\Omega}$. The Dirichlet problem in $\dot{\Omega}$ consists in finding a function u on $\dot{\Omega}$ such that for given functions f, defined in $\dot{\Omega}$, and φ, defined on $\partial\dot{\Omega}$ we have

$$\dot{\Delta}\, u(\dot{x}) = -f(\dot{x}), \quad \dot{x} \in \dot{\Omega}, \qquad \ldots (1.1)$$

and

$$\lim_{\dot{x} \to \dot{y}} u(\dot{x}) = \varphi(\dot{y}), \qquad \forall\, \dot{y} \in \partial\dot{\Omega}. \qquad \ldots (1.2)$$

It is a well-known fact [Doob[5]] that if $\dot{\Omega}$ has compact closure and regular boundary (for example, if $\dot{\Omega}$ is a ball, or a convex polyhedron which is situated above its horizontal face) and f is a Hölder function and φ is a continuous function, then problem (1.1)-(1.2) has unique solution $u(\dot{x})$. This solution can be represented in the form

$$u(\dot{x}) = G_{\dot{\Omega}} f(\dot{x}) + \Pi_{\dot{\Omega}}\, \varphi(\dot{x}), \qquad \ldots (1.3)$$

where

$$G_{\dot{\Omega}} f(\dot{x}) = \int_{\dot{\Omega}} g_{\dot{\Omega}}(\dot{x}, \dot{y}) f(\dot{y})\, d\dot{y}, \qquad \ldots (1.4)$$

and

$$\Pi_{\dot{\Omega}}\, \varphi(\dot{x}) = \int_{\partial\dot{\Omega}} \varphi(\dot{y})\, \Pi_{\dot{\Omega}}(\dot{x}, d\dot{y}), \qquad \ldots (1.5)$$

are the "parabolic" Green potential of the function f and the "parabolic" harmonic function (i.e., a parabolic function) in $\dot{\Omega}$ with boundary values φ, respectively. In (1.4) and (1.5) $g_{\dot{\Omega}}(\dot{x}, \dot{y})$ and $\Pi_{\dot{\Omega}}(\dot{x}, d\dot{y})$ are the "parabolic" Green function and the "parabolic" harmonic measure (i.e., the parabolic measure) of $\dot{\Omega}$, respectively. A detailed exposition of the properties of Green potentials and parabolic functions as well as investigation of the Dirichlet problem for a general $\dot{\Omega}$ can be found in the above mentioned monograph of Doob.

In our short survey we would like to turn our attention to the important probabilistic counterpart of the analytical facts mentioned above. We refer the reader to [Doob[5]] and (Dynkin[6]) (see also[7]). Let $\dot{x}_0 = (x_0, t_0)$ be a point of $\dot{R}^l$, and let $\{\dot{x}(\cdot, \omega), x(0) = x_0\}$ be a Brownian motion in R^l starting from x_0. The process

$$\{\dot{x}(t, \omega), t \in R^+\} := \{(x(t, \omega), t_0 - t), t \in R^+\}$$

with state space $\dot{R}^l$ is called a space-time Brownian motion starting from $\dot{x}_0$. In this definition the space-time Brownian motion moves downward in $\dot{R}^l$, that is, in the direction of decreasing ordinate values. Let $\tau_{\dot{\Omega}}$ be the first exit time of the process $\dot{x}(\cdot, \omega)$ of an open set $\dot{\Omega} \subset \dot{R}^l$. We notice two special cases :

1) if $\dot{\Omega} = \Omega \times R^l$ is a cylinder with the base $\Omega \subset R^l$, then $\tau_{\dot{\Omega}} = \tau_{\Omega}$, where τ_{Ω} is the first exit time of Brownian motion $x(\cdot, \omega)$ of Ω;
2) if $\dot{\Omega} = \Omega \times I$, $I = (a, b)$, is an "interval" with the base $\Omega \subset R^l$, then $\tau_{\dot{\Omega}} = \min\{\tau_{\Omega}, \tau_I\}$ where $\tau_I \le b - a$ is the first exit time of the uniform motion $t \to t_0 - t$ of I. Thus, in particular, $\tau_{\dot{\Omega}} \le b - a$.

As usual we denote by $E_{\dot{x}} F(\omega)$ (resp., $E_x F(\omega)$) the mathematical expectation corresponding to the process $\{\dot{x}(\cdot, \omega), \dot{x}(0) = \dot{x}_0\}$ (resp., $\{x(\cdot, \omega), x(0) = x\}$. We have

$$G_{\dot{\Omega}} f(\dot{x}) = E_{\dot{x}} \left(\int_0^{\tau_{\dot{\Omega}}} f(\dot{x}(s))\, ds \right) \qquad \text{... (1.6)}$$

and
$$\Pi_{\dot{\Omega}} \varphi(\dot{x}) = E_{\dot{x}} [\varphi(\dot{x}(\tau_{\dot{\Omega}})) ; \tau_{\dot{\Omega}} < \infty]. \qquad \text{... (1.7)}$$

In particular, if $\dot{\Omega} = \Omega \times (0, T)$ is an "interval" and $\varphi(x, t) = 0$ for each $x \in \partial\Omega$ and $t \ge 0$, then putting $\varphi(x, 0) := \varphi(x)$ we get an elegant formula

$$u(x, t) = E_x \{\varphi(x(t)); t < \tau_{\Omega}\}, \qquad \text{... (1.8)}$$

which gives the solution $u(x,t)$ of the Cauchy problem in Ω

$$\left.\begin{array}{lll} \partial_t u(x, t) = \frac{1}{2} \Delta u(x, t), & x \in \Omega, & t > 0, \\ u(x, 0) = \varphi(x), & x \in \Omega, & \\ u(x, t) = 0, & x \in \partial\Omega, & t > 0. \end{array}\right\} \qquad \text{... (1.9)}$$

The next fact we would like to mention here is the famous Kac's formula [Kac[1]] (see also [Ito, McKem[8]], [Dyn[6]] and [7]. This formula, in particular, shows how to get solutions of a more general class of Dirichlet problems by using the Brownian motion.

Given a continuous function $\lambda(\dot{x}) \ge 0$, defined a random process

$$M_t := \exp\left\{ - \int_0^t \lambda(\dot{x}(s))\, ds \right\}, \quad t \ge 0,$$

and set

$$u(\dot{x}) := E_{\dot{x}}\left\{ \int_0^{\tau_\Omega} M_s \cdot f(\dot{x}(s))\, ds + M_{\tau_\Omega} \cdot \varphi(\dot{x}(\tau_\Omega)) \right\}, \qquad \text{... (1.10)}$$

where τ_Ω is the first exit time of the space-time Brownian motion $\dot{x}(\cdot, \omega)$, $\dot{x}(0, \omega) = \dot{x}$, of an open regular set $\dot{\Omega} \subset \dot{R}^l$. Then, according to [Kac], the function $u(\dot{x})$, $\dot{x} \in \dot{\Omega}$, gives a solution of the following Dirichlet problem

$$\left.\begin{aligned} &\dot{\Delta} u(\dot{x}) - \lambda(\dot{x})\, u(\dot{x}) = -f(\dot{x}), && \dot{x} \in \dot{\Omega}, \\ &\lim_{\dot{x} \to \dot{y}} u(\dot{x}) = \varphi(\dot{y}), && \dot{y} \in \partial \dot{\Omega}. \end{aligned}\right\} \qquad \text{... (1.11)}$$

In the next section we shall consider a discrete analog of the Kac's formula and its application to approximation on a uniform grid (Sections 3 and 4).

2. Dirichlet Problem : Discrete Case

Let $\dot{\mathbb{Z}}^l$ be the $(l + 1)$-dimensional integer-valued lattice. This lattice consists of points (vectors) of type $\dot{x} = x_1 e_1 + \ldots + x_l e_l + t e$, where $e_1, \ldots, e_l, e$ comprises the orthogonal basis of R^{l+1}, and the coordinates $x_1, \ldots, x_l, t$ are arbitrary integers. Decreasing the t-coordinate by one unit and increasing or decreasing each one of the x-coordrinates by one unit and leaving the other x-coordinates unchanged, we obtain the $2l$ neighbouring lattice points to $\dot{x}$. Let B be a subset of points of the lattice $\dot{\mathbb{Z}}^l$. We call a point $\dot{x} \notin B$ a boundary point for the set B if $\dot{x}$ is a neighbouring point for at least one point of B. The collection of all boundary points of the set B is called the boundary of B, denoting it by ∂B.

Let f be a function defined at the points of the lattice $\dot{\mathbb{Z}}^l$. We set

$$\dot{P} f(\dot{x}) := \frac{1}{2l} \sum_{k=1}^{l} f(\dot{x} \pm e_k - e).$$

It is logical to call $\dot{P}$ the averaging operator. The linear operator $\dot{P} - E$, where E is the unit operator, is the discrete analog of the "parabolic" Laplacian $\dot{\Delta}$. Indeed, for a sufficiently smooth function $f(\dot{x})$ specified over all the space $\dot{R}^l$,

$$\dot{\Delta} f(\dot{x}) = \lim_{h \to 0} \frac{1}{2} h^{-2} \left(\sum_{k=1}^{l} f\left(\dot{x} \pm h e_k - \frac{h^2}{l} e \right) - 2l f(\dot{x}) \right),$$

so that the "parabolic" Laplacian is obtained by passing to the limit from the scaled operator $\dot{P} - E$ as the lattice is infinitely partioned.

Let $\dot{\Omega} \subset \dot{\mathbb{Z}}^l$ be a finite subset and $\lambda(\dot{x})$ be a non-negative function defined on $\dot{\Omega}$. The λ-Dirichlet problem in $\dot{\Omega}$ consists in finding a function $u(\dot{x})$, $\dot{x} \in \dot{\Omega} \cup \partial \dot{\Omega}$ such that for given functions f defined in $\dot{\Omega}$ and φ defined in $\partial \dot{\Omega}$ we have

$$(\dot{P}-E)\,u(\dot{x})-\lambda(\dot{x})\,u(\dot{x})=-f(\dot{x}),\quad \dot{x}\in\dot{\Omega}, \qquad \text{... (2.1)}$$

$$u(\dot{x})=\varphi(\dot{x}),\quad \dot{x}\in\partial\dot{\Omega}. \qquad \text{... (2.2)}$$

First of all we note that if u_1 and u_2 are two solutions of the problem (2.1)-(2.2) then $u_1\equiv u_2$. This fact follows immediately from the following minimum principle.

***Theorem* 2.1** — *Let u be a function on* $\dot{\Omega}\cup\partial\dot{\Omega}$ *such that*

$$(\dot{P}-E)\,u(\dot{x})-\lambda(\dot{x})\,u(\dot{x})\le 0$$

for any $\dot{x}\in\dot{\Omega}$. *If* $u(\dot{x})\ge 0$ *for any* $\dot{x}\in\partial\dot{\Omega}$ *then* $u(\dot{x})\ge 0$ *for any* $\dot{x}\in\dot{\Omega}\cup\partial\dot{\Omega}$.

PROOF : Let $u_*(\dot{x})$ be a function which is defined on $\dot{\Omega}\cup\partial\dot{\Omega}$, strictly positive and satisfies the equation

$$(\dot{P}-E)\,u_*(\dot{x})-\lambda(\dot{x})\,u_*(\dot{x})=0,\quad \dot{x}\in\dot{\Omega}.$$

Existence of such a function will be shown later (see Theorem 2.3). Put $\dot{P}_\lambda:=(1+\lambda)^{-1}\,\dot{P}$ and define the following average operator

$$P^*\,\varphi(\dot{x}):=\frac{1}{u_*(\dot{x})}\,\dot{P}_\lambda\,u_*\cdot\varphi(\dot{x})..$$

It is clear that $P^*\,1(\dot{x})=1$. Consider now the following function

$$\bar{u}(\dot{x}):=u(\dot{x})/u_*(\dot{x}).$$

Then we have $\bar{u}(\dot{x})\ge 0$ for each $\dot{x}\in\partial\dot{\Omega}$, and moreover : for any $\dot{x}\in\dot{\Omega}$

$$\begin{aligned}P^*\,\bar{u}(\dot{x})&=\frac{1}{u_*(\dot{x})}\cdot\frac{1}{1+\lambda(\dot{x})}\cdot\dot{P}\,u(\dot{x})\\ &\le\frac{1}{u_*(\dot{x})}\cdot\frac{1}{1+\lambda(\dot{x})}\cdot(1+\lambda(\dot{x}))\cdot u(\dot{x})\\ &=\frac{u(\dot{x})}{u_*(\dot{x})}=\bar{u}(\dot{x}).\end{aligned}$$

Thus we have

$$P^*\,\bar{u}(\dot{x})\le\bar{u}(\dot{x}),\quad \dot{x}\in\dot{\Omega},$$

$$\bar{u}(\dot{x})\ge 0,\quad \dot{x}\in\partial\dot{\Omega}.$$

Suppose now that there exists $\dot{x}\in\dot{\Omega}$ such that

$$\bar{u}(\dot{x}_0)=\min\{\bar{u}(\dot{x}):\dot{x}\in\dot{\Omega}\cup\partial\dot{\Omega}\}.$$

Then we will have

$$\bar{u}(\dot{x}_0) \geq P^* \bar{u}(\dot{x}_0) \geq \bar{u}(\dot{x}_0).$$

It follows that $\bar{u}(\dot{x}_1) = \bar{u}(\dot{x}_0)$ for each point $\dot{x}_1$ which is neighbouring to $\dot{x}_0$. If one of these points belongs to $\partial\dot{\Omega}$ then the proof is finished, if not then we will repeat our reasoning again. It is clear that after a finite number of steps we will meet the boundary $\partial\dot{\Omega}$. Thus we finally derive that

$$\bar{u}(\dot{x}) \geq \bar{u}(\dot{x}_0) = \bar{u}(\dot{x}_1) = \ldots = \bar{u}(\dot{x}_n) \geq 0$$

for some $\dot{x}_n \in \partial\dot{\Omega}$. Since $u_*(\dot{x}) > 0$ for all $\dot{x} \in \dot{\Omega} \cup \partial\dot{\Omega}$ we find that $u(\dot{x}) \geq 0$ for each $\dot{x} \in \dot{\Omega} \cup \partial\dot{\Omega}$. The proof is now finished.

Next our remark concerns the decomposition $u = u_1 + u_2$ of the solution u of the problem (2.1)-(2.2), where

$$(\dot{P} - E)\, u_1(\dot{x}) - \lambda(\dot{x})\, u_1(\dot{x}) = -f(\dot{x}), \quad \dot{x} \in \dot{\Omega}, \qquad \ldots (2.3)$$

$$u_1(\dot{x}) = 0, \quad \dot{x} \in \partial\dot{\Omega}, \qquad \ldots (2.4)$$

$$(\dot{P} - E)\, u_2(\dot{x}) - \lambda(\dot{x})\, u_2(\dot{x}) = 0, \quad \dot{x} \in \dot{\Omega}, \qquad \ldots (2.5)$$

and

$$u_2(\dot{x}) = \varphi(\dot{x}), \quad \dot{x} \in \partial\dot{\Omega}. \qquad \ldots (2.6)$$

This decomposition is the discrete analog of the decomposition (1.3). Following the same reasonings as in Section 1 of this paper, we give the probabilistic representation of the "discrete" λ-Green potential $u_1 := G^{\lambda}_{\dot{\Omega}} f$ and of the "discrete" λ-parabolic function $u_2 := \prod^{\lambda}_{\dot{\Omega}} \varphi$ in (2.3)-(2.6). In this part of our exposition, we use tools from the monographs [Spitzer][9] and [Dyn Yush][10].

Let $\{x(n), P\}$ be a simple random walk on the lattice $\mathbf{Z}^l$, i.e., a stochastic process with independent identically distributed increments $\Delta x(n) := x(n+1) - x(n)$, $n = 0, 1, \ldots$, and such that

$$P(\Delta x(1) = x) = \begin{cases} \dfrac{1}{2l}, & \text{if } x = \pm e_k, \\ 0, & \text{otherwise.} \end{cases}$$

The process

$$\{\dot{x}(n),\, n = 0, 1, \ldots\} := \{(x(n),\, n_0 - n),\, n = 0, 1, \ldots\}$$

with state space $\dot{\mathbf{Z}}^l$ is called a space-time random walk starting from $\dot{x}(0) = (x(0), n_0)$. In this definition space-time random walk moves downward on $\dot{\mathbf{Z}}^l$, that is, in the direction of decreasing ordinate values. It is easy to see that for each bounded function f

$$E\{f(\dot{x}(1)); \dot{x}(0) = \dot{x}\} = \dot{P} f(\dot{x}),$$

and more generally

$$E\{f(\dot{x}(n)); \dot{x}(0) = \dot{x}\} = \dot{P}^n f(\dot{x}).$$

For each $k = 0, 1, \ldots$ we define the following random variable

$$M_k := \prod_{s=0}^{k} [1 + \lambda(\dot{x}(s))]^{-1},$$

we set also $M_{-1} = 1$.

Let f be a bounded function defined at the points of a lattice $\dot{\mathbb{Z}}^l$. Define the following linear operator

$$G^\lambda f(\dot{x}) := E_{\dot{x}}\left(\sum_{k=0}^{\infty} M_k(f(\dot{x}(k)) \right),$$

where here, and in what follows, we use $E_{\dot{x}}\,\Phi$, instead of $E\{\Phi; \dot{x}(0) = \dot{x}\}$, just to simplify our notions.

First of all we notice that if the function f is supported on a finite subset $\dot{\Omega} \subset \dot{\mathbb{Z}}^l$ then the function $G^\lambda f$ takes finite values. Indeed, we have

$$| G^\lambda f(\dot{x}) | \leq \sup_{\dot{z} \in \dot{\Omega}} | f(\dot{z}) | \, E_{\dot{x}}\left(\sum_{\dot{x}(k) \in \dot{\Omega}}^{\infty} 1 \right).$$

Now we note that $\Sigma_{\dot{x}(k) \in \dot{\Omega}}\, 1$ is exactly the time that spends particle $\dot{x}(\cdot, \omega)$ in the set $\dot{\Omega}$. This time is a bounded random variable, because the particle moves downward in the direction of decreasing ordinate values. We note also, if $\lambda_0 := \inf\{\lambda(\dot{x}) : \dot{x} \in \dot{\mathbb{Z}}^l\} > 0$, then the previous statement holds true without any assumption on supp f. Indeed,

$$| G^\lambda f(\dot{x}) | \leq \sup_{\dot{z} \in \dot{\mathbb{Z}}^l} | f(\dot{z}) | \cdot \sum_{k=0}^{\infty} [1 + \lambda_0]^{-k} < \infty.$$

Next comes our first concerning the problem (2.1)-(2.2).

Theorem 2.2 — *Let the function f be supported on a finite subset* $\dot{\Omega} \subset \dot{\mathbb{Z}}^l$. *Then the function* $u = G^\lambda f$ *satisfies the following equation*

$$(\dot{P} - E)\, u(\dot{x}) - \lambda(\dot{x})\, u(\dot{x}) = -f(\dot{x}), \quad \dot{x} \in \dot{\mathbb{Z}}^l.$$

PROOF : For each $k = 0, 1, \ldots$ denote

$$u(k, \dot{x}) := E_{\dot{x}}\, (M_k f(\dot{x}(k)))$$

and let F_k be the σ-algebra generated by random variables $\{\dot{x}(0), \dot{x}(1), \ldots, \dot{x}(k)\}$.

We have

$$u(k,\dot{x}) := E_{\dot{x}}\,(E_{\dot{x}}(M_k f(\dot{x}(k)) \mid F_1))$$

$$= [1+\lambda(\dot{x})]^{-1}\,E_{\dot{x}}\left(E_{\dot{x}}\left(\prod_{s=1}^{k}\,[1+\lambda(\dot{x}(s))]^{-1} f(\dot{x}(k)) \mid F_1\right)\right)$$

$$= [1+\lambda(\dot{x})]^{-1}\,E_{\dot{x}}\left(E_{\dot{x}(1)}\left(\prod_{s=0}^{k-1}\,[1+\lambda(\dot{x}(s))]^{-1} f(\dot{x}(k-1)) \mid F_1\right)\right)$$

$$= [1+\lambda(\dot{x})]^{-1}\,E_{\dot{x}}(u(k-1,\dot{x}(1))$$

$$= [1+\lambda(\dot{x})]^{-1}\,\dot{P}_{\dot{u}}(k-1,\cdot)\,(\dot{x}).$$

Now we use this trecurrence to get the desired result

$$G^{\lambda} f(\dot{x}) = \sum_{k=0}^{\infty} u(k,\dot{x})$$

$$= u(0,\dot{x}) + [1+\lambda(\dot{x})]^{-1}\,\dot{P}\sum_{k=1}^{\infty} u(k-1,\cdot)\,(\dot{x})$$

$$= [1+\lambda(\dot{x})]^{-1} f(\dot{x}) + [1+\lambda(\dot{x})]^{-1}\,\dot{P}\sum_{k=0}^{\infty} u(k,\cdot)\,(\dot{x})$$

$$= [1+\lambda(\dot{x})]^{-1}\,\{f(\dot{x}) + \dot{P}\,G^{\lambda} f(\dot{x})\}.$$

Thus we finally find

$$[1+\lambda(\dot{x})]\,G^{\lambda} f(\dot{x}) = f(\dot{x}) + \dot{P}\,G^{\lambda} f(\dot{x}).$$

The obtained relation finishes the proof of the theorem.

For a finite set $\dot{\Omega} \subset \dot{\boldsymbol{Z}}^l$ we denote by τ_{Ω} the first exit time of the particle $\dot{x}(\cdot,\omega)$ of $\dot{\Omega}$. For a function φ defined on the set $\dot{\boldsymbol{Z}}^l \setminus \dot{\Omega}$ we define the following linear operator

$$\Pi^{\lambda}_{\dot{\Omega}}\,\varphi(\dot{x}) := E_{\dot{x}}\,\{M_{\tau_{\Omega}-1}\,\varphi(\dot{x}\,(\tau_{\Omega}))\}.$$

According to our agreement $M_{-1} = 1$ and consequently $\Pi^{\lambda}_{\dot{\Omega}}\,\varphi(\dot{x}) = \varphi(\dot{x})$ for each $\dot{x} \in \dot{\boldsymbol{Z}}^l \setminus \dot{\Omega}$ and, in particular, for each $\dot{x} \in \partial\dot{\Omega}$.

Theorem 2.3 — *The function* $u(\dot{x}) := \Pi^{\lambda}_{\dot{\Omega}}\,\varphi(\dot{x})$ *gives the unique solution of the problem (2.5)-(2.6).*

PROOF : As it was mentioned above the function u satisfies the condition (2.6). Thus we have to verify the condition (2.5). Let $\dot{x} \in \dot{\Omega}$, then for a particle starting from the point $\dot{x}$ we have $\tau_{\Omega} \geq 1$. In what follows we will assume that $\dot{x} \in \dot{\Omega}$. We have

$$\Pi^{\lambda}_{\dot\Omega}\varphi(\dot x) = \sum_{k=0}^{\infty} E_{\dot x}\ \{M_{k-1}\,\varphi(\dot x(k)),\ \tau_{\dot\Omega}=k\}$$

$$= \sum_{k=1}^{\infty} E_{\dot x}\ \{M_{k-1}\,\varphi(\dot x(k)),\ \tau_{\dot\Omega}=k\}.$$

For each $k = 0, 1, \ldots$ denote

$$u(\dot x, k) := E_{\dot x}\ \{M_{k-1}\,\varphi(\dot x(k)),\ \tau_{\dot\Omega}=k\}.$$

Using the Markov property of the process $\dot x(\cdot, \omega)$ we will have for $k \geq 1$ that

$$u(\dot x, k) = E_{\dot x}\ \{M_{k-1}\,\varphi(\dot x(k))\,1_{\tau_{\dot\Omega}=k}\}$$

$$= E_{\dot x}\Big(E_{\dot x}\Big\{M_{k-1}\,\varphi(\dot x(k))\,1_{\tau_{\dot\Omega}=k} \mid F_1\Big\}\Big)$$

$$= E_{\dot x}\left(E_{\dot x}\left\{[1+\lambda(\dot x(0))]^{-1}\prod_{s=0}^{k-2}[1+\lambda(\dot x(s+1))]^{-1}\,\varphi(\dot x(k-1+1))\,1_{\tau_{\dot\Omega}-1=k-1} \mid F_1\right\}\right)$$

$$= [1+\lambda(\dot x)]^{-1}\,E_{\dot x}\Big(E_{\dot x(1)}\Big\{M_{k-2}\,\varphi(\dot x(k-1))\,1_{\tau_{\dot\Omega}=k-1}\Big\}\Big)$$

$$= [1+\lambda(\dot x)]^{-1}\,E_{\dot x}\,(u(\dot x(1), k-1))$$

$$= [1+\lambda(\dot x)]^{-1}\,\dot P u(\cdot, k-1)\,(\dot x).$$

Now we use this recurrence to get the final result. Indeed, for $\dot x \in \dot\Omega$ we have

$$\Pi^{\lambda}_{\dot\Omega}\varphi(\dot x) = \sum_{k=1}^{\infty} u(\dot x, k)$$

$$= \sum_{k=1}^{\infty} [1+\lambda(\dot x)]^{-1}\,\dot P u\,(\cdot, k-1)\,(\dot x)$$

$$= [1+\lambda(\dot x)]^{-1}\,\dot P \sum_{k=1}^{\infty} u\,(\cdot, k-1)\,(\dot x)$$

$$= [1+\lambda(\dot x)]^{-1}\,\dot P \sum_{k=0}^{\infty} u\,(\cdot, k)\,(\dot x)$$

$$= [1+\lambda(\dot x)]^{-1}\,\dot P\,\Pi^{\lambda}_{\dot\Omega}\varphi(\dot x).$$

Thus we have proved that for each $\dot x \in \dot\Omega$ the following relation holds true

$$\Pi^{\lambda}_{\dot\Omega}\varphi(\dot x) = [1+\lambda(\dot x)]^{-1}\,\dot P\,\Pi^{\lambda}_{\dot\Omega}\varphi(\dot x).$$

Clearly, from this last relation it follows the statement of the theorem.

We again fix a finite set $\dot{\Omega} \subset \dot{\mathbf{Z}}^l$ and we introduce the following linear operator.

$$G^{\lambda}_{\dot{\Omega}} f(\dot{x}) := E_{\dot{x}}\left(\sum_{k=0}^{\tau_{\Omega}-1} M_k f(\dot{x}(k)) \right)$$

(of course, we set $\Sigma_{k \in \emptyset} = 0$). Thus we see that $G^{\lambda}_{\dot{\Omega}} f(\dot{x}) = 0$ for each $\dot{x} \notin \dot{\Omega}$ and, in particular for $x \in \partial\dot{\Omega}$.

Our next result concerns the relation between operators G^{λ}, $G^{\lambda}_{\dot{\Omega}}$ and $\Pi^{\lambda}_{\dot{\Omega}}$. This relation is a discrete analog of the well-known Dynkin's formula for Markov processes (see [Dyn][6]).

Theorem 2.4 — *The following relation holds true*

$$G^{\lambda} f(\dot{x}) - \Pi^{\lambda}_{\dot{\Omega}} G^{\lambda} f(\dot{x}) = G^{\lambda}_{\dot{\Omega}} f(\dot{x}), \quad \dot{x} \in \dot{\Omega}.$$

PROOF : We have

$$G^{\lambda} f(\dot{x}) = E_{\dot{x}}\left(\sum_{k=0}^{\infty} M_k f(\dot{x}(k)) \right)$$

$$= E_{\dot{x}}\left(\sum_{k=0}^{\tau_{\Omega}-1} M_k f(\dot{x}(k)) \right) + E_{\dot{x}}\left(\sum_{k=\tau_{\Omega}}^{\infty} M_k f(\dot{x}(k)) \right)$$

$$=: \; G^{\lambda}_{\dot{\Omega}} f(\dot{x}) + I.$$

Denote by $F_{\tau_{\Omega}}$ the σ-algebra of events A such that $A \cap \{\tau_{\dot{\Omega}} \le k\} \subset F_k$ for each $k \ge 1$ (see [Spitz][9], [Shir][11]). Using now the strong Markov property of the process $\dot{x}(\cdot, \omega)$ we investigate the function I. We have

$$I = E_{\dot{x}}\left(\sum_{k=\tau_{\Omega}}^{\infty} M_k f(\dot{x}(k)) \right) = E_{\dot{x}}\left(\sum_{k=0}^{\infty} M_{k+\tau_{\Omega}} f(\dot{x}(k+\tau_{\dot{\Omega}})) \right)$$

$$= E_{\dot{x}}\left(E_{\dot{x}}\left(\sum_{k=0}^{\infty} \prod_{s=0}^{k+\tau_{\Omega}} [1+\lambda(\dot{x}(s))]^{-1} f(\dot{x}(k+\tau_{\dot{\Omega}})) \mid F_{\tau_{\Omega}} \right)\right)$$

$$= E_{\dot{x}}\left(E_{\dot{x}}\left(\sum_{k=0}^{\infty} \prod_{s=0}^{\tau_{\Omega}-1} [1+\lambda(\dot{x}(s))]^{-1} \cdot \right.\right.$$

$$\left.\left. \times \prod_{s=\tau_{\Omega}}^{k+\tau_{\Omega}} [1+\lambda(\dot{x}(s))]^{-1} f(\dot{x}(k+\tau_{\dot{\Omega}})) \mid F_{\tau_{\Omega}} \right)\right)$$

$$= E_{\dot{x}}\left(\prod_{s=0}^{\tau_\Omega - 1} [1+\lambda(\dot{x}(s))]^{-1} \right.$$

$$\left. E_{\dot{x}}\left\{ \sum_{k=0}^{\infty} \prod_{s=0}^{k} [1+\lambda(\dot{x}(s+\tau_\Omega))]^{-1} f(\dot{x}(k+\tau_\Omega)) \mid F_{\tau_\Omega} \right\} \right)$$

$$= E_{\dot{x}}\left(M_{\tau_\Omega - 1} E_{\dot{x}(\tau_\Omega)}\left\{ \sum_{k=0}^{\infty} M_k f(\dot{x}(k)) \right\} \right)$$

$$= E_{\dot{x}}\,(M_{\tau_\Omega - 1}\, G^\lambda\,(\dot{x}(\tau_\Omega))) = \Pi^\lambda_\Omega G^\lambda f(\dot{x}).$$

Thus the proof of the theorem is finished.

Theorem 2.5 — *The function* $u(\dot{x}) := G^\lambda_\Omega f(\dot{x})$ *gives the unique solution of the problem (2.3)-(2.4).*

PROOF : It follows immediately from Theorems 2.2, 2.3, and 2.4.

Now we collect the obtained results in the following theorem.

Theorem 2.6 — *Let* $\dot{\Omega} \subset \dot{\mathbf{Z}}^l$ *be a finite set. Then the problem (2.1)-(2.2) has unique solution u which can be represented by the formula*

$$u(\dot{x}) = E_{\dot{x}}\left\{ \sum_{k=0}^{\tau_\Omega - 1} M_k f(\dot{x}(k)) + M_{\tau_\Omega - 1}\, \varphi\,(\dot{x}(\tau_\Omega)) \right\},$$

where $M_k = \Pi^k_{s=0}\,[1+\lambda(\dot{x}(s))]^{-1}$, $k = 0, 1, \ldots, M_{-1} = 1$, *and* τ_Ω *is the first exit time of the process* $\dot{x}(\cdot, \omega)$ *from the set* $\dot{\Omega}$.

As an important particular case of Theorem 2.6 we mention here the "discrete" Cauchy problem.

For a function $u(\dot{x}) = u(t, x)$ specified on the lattice $\dot{\mathbf{Z}}^l = \mathbf{Z}^1 \times \mathbf{Z}^l$ we define the following operators

$$\partial u(t, x) := u(t+1, x) - u(t, x)$$

and
$$Pu(t, x) := \frac{1}{2l} \sum_{k=1}^{l} u(t, x \pm e_k).$$

Theorem 2.7 — *Let* Ω *be a subset of* $\mathbf{Z}^l$ *and* φ *be a bounded function defined on* Ω. *The "discrete" Cauchy problem with equations*

$$\partial u(t, x) = (P - E)\, u(t, x) - \lambda(x)\, u(t, x), \quad x \in \Omega, \qquad \ldots (2.7)$$

$$u(0, x) = \varphi(x), \quad x \in \Omega, \qquad \ldots (2.8)$$

and
$$u(t, x) = 0, \quad x \in \partial\Omega, \quad t > 0 \qquad \ldots (2.9)$$

has a unique solution, which can be expressed in the following form

$$u(t, x) = E_x(M_{t-1}\, \varphi(x(t)), \quad t < \tau_\Omega),$$

where $x(\cdot, \omega)$ *is the simple random walk in* $\mathbf{Z}^l$ *and* τ_Ω *is the first exit time of* $x(\cdot, \omega)$ *from the set* Ω *(see (1.8)).*

Now we come back to general situation. For a set $\dot{\Omega} \subset \dot{\mathbf{Z}}^l$ and a function u defined on $\dot{\Omega}$ we set

$$\| u \|_{\dot{\Omega}} := \sup \; \{ | u(\dot{x}) | := \dot{x} \in \dot{\Omega} \}.$$

An auxiliary result that is needed for later follows.

Theorem 2.8 — *For every function* u *on* $\dot{\Omega} \cup \partial\dot{\Omega}$ *the following inequality holds true*

$$\| u \|_{\dot{\Omega} \cup \partial\dot{\Omega}} \le C(\dot{\Omega}, \lambda) \, \| (\dot{P} - E)u - \lambda u \|_{\dot{\Omega}} + \| u \|_{\partial\dot{\Omega}}, \qquad \text{... (2.10)}$$

where $\qquad C(\dot{\Omega}, \lambda) := \| G^\lambda_{\dot{\Omega}} 1 \|_{\dot{\Omega}}.$

PROOF : Define the following functions

$$f(\dot{x}) := -(\dot{P} - E)\, u(\dot{x}) + \lambda(\dot{x})\, u(\dot{x}), \quad \dot{x} \in \dot{\Omega},$$

and $\qquad \varphi(\dot{x}) := u(\dot{x}), \qquad \dot{x} \in \partial\dot{\Omega}.$

Then we have

(i) $(\dot{P} - E)\, u(\dot{x}) - \lambda(\dot{x})\, u(\dot{x}) = -f(\dot{x}), \quad x \in \dot{\Omega}$

and

(ii) $u(\dot{x}) = \varphi(\dot{x}), \qquad \dot{x} \in \partial\dot{\Omega}.$

By Theorems 2.3 and 2.5 we get that

$$u(\dot{x}) = G^\lambda_{\dot{\Omega}} f(\dot{x}) + \Pi^\lambda_{\dot{\Omega}}\, \varphi(\dot{x}).$$

From this identity we immediately find that

$$\| u \|_{\dot{\Omega} \cup \partial\dot{\Omega}} \le \| G^\lambda_{\dot{\Omega}} 1 \|_{\dot{\Omega}} + \| f \|_{\dot{\Omega}} + \| \Pi^\lambda_{\dot{\Omega}}\, \varphi \|_{\dot{\Omega} \cup \partial\dot{\Omega}}$$

$$\le C\, (\dot{\Omega}, \lambda) \, \| (\dot{P} - E)u - \lambda u \|_{\dot{\Omega}} + \| \varphi \|_{\partial\dot{\Omega}}$$

$$= C\, (\dot{\Omega}, \lambda) \, \| (\dot{P} - E)u - \lambda u \|_{\dot{\Omega}} + \| u \|_{\partial\dot{\Omega}}.$$

The proof is finished.

The following result turns out to be useful in our further considerations.

Theorem 2.9 — *Let* $\dot{\Omega} = \Omega \times I$ *be an interval, where*

$$\Omega = \{ (x_1, ..., x_l) \in \mathbf{Z}^l : 1 \le x_i \le N, \;\; i = 1, \; ..., \; l \},$$

$$I = \{ 1, \; 2, \; ..., \; T \}, \; T \in \mathbf{N},$$

and let also $\lambda_* := \max \; \{ \lambda(\dot{x}) : \dot{x} \in \dot{\Omega} \}$. *Then we will have*

$$\frac{1}{2} e^{-T\lambda_*} \min\left\{ T, \frac{2}{\pi^2}(N+1)^2 \right\} \le C(\dot\Omega, \lambda) \le \min\left\{ T, \frac{l}{4}(N+1)^2 \right\}.$$

PROOF : We get $c(\dot\Omega) := \max\ \{E_{\dot x}(\tau_{\dot\Omega}) : \dot x \in \dot\Omega\}$ and prove that the following inequality

$$e^{-T\lambda_*} c(\dot\Omega \le C(\dot\Omega, \lambda) \le c(\dot\Omega) \tag{*}$$

holds true. Indeed, we have

$$C(\dot\Omega, \lambda) = \max_{\dot x \in \dot\Omega} E_{\dot x}\left\{ \sum_{k=0}^{\tau_{\dot\Omega}-1} M_k \right\} \le \max_{\dot x \in \dot\Omega} E_{\dot x}\, \tau_{\dot\Omega} = c(\dot\Omega).$$

On the other hand if we put $(1 + \lambda_*)^{-1} := q$, then we will have

$$\begin{aligned} C(\dot\Omega, \lambda) &\ge \max_{\dot x \in \dot\Omega} E_{\dot x}\{q + \ldots + q^{\tau_{\dot\Omega}}\} \\ &\ge \max_{\dot x \in \dot\Omega} E_{\dot x}\{\tau_{\dot\Omega}\, q^{\tau_{\dot\Omega}}\} \\ &\ge \max_{\dot x \in \dot\Omega} E_{\dot x}\{\tau_{\dot\Omega}(1+\lambda_*)^{-\tau_{\dot\Omega}}\} \\ &\ge \max_{\dot x \in \dot\Omega} E_{\dot x}\{\tau_{\dot\Omega}(1+\lambda_*)^{-T}\} \\ &\ge e^{-T\lambda_*} \max_{\dot x \in \dot\Omega} E_{\dot x}(\tau_{\dot\Omega}) = e^{-T\lambda_*} c(\dot\Omega), \end{aligned}$$

where on the last step of our estimates we used the elementary inequalities log $(1 + \lambda_*) \le \lambda_*$ and $\tau_{\dot\Omega} \le T$. Thus the inequality (*) is proved.

On the next step of our reasonings we prove the following inequality

$$\frac{1}{2} \min\ \{T, c(\Omega)\} \le c(\dot\Omega) \le \min\ \{T, c(\Omega)\}, \tag{**}$$

where $c(\Omega) := \max_{x\in\Omega} E_x(\tau_\Omega)$ and τ_Ω is the first exist time of simple random walk $x(\cdot, \omega)$ from Ω. We observe for a process $\dot x(\cdot, \omega)$ with $\dot x(0) = \dot x$, where $\dot x = (x, t)$, that we have $\tau_{\dot\Omega} = \min\ \{\tau_\Omega, t\}$. It implies the inequality

$$E_{\dot x}(\tau_{\dot\Omega}) \le \min\ \{t,\ E_x(\tau_\Omega)\},$$

from which the right-hand side of (**) follows. To prove the left-hand side of (**) we consider the functions $u(x) := E_x(\tau_\Omega)$ and $\bar u(\dot x) := tu(x)$. It is clear that $\bar u(\dot x) = 0$ for any $\dot x \in \partial\dot\Omega$. Applying to function $\bar u$ inequality (2.10) with $\lambda \equiv 0$ we get

$$\| \bar u \|_{\dot\Omega} \le c(\dot\Omega) \| (\dot P - E)\bar u \|_{\dot\Omega}. \tag{***}$$

Now we note that $\| \bar u \|_{\dot\Omega} = T \cdot \| u \|_\Omega = T \cdot c(\Omega)$. To calculate $(\dot P - E)\bar u$ we observe that τ_Ω coincides with the first exit time from cylinder $\dot\Omega_c := \Omega \times (-\infty, \infty)$. Thus the function

$$u_c(\dot{x}) := u(x) = E_x(\tau_\Omega) = E_{\dot{x}}(\tau_{\dot{\Omega}_c})$$

satisfies the equation

$$(\dot{P} - E)\, u_c(\dot{x}) = -1, \quad \dot{x} \in \dot{\Omega}_c.$$

Using the facts mentioned above we get

$$(\dot{P} - E)\, \bar{u}(\dot{x}) = \dot{P}\, \bar{u}(\dot{x}) - \bar{u}(\dot{x}) = (t-1)\, \dot{P}\, u_c(\dot{x}) - t u_c(\dot{x})$$

$$= (t-1)\,(u_c(\dot{x}) - 1) - t u_c(\dot{x}) = -\,(u_c(\dot{x}) + (t-1)).$$

Thus we finally find

$$\| (\dot{P} - E)\, \bar{u} \|_{\dot{\Omega}} \le (\| u \|_\Omega + T) = (c(\Omega) + T). \qquad (****)$$

Estimates (***) and (****) imply that

$$Tc(\Omega) \le c(\dot{\Omega})\,(T + c(\Omega)),$$

and consequently

$$c(\dot{\Omega}) \ge \frac{T \cdot c(\Omega)}{T + c(\Omega)} \ge \frac{1}{2} \min\, \{T, c(\Omega)\}.$$

Thus the left-hand side of (**) follows. Now it remains to find upper and lower bounds of the constant $c(\Omega) = \max_{x \in \Omega} E_x(\tau_\Omega)$. Let $\Omega_1 := \{x \in \mathbb{Z}^l : 1 \le x_1 \le N\}$ be a slab. Since $\Omega \subset \Omega_1$ we have $\tau_\Omega \le \tau_{\Omega_1}$ and consequently

$$E_x(\tau_\Omega) \le E_x(\tau_{\Omega_1}).$$

Now we note that the function

$$\bar{u}_1(\dot{x}) := E_{\dot{x}}(\tau_{\dot{\Omega}_1})$$

satisfies the equation $(\dot{P} - E)\, \bar{u}_1 = -1$ in a space-time slab $\dot{\Omega}_1 := \Omega_1 \times (-\infty, \infty)$ and has zero boundary values. Consider the function $n(\dot{x}) := l x_1\, ((N+1) - x_1)$. It is easy to see that this function satisfies the equation $(\dot{P} - E)n = -1$ in $\bar{\dot{\Omega}}_1$ and has zero boundary values. Thus by unicity $\bar{u}_1(\dot{x}) = n(\dot{x})$ for all $\dot{x} \in \dot{\Omega}_1$, and consequently

$$c(\Omega) \le \max\, \{\bar{u}_1(\dot{x}) : \dot{x} \in \dot{\Omega}_1\} = \max\, \{n(\dot{x}) : \dot{x} \in \dot{\Omega}_1\} \le l \left(\frac{N+1}{2} \right)^2,$$

which gives us the upper bound of $c(\Omega)$. To find the lower bound of $c(\Omega)$ we use a similar method. Namely, we consider the function $u(s) := \sin \dfrac{\pi s}{N+1}$ and denote

$$\bar{u}(\dot{x}) := \prod_{k=1}^{l} u(x_k), \quad x \in \Omega.$$

It is clear that $\bar{u}(\dot{x}) = 0$ for every $\dot{x} \in \partial\dot{\Omega}_c$. Let P_i, $i = 1, \ldots, l$ be the one-dimensional average operator

$$P_i \bar{u}(\dot{x}) := \frac{1}{2} \{\bar{u}(\dot{x} + e_i) + \bar{u}(\dot{x} - e_i)\}.$$

It is clear that

$$(\dot{P} - E) \bar{u}(\dot{x}) = \frac{1}{l} \sum_{i=1}^{l} (P_i - E) \bar{u}(\dot{x})$$

$$= \frac{1}{l} \sum_{i=1}^{l} (P_i - E) u(x_i) \prod_{k \neq i} u(x_k).$$

Now we compute $(P_i - E) u(x_i)$ for $1 \leq x_i \leq N$ and for $1 \leq i \leq l$. We have

$$(P_i - E) u(x_i) = -2 \sin^2 \frac{\pi}{2(N+1)} u(x_i),$$

and thus we find that for $\dot{x} \in \dot{\Omega}_c$

$$(\dot{P} - E) \bar{u}(\dot{x}) = -2 \sin^2 \frac{\pi}{2(N+1)} \cdot \bar{u}(\dot{x}).$$

To get the lower bound of $c(\Omega) = c(\dot{\Omega}_c)$ it remains to apply the inequality (2.10) with $\lambda \equiv 0$,

$$0 < \| \bar{u} \|_{\Omega_c} \leq c(\Omega) \cdot 2 \sin^2 \frac{\pi}{2(N+1)} \| \bar{u} \|_{\Omega_c}$$

$$\leq c(\Omega) \frac{\pi^2}{2(N+1)^2} \| \bar{u} \|_{\Omega_c}.$$

Thus we finally find

$$c(\Omega) \geq \frac{2}{\pi^2} (N+1)^2,$$

which gives us the lower bound of $c(\Omega)$.

Inequalitites (*) and (**) together with the upper and lower bounds of $c(\Omega)$ give us the desired result.

3. Approximation on the Grid

We consider the following Dirichlet problem on the interval $\dot{\Omega} := \{\dot{x} \in \dot{R}^l : 0 < x_i < 1,$ $0 < t < \infty,\ i = 1, \ldots, l\}$

$$\left.\begin{aligned} \dot{\Delta}\, u(\dot{x}) - \lambda(\dot{x})\, u(\dot{x}) &= -f(\dot{x}), \quad \dot{x} \in \dot{\Omega}, \\ u(\dot{x}) &= \varphi(\dot{x}), \quad \dot{x} \in \partial\dot{\Omega}, \end{aligned}\right\} \quad \text{... (3.1)}$$

with φ and $\lambda \geq 0$ continuous functions and f a bounded locally Hölder function. In what follows we restrict our treatment to problem (3.1) for which $u \in C^{(2)}(\bar{\dot{\Omega}})$.

Let $h := 1/n$ with $n \in \mathbb{N}$ and $k := h^2/l$. An approximate solution u_h, defined on the grid $\bar{\dot{\Omega}}_h = \dot{\Omega}_h \cup \partial\dot{\Omega}_h$, where

$$\bar{\dot{\Omega}}_h := \{\dot{x} = (x_1, ..., x_l, t) \in \dot{R}^l : x_i = k_i h, \ t = jk,$$

$$k_i = 0, ..., n, \ i = 1, ..., l, \ j = 0, 1, ...\},$$

and

$$\dot{\Omega}_h := \bar{\dot{\Omega}}_h \cap \dot{\Omega}, \quad \partial\dot{\Omega}_h := \bar{\dot{\Omega}}_h \setminus \dot{\Omega}_h,$$

is obtained as the solution of the discrete counterpart to (3.1)

$$\left.\begin{aligned} \dot{\Delta}_h\, u_h(\dot{x}) - \lambda(\dot{x})\, u_h(\dot{x}) &= -f(\dot{x}), \quad \dot{x} \in \dot{\Omega}_h, \\ u_h(\dot{x}) &= \varphi(\dot{x}), \quad \dot{x} \in \partial\dot{\Omega}_h. \end{aligned}\right\} \quad \text{... (3.2)}$$

Here the "discrete" parabolic Laplacian $\dot{\Delta}_h$ is given by

$$\dot{\Delta}_h\, u(\dot{x}) := \frac{1}{2} h^{-2} \left(\sum_{k=1}^{l} u(\dot{x} \pm h e_k - k \cdot e) - 2l u(\dot{x}) \right),$$

where $\qquad k = k(h) := h^2/l.$

We denote by $\dot{\mathbb{Z}}_h^l := \{\dot{x} \in \dot{R}^l : x = zh, \ t = nk, \ z \in \mathbb{Z}^l, \ n \in \mathbb{Z}^1\}$, and we consider the space-time random walk on $\dot{\mathbb{Z}}_h^l$

$$\{\dot{x}(n), \ n = 0, 1, \ldots\} = \{(x(n), \ t - nk), \ n = 0, 1, ...\}$$

starting from point $\dot{x} = (x, t)$, where $\{x(n), \ n = 0, 1, ...\}$ is a simple random walk on the h-lattice $h\mathbb{Z}^l$ starting from point $x(0) = x$. Corresponding to $\{\dot{x}(n), \ n = 0, 1, \ldots\}$ values, functions and operators are attached to the index h. Thus,. for example, the average operator $\dot{P}_h$ takes the following form

$$\dot{P}_h\, u(\dot{x}) = E_{\dot{x}}\, \{u(\dot{x}(1))\} = \frac{1}{2l} \sum_{k=1}^{l} u(\dot{x} \pm h e_k - k e).$$

It is easy to see that with these notations the discrete parabolic Laplacian $\dot{\Delta}_h$ takes the following form, $\dot{\Delta}_h = l h^{-2} (\dot{P}_h - E)$. Now applying Theorem 2.6 we see that the problem (3.2) has a unique solution u_h which can be represented by the form

$$u_h(\dot{x}) = E_{\dot{x}} \left\{ k \sum_{n=0}^{\tau_{\Omega_h}-1} M_n f(\dot{x}(n)) + M_{\tau_{\Omega_h}-1}\, \varphi(\dot{x}(\tau_{\Omega_h})) \right\},$$

where $k = h^2/l$ and

$$M_n = \prod_{s=0}^{n} [1 + k\lambda(\dot{x}(s))]^{-1} \sim \exp\left\{ -k \cdot \sum_{s=0}^{n} \lambda(\dot{x}(s)) \right\}, \quad k \downarrow 0.$$

It follows by Donsker's extension of the De Moive-Laplace limit theorem (see [Don[12]], [Ito, McKen[8], 1.10]) that, as $h \downarrow 0$,

$$u_h(\dot{x}) \to u(\dot{x}) := E_{\dot{x}} \left\{ \int_0^{\tau_\Omega} \mathcal{M}_s f(\dot{x}(s))\, ds + \mathcal{M}_{\tau_\Omega} \varphi(\dot{x}(\tau_\Omega)) \right\},$$

where $\dot{x}(s)$ is the space-time Brownian motion starting from point $\dot{x}(0) = \dot{x}$, τ_Ω is the first exit time of $\dot{x}(s)$ from domain $\dot{\Omega}$ and

$$\mathcal{M}_s := \exp\left\{ - \int_0^s \lambda(\dot{x}(p))\, dp \right\}, \quad s > 0.$$

According to Section 1 of our exposition $u(\dot{x})$ is exactly the solution of the Dirichlet problem (3.1) (the Kac's formula).

To estimate the rate of convergence $u_h \to u$ as $h \downarrow 0$ we will use the representation

$$u_h(\dot{x}) = k\, G^{k\lambda}_{\Omega_h} f(\dot{x}) + \Pi_{\Omega_h} \varphi(\dot{x}) \qquad \text{... (3.3)}$$

(see Theorems 2.3 and 2.5) as well as the important inequality (2.10), which now takes the following form

$$\| u \|_{\Omega_{h,T}} \le \min\left\{ T, \frac{1}{4} \right\} \| \dot{\Delta}_h u - \lambda u \|_{\Omega_{h,T}} + \| u \|_{\partial\Omega_{h,T}}, \qquad \text{... (3.4)}$$

where $\dot{\Omega}_{h,T} := \{\dot{x} \in \dot{\Omega}_h : t \le T\}$. Indeed, by application of (2.10) and Theorem 2.9 we get

$$\begin{aligned} \| u \|_{\Omega_{h,T}} &\le C(\dot{\Omega}_{h,T}, k\lambda) \| (\dot{P}_h - E)u - k\lambda u \|_{\Omega_{h,T}} + \| u \|_{\partial\Omega_{h,T}} \\ &\le \min\left\{ \frac{T}{k}, \frac{1}{4} n^2 \right\} k \| \dot{\Delta}_h u - \lambda u \|_{\Omega_{h,T}} + \| u \|_{\partial\Omega_{h,T}} \\ &= \min\left\{ T, \frac{1}{4} \right\} \| \Delta_h u - \lambda u \|_{\Omega_{h,T}} + \| u \|_{\partial\Omega_{h,T}}. \end{aligned}$$

Next we apply inequality (3.4) to $u_h - u$, where u_h and u are the solutions of the problems (3.2) and (3.1), respectively, and we obtain the following error estimate

$$\| u_h - u \|_{\Omega_{h,T}} \le \min\left\{ T, \frac{1}{4} \right\} \| \dot{\Delta}_h (u_h - u) - \lambda(u_h - u) \|_{\Omega_{h,T}}$$

$$= \min\left\{ T, \frac{1}{4} \right\} \| (\dot{\Delta}_h u_h - \lambda u_h) - (\dot{\Delta}_h u - \lambda u) \|_{\Omega_{h,T}}$$

$$= \min\left\{ T, \frac{1}{4} \right\} \| -f - (\dot{\Delta}_h u - \lambda u) \|_{\Omega_{h,T}}$$

$$= \min\left\{ T, \frac{1}{4} \right\} \| (\dot{\Delta} u - \lambda u) - (\dot{\Delta}_h u - \lambda u) \|_{\Omega_{h,T}}$$

$$= \min\left\{ T, \frac{1}{4} \right\} \| \dot{\Delta} u - \dot{\Delta}_h u \|_{\Omega_{h,T}},$$

i.e., we have gotten

$$\| u_h - u \|_{\Omega_{h,T}} \le \min\left\{ T, \frac{1}{4} \right\} \| \dot{\Delta} u - \dot{\Delta}_h u \|_{\Omega_{h,T}}. \qquad \text{... (3.5)}$$

Now the rate of convergence of $u_h \to u$ as $h \downarrow 0$ can be measured via the partial moduli of f continuity ω_1 and $\omega_{2,i}$, $i = 1, \ldots, l$, where

$$\omega_1(\delta, f; \overline{\Omega}) := \sup \{ | f(\dot{x} + \theta e) - f(\dot{x}) | : \dot{x}, \dot{x} + \theta e \in \overline{\Omega}, \ |\theta| < \delta \},$$

$$\omega_{2,i}(\delta, f; \overline{\Omega}) := \sup \{ | f(\dot{x} + \theta e_i) - 2f(\dot{x}) + f(\dot{x} - \theta e_i) | :$$

$$\dot{x}, \dot{x} \pm \theta e_i \in \overline{\Omega}, \ |\theta| < \delta \}.$$

Theorem 3.1 — *Suppose that the solution u of the problem (3.1) satisfies $u \in C^{(2)}(\overline{\Omega})$, then for the solution u_h of the problem (3.2) the following inequality hold true :*

$$\| u_h - u \|_{\Omega_{h,T}} \le \min\left\{ T, \frac{1}{4} \right\} \left[\frac{1}{4} \sum_{i=1}^{l} \omega_{2,i}(h, \partial^2_{x_i} u; \overline{\Omega}_{h,T}) + \frac{1}{2} \sum_{i=1}^{l} \omega_1(k, \partial^2_{x_i} u; \overline{\Omega}_{h,T}) + \omega_1(k, \partial_t u; \overline{\Omega}_{h,T}) \right] \qquad \text{... (3.6)}$$

PROOF : For $u \in C^{(2)}(\overline{\Omega})$ and $k = h^2/l$ we have

$$\dot{\Delta}_h u(x,t) = \frac{1}{2} h^{-2} \sum_{i=1}^{l} [u(x + he_i, t-k) + u(x - he_i, t-k) - 2u(x,t)]$$

$$= \frac{1}{2} h^{-2} \left\{ \sum_{i=1}^{l} [u(x + he_i, t-k) + u(x - he_i, t-k) - 2u(x, t-k)] \right\}$$

$$- \frac{1}{k}[u(x,t) - u(x,t-k)] := \sum_{i=1}^{l} \Delta_{h,i}\, u(x,t-k) - \partial_{k,t}\, u(x,t).$$

By appropriate Taylor expansion, we obtain

$$\left| \Delta_{h,i}\, u(x,t-k) - \frac{1}{2}\partial^2_{x_i} u(x,t-k) \right| \le \frac{1}{4}\omega_{2,i}(h, \partial^2_{x_i} u;\ \overline{\Omega}_{h,T}),$$

and

$$|\partial_{k,t}\, u(x,t) - \partial_t u(x,t)| \le \omega_1(k, \partial_t u; \overline{\Omega}_{h,T}).$$

Thus we obtain the following estimate

$$|\dot{\Delta}_h u(x,t) - \dot{\Delta} u(x,t)| \le \frac{1}{4}\sum_{i=1}^{l} \omega_{2,i}(h, \partial^2_{x_i} u;\ \overline{\Omega}_{h,T})$$

$$+ \frac{1}{2}\sum_{i=1}^{l} \omega_1(k, \partial^2_{x_i} u;\ \overline{\Omega}_{h,T}) + \omega_1(k, \partial_t u; \overline{\Omega}_{h,T}).$$

Finally we apply this estimate to (3.5) to get the desired result.

Remark 3.1 : The estimate (3.6) is sharp, i.e., there exists a function u such that

$$\liminf_{h \to 0} \| u - u_h \|_{\Omega_{h,T}} / R(h,u) > 0, \qquad \text{... (3.7)}$$

where $R(h, u)$ is the right-hand side of the inequality (3.6). Indeed, choose $u(\dot{x}) := x_1^4$ and compute both sides of the inequality (3.6). We will have

$$\dot{\Delta}\, u(\dot{x}) := 6x_1^2,$$

and

$$\dot{\Delta}_h\, u(\dot{x}) := 6x_1^2 + h^2.$$

That is,

$$\dot{\Delta}\, u(\dot{x}) - \dot{\Delta}_h\, u(\dot{x}) = -h^2.$$

Now we apply (3.3) to the function $u - u_h$, which takes zero values on the boundary $\partial\dot{\Omega}_h$, we get

$$u(\dot{x}) - u_h(\dot{x}) = -k\, G^{k\lambda}_{\Omega_h}[(\dot{\Delta}_h u - \lambda u) - (\dot{\Delta}_h u_h - \lambda u_h)](\dot{x})$$

$$= -k\, G^{k\lambda}_{\Omega_h}[(\dot{\Delta}_h u - \lambda u) - (\dot{\Delta} u - \lambda u)](\dot{x})$$

$$= -k\, G^{k\lambda}_{\Omega_h}(\dot{\Delta}_h u - \dot{\Delta} u)(\dot{x}) = -kh^2\, G^{k\lambda}_{\Omega_h} 1(\dot{x}). \qquad \text{... (3.8)}$$

Thus by (3.8) and Theorem 2.9 we have

$$\| u - u_h \|_{\Omega_{h,T}} \geq kh^2 \frac{1}{2} e^{-\frac{T}{k} \cdot k\lambda_*} \min \left\{ \frac{T}{k}, \frac{2}{\pi^2} h^{-2} \right\}$$

$$= \frac{1}{2} h^2 e^{-T\lambda_*} \min \left\{ T, \frac{2}{\pi^2 l} \right\}. \qquad \text{... (3.9)}$$

On the other hand,

$$R(h, u) = \min \left\{ T, \frac{1}{4} \right\} \cdot \frac{1}{4} \sum_{i=1}^{l} \omega_{2,i} (h, \partial_{x_i}^2 u; \overline{\Omega}_{h,T})$$

$$= \min \left\{ T, \frac{1}{4} \right\} \cdot \frac{1}{4} \omega_{2,1} (h, \partial_{x_1}^2 u; \overline{\Omega}_{h,T}) = \min \left\{ T, \frac{1}{4} \right\} 6h^2. \qquad \text{... (3.10)}$$

Therefore, (3.9) and (3.10) imply (3.7).

4. Sharpness of the Error Estimate

As it was mentioned in Remark 3.1 the error estimate (3.6) is sharp, i.e., there exists a function u such that

$$\| u - u_h \|_{\Omega_{h,T}} \approx R(h, u) \quad \text{as} \quad h \downarrow 0,$$

where $R(h, u)$ is the right-hand side of the inequality (3.6).

The fact that (3.6) is sharp with regard to the rate of convergence is now established in connection to general Lipschitz classes, determined by an abstract modulus of continuity, i.e., by a function $\omega(t)$, continuous on $[0, +\infty)$ such that

$$0 = \omega(0) < \omega(s) \leq \omega(s+t) \leq \omega(s) + \omega(t), \quad s, t > 0.$$

Here we follow the same technique that was applied[2-4]. These were articles devoted to the elliptic and parabolic Dirichlet problems with $\lambda(\dot{x}) \equiv 0$ in dimensions $l = 1, 2$ and $l = 1$, respectively. Our reasoning is based on the following variant of the uniform boundness principle [Dick[13]]. For a Banach space $(X, \| \cdot \|)$ let X^* be the set of sublinear bounded functionals on X. We have

Theorem 4.1 — *Suppose that for given* $\{T_n\}_{n \in N} \subset X^*$ *and* $\{S_\delta\}_{\delta > 0} \subset X^*$ *there are* $\{g_n\}_{n \in N} \subset X$ *such that*

$$\| g_n \| \leq c_1, \quad n = 1, 2, \ldots \qquad \text{... (4.1)}$$

$$\liminf_{n \to \infty} | T_n g_n | > 0 \qquad \text{... (4.2)}$$

$$| S_\delta g_n | \leq c_2 \min \left\{ 1, \frac{\sigma(\delta)}{\varphi_n} \right\}, \quad n = 1, 2, \ldots, \qquad \text{... (4.3)}$$

where $\sigma(\delta)$ *is a strictly positive function on* $(0, \infty)$ *and* $\{\varphi_n\}_{n \in N}$ *is a strictly decreasing real sequence with* $\lim_{n \to \infty} \varphi_n = 0$*. Then for each modulus of continuity* ω *as above, satisfying*

$$\lim_{t \to 0} \frac{\omega(t)}{t} = \infty, \qquad \text{... (4.4)}$$

there exists an element $u_\omega \in X$ *such that*

$$| S_\delta u_\omega | \le c_\omega \cdot \omega(\sigma(\delta)), \quad 0 < \delta < 1, \qquad \text{... (4.5)}$$

$$\lim \inf | T_n u_\omega | / \omega(\varphi_n) > 0. \qquad \text{... (4.6)}$$

Next comes our optimal result.

Theorem 4.2 — *For every modulus of continuity* ω *there exists a function* $u_\omega \in C^{(2)}(\overline{\Omega})$*, such that*

$$R(\delta, u_\omega) \le c_\omega\, \omega(\delta^2), \quad 0 < \delta < 1 \qquad \text{... (4.7)}$$

and

$$\liminf_{h \to 0} \| u_\omega - u_{\omega, h} \|_{\dot{\Omega}_{h,T}} / \omega(h^2) > 0. \qquad \text{... (4.8)}$$

PROOF : To apply Theorem 4.1 we denote by

$$X := C^{(2)}(\overline{\Omega}),$$

$$T_n u := \| u - u_h \|_{\overline{\Omega}_{h,T}}, \quad h = 1/n,$$

$$S_\delta u := R(\delta, u), \quad 0 < \delta < 1$$

and

$$g_n(\dot{x}) := n^{-2} \sum_{i=1}^{l} \sin^2 \pi n x_i, \quad \dot{x} = (x, t), \quad x = (x_1, \ldots, x_l) \in \Omega.$$

Then (4.1) is satisfied with $c_1 = \pi^2 l$. We observe that $g_n(\dot{x}) = g_{n,h}(\dot{x})$ for $\dot{x} \in \partial\dot{\Omega}_{h,T}$, $h = 1/n$, and $\dot{\Delta}\, g_n(\dot{x}) = \pi^2 l$ and $\dot{\Delta}_h\, g_n(\dot{x}) = 0$ for all $\dot{x} \in \dot{\Omega}_{h,T}$. Then (3.3) implies

$$\begin{aligned} T_n g_n &= \| g_n - g_{n,h} \|_{\Omega_{h,T}} \\ &= k \| G^{k\lambda}_{\Omega_{h,T}} [\dot{\Delta}_h (g_n - g_{n,h}) - \lambda (g_n - g_{n,h})] \|_{\Omega_{h,T}} \\ &= k \| G^{k\lambda}_{\Omega_{h,T}} [(\dot{\Delta}_h (g_n - \lambda g_n) - (\dot{\Delta}_h g_{n,h} - \lambda g_{n,h})] \|_{\Omega_{h,T}} \\ &= k \| G^{k\lambda}_{\Omega_{h,T}} [(\dot{\Delta}_h g_n - \lambda g_n) - (\dot{\Delta}\, g_n - \lambda g_n)] \|_{\Omega_{h,T}} \\ &= k \| G^{k\lambda}_{\Omega_{h,T}} (\dot{\Delta}_h g_n - \dot{\Delta}\, g_n) \|_{\Omega_{h,T}} \\ &= k\pi^2 l \| G^{k\lambda}_{\Omega_{h,T}} 1 \|_{\Omega_{h,T}} \end{aligned}$$

$$\geq k\pi^2 l \cdot \frac{1}{2} \cdot e^{-\frac{T}{k} \cdot k\lambda_*} \min\left\{\frac{T}{k}, \frac{2}{\pi^2} h^{-2}\right\}$$

$$= e^{-T\lambda_*} \min\left\{1, \frac{\pi^2 lT}{2}\right\}.$$

The last inequality comes from Theorem 2.9, and hence condition (4.2) is satisfied. To verify the condition (4.3) we observe that

$$S_\delta g_n \leq \frac{1}{2}\pi^2 l$$

and

$$S_\delta g_n \leq \frac{\delta^2}{16} \sum_{i=1}^{l} \| \partial^4_{x_i} g_n \|_{\Omega_{h,T}} \leq \frac{\delta^2 n^2 \pi^4 l}{2}.$$

These upper bounds to $S_\delta g_n$ yield (4.3) with $\sigma(\delta) := \pi^2\delta^2$ and $\varphi_n := n^{-2}$.

Thus we are able to apply Theorem 4.1 and (4.7)-(4.8) are established.

REFERENCES

1. M. Kac, On some connections between probability theory and differential and integral equations, *Proc. Second Berkeley Symposium on Math. Stat. Probability*, 189-215. Univ. California Press, (1951).
2. B. Büttgenbach, H. Esser and R. S. Nessel, *Numer. Funct. Anal. Optim.*, **12** (1991), 285-98.
3. B. Büttgenbach, H. Esser, G. Lüttgens and R. J. Nessel, *J. Comp. appl. Math.* **44** (1992), 331-37.
4. H. Esser, St. J. Goebbels, G. Lüttgens and R. J. Nessel, Sharp error bounds for the Crank-Nicolson and Saulyev difference scheme in connection with an initial boundary value problem for the inhomogeneous heat equation, to appear : Special Issue on *Concrete Analysis, J. Comp. Math. Appl.* (1995).
5. J. L. Doob, *Classical Potential Theory and its Probabilistic Counterpart*, Springer-Verlag, New York, (1984).
6. E. B. Dynkin, *Markov Processes*, Springer-Verlag, Berlin-Heidelberg, New York (1965).
7. R. M. Blumenthal and R. K. Getoor, *Markov Processes and Potential Theory*, Academic Press, New York and London (1968).
8. K. Ito and M. P. McKean, *Diffusion Processes and Their Sample Path*, Berlin, Springer-Verlag (1974).
9. F. Spitzer, *Principles of Random Walk*, Graduate Texts in Math., 34, Springer-Verlag, New York (1976).
10. E. B. Dynkin and A. A. Yushkevich, *Markov Processes : Theorems and Problems*, Plenum Press, New York (1969).
11. A. Shiryaev, *Probability*, Springer-Verlag, New York, Berlin (1984).
12. M. D. Donsker, *Mem. Amer. math. Soc.*, No. **6** (1951).
13. W. Dickmeis, R. J. Nessel and E. Van Wickeren, *Jahr. Deutsch. Math. Verein.* **89** (1987), 105-34.