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ENUMERATION OF BIPARTITE SELF-COMPLEMENTARY GRAPHS

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A bipartite graph is a graph (simple) G where the vertex set V can be decomposed into two subsets V_1 and V_2 such that each edge of G joins a vertex of V_1 to a vertex of V_2. Given a bipartite graph $G(V_1, V_2)$ its bipartite — complement is defined as the bipartite graph $\overline{G}(V_1, V_2)$ whose vertex set is $V(G)$ and the edge set is $\{uv \mid u \in V_1, v \in V_2 \text{ and } uv \notin E(G)\}$. A bipartite graph will be said to be bipartite self-complementary if it is isomorphic to its bipartite — complement. In this paper we obtain generating functions to enumerate the bipartite self-complementary graphs with a given bipartition.

Key Words : Bipartite Graph; Self-Complementary Graph

1. INTRODUCTION

Formulae for the numbers of self-complementary graphs and digraphs were obtained by Read[1] using de Bruin's extension[2] of Polya's theorem. Parthasarathy and Sridharan[3] obtained generating functions for these graphs in terms of their partitions. In this paper, we obtain generating functions for bipartite self-complementary graphs in terms of partitions.

All graphs are assumed to have no multiple edges or loops. We refer the reader to Harary[4] for notations in graph theory. A graph G is said to be bipartite if its vertex set V can be partitioned into two subsets V_1 and V_2 such that every edge joins a vertex of V_1 to a vertex of V_2 and is denoted by $G(V_1, V_2)$. Gangopadhyay[5] introduced the notion of bipartite — complement of a graph as follows : Given a bipartite graph $G(V_1, V_2)$ its bipartite — complement is defined to be the bipartite graph $\overline{G}$ whose vertex set is $V(G)$ and the edge set $E(\overline{G})$ is $\{uv \mid u \in V_1, v \in V_2 \text{ \& } uv \notin E(G)\}$. A bipartite graph G is said to be bipartite self-complementary if G is isomorphic to $\overline{G}$.

2. Bipartite Graphs $G(m, n)$ with $m \neq n$

Let X and Y be two sets. Let $D = X \times Y$ be the cartesian product of X and Y and $R = \{0, 1\}$. Then there is a correspondence between functions from D to R and the bipartite graphs on $X \cup Y$. Here we consider the case when $|X| \neq |Y|$. For the permutation group K acting on D we take $S_m \times S_n$, $|X| = m$, $|Y| = n$. Isomorphic bipartite graphs on $X \cup Y$ correspond to equivalent functions from D into R.

The generating function for the bicoloured graphs is given by Polya's theorem as

$$\sum_{f \in \mathcal{F}} W(f) \equiv S_1(W) = \frac{1}{m!\, n!} \sum_{k \in S_m \times S_n} \sum_{f}^{(k)} W(f),$$

where

$$W(f) = 0 \prod_{x_i \in X} \prod_{y_j \in Y} (u_i v_j)^{f(x_i, y_j)} \equiv 0 W_0 f\ ;$$

the effect of 0 is to replace $W_0(f)$ by the leading term of the symmetric function product $(\pi_m)_u (\pi_n)_v$ corresponding to the bipartition (π_m/π_n) of the bipartite graph represented by f.

Under the weight function isomorphic graphs have equal weights but a graph and its bipartite — complement may have, in general, different weights. To obtain the same weight for a graph and its bipartite — complement we use the modified weight function

$$W_1(f) = \tilde{0} \prod_{x_i \in X} \prod_{y_j \in Y} (u_i v_j)^{f(x_i, y_j)}$$

where $\tilde{0}$ acts as follows :

$$\tilde{0}\ (u_1^{s_1} u_2^{s_2} \ldots u_m^{s_m} / v_1^{t_1} v_2^{t_2} \ldots v_n^{t_n})$$

$$= u_1^{s_{\nu_1}} u_2^{s_{\nu_2}} \ldots u_m^{s_{\nu_m}} / v_1^{t_{\mu_1}} v_2^{t_{\mu_2}} \ldots v_n^{t_{\mu_n}} \text{ if } \pi_\nu \prec \bar{\pi}_\nu$$

$$= u_1^{S_{\nu_1}} \ldots u_m^{S_{\nu_m}} / v_1^{T_{\mu_1}} \ldots v_n^{T_{\mu_n}} \text{ if } \bar{\pi}_\nu \prec \pi_\nu$$

where $\pi_\nu = (s_{\nu_1}, \ldots, s_{\nu_m} / t_{\mu_1}, \ldots, t_{\mu_n})$ and $\bar{\pi}_\nu = (S_{\nu_1}, \ldots, S_{\nu_m} / T_{\mu_1}, \ldots, T_{\mu_n})$, with $S_{\nu_i} = n - s_{\nu_i}$ and $T_{\mu_i} = m - t_{\mu_i}$ are monotonic nonincreasing rearrangements of $(s_1, \ldots, s_m / t_1, \ldots, t_n)$ and $\prec$ is defined as follows. Let $\pi_1 = (d_1, \ldots, d_m / e_1, \ldots, e_n)$, $\pi_2 = (f_1, \ldots, f_m / g_1, \ldots, g_n)$. $\pi_1 \prec \pi_2$ if there exists N_1, N_2 such that $d_i = f_i$, $i = 1, \ldots, N_1$, $e_j = g_j$, $j = 1, 2, \ldots, N_2$ and $d_{N_1+1} > f_{N_1+1}$ and $e_{N_2+1} > g_{N_2+1}$.

We now require the general form of a counting theorem due to Harary and Palmer[6]. Take $G = S_m \times S_n$, $H = S_2$ acting on $\{0, 1\}$ and change K to the power group H^G i.e., $S_2^{S_m \times S_n}$. We have here the Harary-Palmer situation in which the generating function by equivalence classes is given by

$$S_2(W) = \frac{1}{2}\frac{1}{m!\,n!} \sum_{k \in S_m \times S_n} \sum_{f}^{(k,k')} W(f)$$

$$= \frac{1}{2}\frac{1}{m!\,n!}\left[\sum_{f}^{(k,k_1')} W(f) + \sum_{f}^{(k,k_2')} W(f) \right]$$

where $k_1' = (0)\ (1)$, $k_2' = (01)$ and $W(f)$ is any weight function which gives equal weight to equivalent functions.

With W_1 for the weight function, $S_1(W_1)$ counts a bipartite non-self complementary graph and its bipartite—complement as two different graphs with the same weight while $S_2(W_1)$ counts them as one graph. A bipartite—self complementary graph is counted as just one graph with the same weight in $S_1(W_1)$ and $S_2(W_1)$. Hence, the expression $2S_2(W_1) - S_1(W_1)$ enumerates bipartite-selfcomplementary graphs since for such graphs the partitions π_V and $\overline{\pi}_V$ are equal. Hence, the generating function is

$$2S_2(W_1) - S_1(W_1) = \frac{1}{m!\,n!} \sum_{k \in S_m \times S_n} \sum_{f}^{(k,k_2')} W_1(f).$$

To evaluate the generating function we set

$$\chi(k) = \sum_{f}^{(k,k_2')} \prod_{x_i \in X} \prod_{y_j \in Y} (u_i v_j)^{f(x_i, y_j)}$$

for $k \in S_m \times S_n$. Each $k \in S_m \times S_n$ corresponds to a pair (g, h) where $g \in S_m$, $h \in S_n$ are permutations acting on X and Y respectively. The restriction on the summation requires that $f(d) = (0, 1)\ f(kd)$ for $d = (x_i, y_j) \in X \times Y$. This implies that

$$f(d) = f(k^2 d) = f(k^4 d) = \ldots,\quad f(kd) = f(k^3 d) = \ldots,\quad f(d) \neq f(kd).$$

Hence if $k = (g, h)$, then the contribution of k to $\Sigma\chi(k)$ is non-zero only if both g and h do not contain an odd cycle (simultaneously). Thus we must have that either all the cycles of g are even or all the cycles of h are even. This also implies that both m and n cannot be odd. So without loss we assume that n is even. When m is odd we consider only those permutations of S_n which have all even cycles. Thus let $g = 1^{\lambda_1} 2^{\lambda_2} \ldots m^{\lambda_m}$ and $h = 2^{\mu_2} 4^{\mu_4} \ldots n^{\mu_n}$. We compute $\chi(k)$ for a given pair (g, h) $\in S_m \times S_n$ as follows. The elements of $D = X \times Y$ on which k acts may be obtained by considering all cycle pairs (α, β) of k. Each cycle pair (α, β) of k contributes to $\chi(k)$. If lengths of α and β are p and q respectively ($p \neq q, p$ odd, q even) then $\alpha \times \beta$ forms (p, q) cycles of k of length $[p, q]$ each where (p, q) denotes the gcd and $[p, q]$ the lcm of p and q.

Let $p_{i_1} p_{i_2} \ldots p_{i_p}$ be the ith cycle of length p in g and $q_{j_1} q_{j_2} \ldots q_{j_q}$ be the jth cycle of length q in h. Let

$$U_{i_p} = u_{i_1} u_{i_2} \ldots u_{i_p}$$

$$V_{1j_q} = v_{j_1} v_{j_3} \ldots v_{j_{q-1}}, \quad V_{2j_q} = v_{j_2} v_{j_4} \ldots v_{j_q}.$$

Then the contribution to $\chi(k)$ from these $[p, q]$-cycles of (α, β) is $U_{i_p}^{b/2} [V_{1j_q}^a + V_{2j_q}^a]$ where $a = p/(p, q)$, $b = q/(p, q)$. There are in all (p, q) such cycles. Thus the contibution from this pair (i.e. ith cycle of length p and jth cycle of length q) is

$$[U_{i_p}^{b/2} \{V_{1_{jq}}^a + V_{2_{jq}}^a\}]^{(p, q)}.$$

Now as $g = 1^{\lambda_1} 2^{\lambda_2} \ldots m^{\lambda_m}$ and $h = 2^{\mu_2} \ldots n^{\mu_n}$ the factor corresponding to odd cycles of g is

$$\prod_{p \text{ odd}} \prod_{q \text{ even}} \prod_{i=1}^{\lambda_p} \prod_{j=1}^{\mu_q} [U_{ip}^{b/2} (V_{1_{jq}}^a + V_{2_{jq}}^a)]^{(p, q)}.$$

If p and q are both even, the corresponding factor of $\chi(k)$ becomes

$$[(U_{1_{ip}}^b V_{1_{jq}}^a + U_{2_{ip}}^b V_{2_{jq}}^a)(U_{1_{ip}}^b V_{2_{jq}}^a + U_{2_{ip}}^b V_{1_{jq}}^a)]^{(p, q)/2}.$$

[This product covers both $p = q$ and $p \neq q$. If $p = q$ the corresponding term becomes

$$[(U_{1_{ip}} V_{1_{jq}} + U_{2_{ip}} V_{2_{jq}})(U_{1_{ip}} V_{2_{jq}} + U_{2_{ip}} V_{1_{jq}})]^{p/2} \;]$$

Hence if $g = 1^{\lambda_1} 2^{\lambda_2} \ldots m^{\lambda_m}$, $h = 2^{\mu_2} 4^{\mu_4} \ldots n^{\mu_n}$, $k = (g, h)$ then

$$\chi(k) = \prod_{p \text{ odd}} \prod_{q \text{ even}} \prod_{i=1}^{\lambda_p} \prod_{j=1}^{\mu_q} [U_{ip}^{b/2} (V_{1_{jq}}^a + V_{2_{jq}}^a)]^{(p, q)}$$

$$\times \prod_{p \text{ even}} \prod_{q \text{ even}} \prod_{i=1}^{\lambda_p} \prod_{j=1}^{\mu_q} [(U_{1_{ip}}^b V_{1_{jq}}^a + U_{2_{ip}}^b V_{2_{jq}}^a)(U_{1_{ip}}^b V_{2_{jq}}^a + U_{2_{ip}}^b V_{1_{jq}}^a)]^{(p, q)/2}$$

Using the above method the following generating functions are obtained for graphs on (3, 2) vertices and (3, 4) vertices respectively.

(3, 2) Vertices :

$$u_1 u_2 u_3 v_1^3 + u_1 u_2 u_3 v_1^2 v_2 + u_1^2 u_2 v_1^2 v_2.$$

The corresponding graphs are shown in Figure 1.

(3, 4) Vertices :

$$(u_1 u_2 u_3)^2 v_1^3 v_2^3 + (u_1 u_2 u_3)^2 v_1^3 v_2^2 v_3 + 2(u_1 u_2 u_3)^2 v_1^2 v_2^2 v_3 v_4$$
$$+ u_1^4 u_2^2 v_1^2 v_2^2 v_3 v_4 + u_1^3 u_2^2 u_3 v_1^3 v_1^2 v_3 + u_1^3 u_2^2 u_3 v_1^2 v_2^2 v_3 v_4.$$

The corresponding graphs are shown in Figure 2.

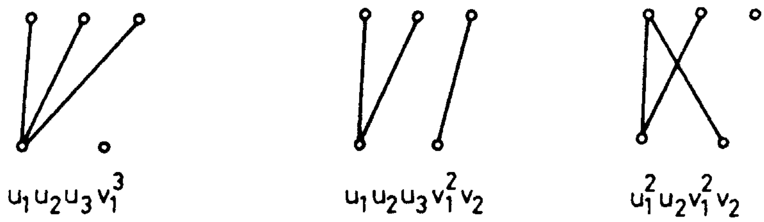

FIG. 1. Bipartite self complementary graphs of type (3, 2)

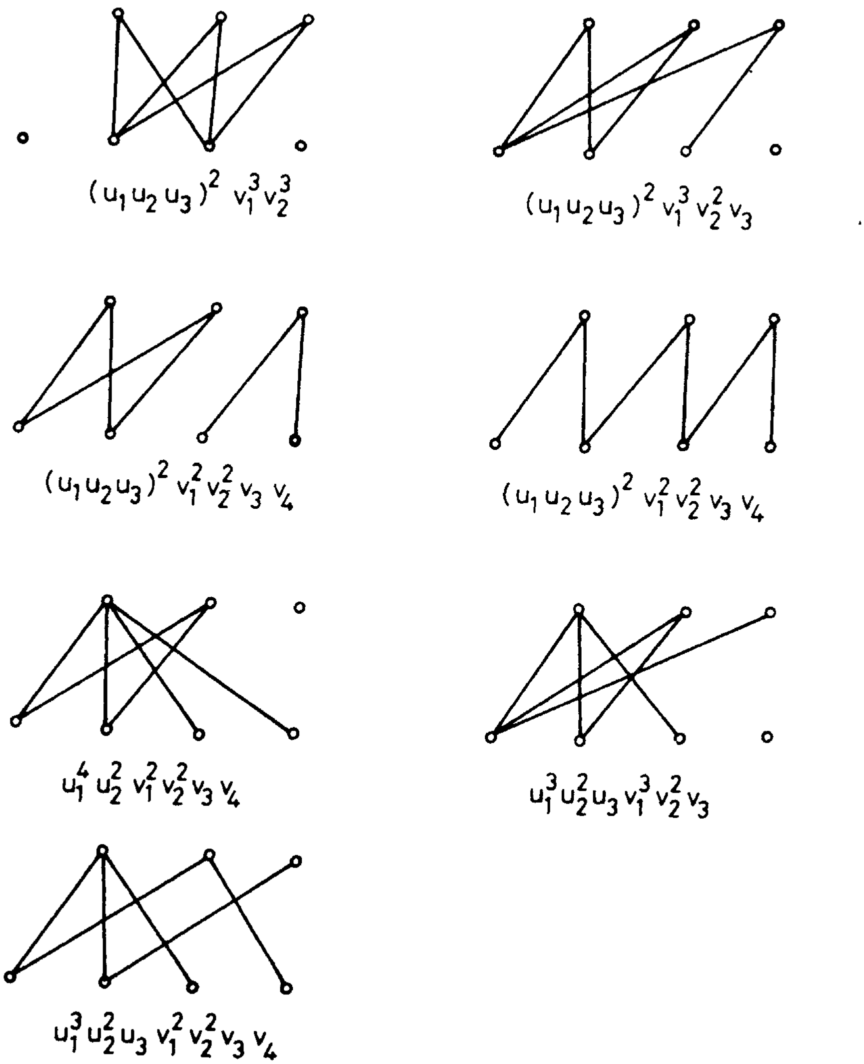

FIG. 2. Bipartite self-complementary graphs of type (3, 4)

3. Bipartite Graphs $G_{m,n}$ with $m = n$

The changes required for this case from the previous one are in the weight function and the permutation group acting on the domain set $D = X \times Y$. To count the number of bipartite graphs with given degree sequence we define the weight function by

$$W(f) = \bar{0}\ W_0(f) = \bar{0} \prod_{x_i \in X} \prod_{y_j \in Y} (u_i v_j)^{f(x_i y_j)}$$

where the effect of $\bar{0}$ is to replace $W_0(f)$ by the leading term of the symmetric function product $(\pi_n^1)_u \times (\pi_n^2)_v$ corresponding to the bipartition (π_n^1, π_n^2) of the graph represented by f if $\pi_n^1 > \pi_n^2$; if however $\pi_n^2 > \pi_n^1$, $\bar{0}$ replaces $W_0(f)$ by the leading term $(\pi_n^2)_u \times (\pi_n^1)_v$. Here π_n^1 and π_n^2 denote two n-part partitions of the same number of lines.

Under the weight function $\bar{0}$, isomorphic bipartite graphs have equal weights but a graph and its bipartite complement have in general different weights. To obtain the same weight for a graph and its bipartite-complement, we use the modified weight function

$$W_1(f) = \tilde{0}\left[\ \bar{0}\left(\prod_{x_i \in X} \prod_{y_j \in Y} (u_i v_j)^{f(x_i y_j)}\right)\right]$$

where $\tilde{0}$ $(\bar{0})$ is as follows :

Given π_n^1/π_n^2 two n-part partitions of the same number of lines, if $\pi_n^1 \succ \pi_n^2$ attach π_n^1 to the X's and π_n^2 to the Y's. If, however, $\pi_n^2 \succ \pi_n^1$ attach π_n^1 to the Y's and π_n^2 to the X's, after having fixed these, call the resulting sequence as $\pi^* = (s_1, \dots, s_n/t_1, \dots, t_n)$. Then operate $\tilde{0}$, i.e. consider

$$\tilde{0}\,(u_1^{s_1}, \dots, u_n^{s_n}/v_1^{t_1}, \dots, v_n^{t_n})$$

$$= u_1^{s_{\nu_1}} \cdot u_2^{s_{\nu_2}} \dots u_n^{s_{\nu_n}} \times v_1^{t_{\mu_1}} \cdot v_2^{t_{\mu_2}} \dots v_n^{t_{\mu_n}} \qquad \text{if} \quad \pi_\nu^* \geq \bar{\pi}_\nu^*$$

$$= u_1^{S_{\nu_1}} \cdot u_2^{S_{\nu_2}} \dots u_n^{S_{\nu_n}} \times v_1^{T_{\mu_1}} \cdot v_2^{T_{\mu_2}} \dots v_n^{T_{\mu_n}} \qquad \text{if} \quad \bar{\pi}_\nu^* \geq \pi_\nu^*$$

where $\pi_\nu^* = (s_{\nu_1}, s_{\nu_2}, \dots, s_{\nu_n}/t_{\nu_1}, t_{\nu_2}, \dots, t_{\mu_n})$ and

$$\bar{\pi}_\nu^* = (S_{\nu_1}, S_{\nu_2}, \dots, S_{\nu_n}/T_{\mu_1}, T_{\mu_2}, \dots, T_{\mu_n})$$

$$S_{\nu_i} = n - s_{\nu_i}, \quad T_{\mu_i} = n - t_{\mu_i}$$

are monotonic nonincreasing rearrangements of $(s_1, s_2, ..., s_n/t_1, t_2, ..., t_n)$ and $\succ$ is defined as in section 2.

Harary[7] has shown that the permutation group K appropriate for this case is the exponentiation group $S_n^{S_2}$ which is the line group of the complete bipartite graph K_{nn}. The point-group or automorphism group of K_{nn} is the composition group $S_2[S_n]$.

In Harary's notation these groups can be written as

$$S_2[S_n] = (S_n \cdot S_n) \bigcup r\,(S_n \cdot S_n)$$

$$S_n^{S_2} = (S_n \times S_n) \bigcup \rho\,(S_n \times S_n)$$

Here $S_n \cdot S_n$ and $S_n \times S_n$ are the direct product and Cartesian product of two copies of S_n acting on X and Y. The set $r(S_n \cdot S_n)$ consists of $(n!)^2$ permutations on $X \bigcup Y$ of the form $r(g, h)$ where $g, h \in S_n$ and the effect of r is to interchange corresponding elements x_i and y_i of X and Y. Each of these permutations is obtained by interposing the elements of two permutations of the two copies of S_n. Thus, corresponding to each permutation $g \in S_n$ with cycle structure $(\lambda) = 1^{\lambda_1} 2^{\lambda_2} \ldots n^{\lambda_n}$, there are $n!$ permutations of $r(S_n \cdot S_n)$ with cycle structure $(2\lambda) = 2^{\lambda_1} 4^{\lambda_2} \ldots (2n)^{\lambda_n}$. The set of permutations $\rho(S_n \times S_n)$ are those induced on the elements of $D = X \times Y$ by the members of $r(S_n \cdot S_n)$.

Using Polya's theorem with the above specifications we obtain the required generating function as

$$2S_2(W_1) - S_1(W_1) = \frac{1}{2(n!)^2}\left[\sum_{k \in S_n \times S_n} \sum_{f}^{(k, k_2')} W_1(f) + \sum_{k \in \rho(S_n \times S_n)} \sum_{f}^{(k, k_2')} W_1(f)\right]$$

The first term is as in section 2. For the second sum we set

$$\chi(k) = \sum_{f}^{(k, k_2')} \prod_{x_i \in X} \prod_{y_j \in Y} (u_i v_j)^{f(x_i, y_j)} \text{ for } k \in \rho(S_n \times S_n).$$

Here also, as in section 2, the restriction

$$f(kd) = f(k^3 d), ..., f(k^2 d) = f(k^4 d) = ..., f(d) \neq f(kd)$$

implies that in $r(g, h)$ g cannot have any odd cycle. Thus the cycle structure of g is of the form $(\lambda) = 2^{\lambda_2} 4^{\lambda_4} \ldots$ and hence that of $r(g, h)$ is $(2\lambda) = 4^{\lambda_2} 8^{\lambda_4} \ldots$.

Straightforward computations give the expression for $\chi(k)$ as

$$\chi(k) = \prod_{p=1}^{n} \prod_{q=1}^{n} \prod_{i=1}^{\lambda_p} \prod_{j=1}^{\lambda_q} (U^b_{p_i} V^a_{q_j} + U^a_{q_j} V^b_{p_i})^{(p,q)}$$

$$\times \prod_{p=1}^{n} \prod_{i=1}^{\lambda_p} \prod_{j=1}^{\lambda_p} (U_{p_i} V_{p_j} + U_{p_j} V_{p_i})^p$$

$$\times \prod_{i=1}^{n} \prod_{i=1}^{\lambda_p} (2U_{p_i} V_{p_i})^{p/2}.$$

Using the above method the following generating function is obtained for graphs on (4, 4) vertices.

$$u_1^4 u_2^4 (v_1 v_2 v_3 v_4)^2 + u_1^3 u_2^3 u_3 u_4 v_1^3 v_2^3 v_3 v_4 + 2(u_1 u_2 u_3 u_4)^2 (v_1 v_2 v_3 v_4)^2$$
$$+ u_1^4 u_2^3 u_3 (v_1 v_2 v_3 v_4)^2 + u_1^4 u_2^2 u_3^2 (v_1 v_2 v_3 v_4)^2 + 2u_1^3 u_2^3 u_3 u_4 (v_1 v_2 v_3 v_4)^2$$
$$+ 2u_1^3 u_2^2 u_3^2 u_4 (v_1 v_2 v_3 v_4)^2 + u_1^4 u_2^3 u_3 v_1^3 v_2^2 v_3^2 v_4 + u_1^4 u_2^2 u_3^2 v_1^3 v_2^3 v_3 v_4$$
$$+ 2u_1^3 u_2^2 u_3^2 u_4 v_1^3 v_2^2 v_3^2 v_4 + u_1^4 u_2^2 u_3^2 v_1^3 v_2^2 v_3^2 v_4 + u_1^3 u_2^3 u_3 u_4 v_1^3 v_2^2 v_3^2 v_4.$$

The corresponding graphs are shown in Figure 3.

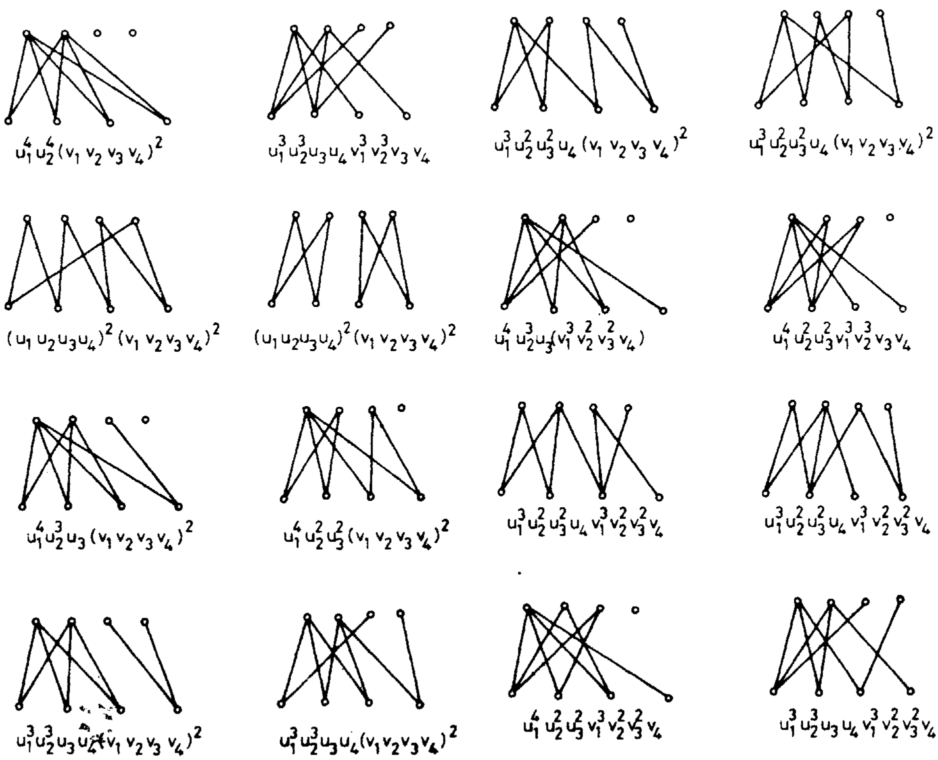

FIG. 3. Bipartite self complementary graphs of type (4; 4)

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