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ON WEIGHTED COMPOSITION OPERATORS ON NON-LOCALLY CONVEX FUNCTION SPACES

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For a system V of weights on a topological space X and a Hausdorff topological vector space E, let $FV_b(X, E)$ and $FV_0(X, E)$ denote the weighted spaces of E-valued functions f on X for which $v.f(X)$ is bounded in E and $v.f$ vanishes at infinity on X, respectively. Let $\pi: X \to B(E)$ and $T: X \to X$ be two mappings, where $B(E)$ is the space of all continuous linear mappings from E into itself. In this paper, we present necessary and sufficient conditions for the pair (π, T) to induce weighted composition operators on $FV_b(X, E)$ and $FV_0(X, E)$. These results extend the recent work of Singh and Singh[4] to a non-locally convex setting.

Key Words : System of Weights, Locally Bounded Space, Weighted Topology, Weighted Composition Operator.

INTRODUCTION

The contents of this paper pertain to the theory of weighted composition operators on weighted spaces of vector-valued functions which are recently studied by Singh and Summers[1], Singh and Manhas[2,3], Singh and Singh[4], and Khan and Taheem[5]. For details, we refer to the recent survey article of Singh and Singh[6]. This paper is a continuation of our earlier paper[7] in which we have studied weighted composition operators $W_{\pi,T}(W_{\theta,T})$ on non-locally convex spaces $FV_b(X, E)$ and $FV_0(X, E)$ induced by $\pi: X \to E$ ($\theta: X \to \mathbb{C}$) and $T: X \to X$. The purpose of this paper is to characterize those weighted composition operators on $FV_b(X, E)$ and $FV_0(X, E)$ which are induced by $\pi: X \to B(E)$ and $T: X \to X$. These results extend results of Singh and Singh[4] to the spaces $FV_b(X, E)$ and $FV_0(X, E)$ of E-valued functions on X which are not necessarily continuous, and where X is just a topological space and E is a topological vector space or a locally bounded space which is not necessarily locally convex.

2. PRELIMINARIES

Throughout this paper we shall assume, unless stated otherwise, that X is a Hausdorff topological space and E is a non-trivial Hausdorff topological vector space with a base $\mathcal{N}$ of balanced "shrinkable" neighbourhoods of 0. (A neighbourhood G of 0 in E is called shrinkable (cf. Klee[8]) if $r\overline{G} \subset$ int. G for $0 \leq r < 1$. By Klee[8], Theorems 5 and 6), every Hausdorff topological vector space has a base of shrinkable neighbourhoods of 0 and also the Minkowski functional p_G of any such neighbourhood G is continuous.

Let $S^+(X)$ denote the set of all non-negative upper semi-continuous functions on X, and let $S_0^+(X)$ ($S_c^+(X)$) be the subset of $S^+(X)$ consisting of those functions which vanish at infinity (have compact support). A Nachbin family on X is a subset V of $S^+(X)$ such that for every $u, v \in V$ and $\alpha > 0$, there exists $w \in V$ so that $\alpha u, \alpha v \leq w$ (pointwise on X). If corresponding to each $x \in X$, there exists $v_x \in V$ such that $v_x(x) \neq 0$, then we write $V > 0$. A Nachbin family V on X such that $V > 0$ is called a system of weights on X. Let $F(X, E)$ be the vector space of all E-valued functions on X and let $F_b(X, E)$ ($F_0(X, E)$) denote the subspace of $F(X, E)$ consisting of those functions which are bounded (vanish at infinity).

For a system V of weights on X, let $FV_b(X, E)$ ($FV_0(X, E)$) be the subspace of $F(X, E)$ consisting of those functions f such that $v.\ f \in F_b(X, E)$ ($F_0(X, E)$) for all $v \in V$. On $FV_b(X, E)$, we shall consider the weighted topology w_V which is a linear Hausdorff topology having a base of neighbourhoods of 0 consisting of all sets of the form :

$S(v, G) = \{f \in FV_b(X, E) : v(x)f(x) \in G \text{ for all } x \in X\}$, where $v \in V$ and $G \in \mathcal{N}$. The space $FV_b(X, E)$ endowed with w_V is called a weighted space. The space $FV_0(X, E)$, being subspace of $FV_b(X, E)$, is equipped with the topology induced by $FV_b(X, E)$.

The following are some instances of weighted spaces :

(i) If $V = K^+(X)$, the set of all non-negative constant functions on X, then $FV_b(X, E) = F_b(X, E)$ and $FV_0(X, E) = F_0(X, E)$. The totpology w_V in this case is the topology σ of uniform convergence.

(ii) If $V = \{\alpha\chi_K : \alpha \geq 0 \text{ and } K \subset X, K \text{ compact}\}$, where χ_K denotes the characteristic function of K, and $V = S_c^+(X)$, then $FV_b(X, E) = FV_0(X, E) = F(X, E)$ and w_V is the compact-open topology k.

(iii) If $V = S_0^+(X)$, then $FV_b(X, E) = FV_0(X, E) = F_b(X, E)$ and w_V in this case is called the substrict topology β_0.

For more details on weighted spaces, we refer to Nachbin[9], Summers[16], Prolla[10], Bierstedt[11], and Khan[12,13].

Let $B(E)$ denote the vector space of all continuous linear mappings from E into itself, endowed with the linear topology which has a base of neighbourhoods of 0 consisting of all sets of the form:

$$U(B, G) = \{A \in B(E) : A(B) \subseteq G\},$$

where B is a bounded subset of E and $G \in \mathcal{N}$. The convergence in this topology is the uniform convergence on bounded subsets of E. For details about such topologies, we refer to Köthe[14].

Let $\pi : X \to B(E)$ and T be a selfmap on X. Then a weighted composition operator on $FV_b(X, E)$ ($FV_0(X, E)$) induced by the pair (π, T), denoted by $W_{\pi, T}$, is a continuous linear map from $FV_b(X, E)$ ($FV_0(X, E)$) into itself and is given by

$$W_{\pi, T} f(x) = \pi(x)\,(f(T(x)))$$

for all $f \in FV_b(X, E)$ ($FV_0(X, E)$) and $x \in X$. If $\pi(x) = I$, the identity mapping on E, for all $x \in E$, we write $W_{\pi, T}$ as C_T and call it the composition operator on $FV_b(X, E)$ ($FV_0(X, E)$) induced by T. In case T is the identity map on X, we write $W_{\pi, T}$ as M_π and call it the multiplication operator on $FV_b(X, E)$ ($FV_0(X, E)$) induced by π. For details of such operators on weighted spaces, we refer to the monograph of Singh and Manhas[15].

3. Weighted Composition Operators

Before characterizing weighted composition operators on $FV_b(X, E)$, we first present the following proposition :

Proposition 3.1 — Let $\mathcal{N}$ be a base of neighbourhoods of 0 on a Hausdorff topological vector space E. Suppose $\pi : X \to B(E)$ and $T : X \to X$ are two mappings such that $W_{\pi, T} : FV_0(X, E) \to FV_b(X, E)$ is a weighted composition operator. Then for every $v \in V$ and $G \in \mathcal{N}$, there exists $u \in V$ and $H \in \mathcal{N}$ such that

$$v(x) p_G(\pi(x)t) < u \circ T(x) p_H(t) \text{ for all } x \in X \text{ and } t \in E.$$

Proof : We may assume that $\mathcal{N}$ consists of closed, balanced, and shrinkable sets.

Let $v \in V$ and $G \in \mathcal{N}$. By continuity of $W_{\pi, T}$, there exists $u \in V$ and $H \in \mathcal{N}$ such that

$$W_{\pi, T}\,(S(u, H) \cap FV_0(X, E)) \subseteq S(v, G). \qquad \ldots (1)$$

Claim : $v(x)\, p_G(\pi(x)t) \leq 3\ u \circ T(x) p_H(t)$ for all $x \in X$ and $t \in E$. Assume that for some $x_0 \in X$ and $t_0 \in E$,

$$v(x_0) p_G(\pi(x_0)t_0) > 3\ u \circ T(x_0) p_H(t_0).$$

Define $h : X \to E$ as follows

$$h(x) = \begin{cases} 3t_0 & \text{for } x = T(x_0) \\ 0 & \text{otherwise.} \end{cases}$$

Then $h \in FV_0(X, E)$. Let $\alpha = v(x_0)p_G(\pi(x_0)t_0)$ and put $f = \frac{h}{\alpha}$. Since p_H is homogenous, for any $x \in X$,

$$p_H(u(x)f(x)) \leq \frac{3}{\alpha} u \circ T(x_0)\, p_H(t_0) < 1,$$

or equivalently, $u(x)f(x) \in H$. So $f \in S(u, H)$ and hence, by (1), $W_{\pi,T} f \in S(v, G)$. This implies that, for any $x \in X$,

$$p_G(v(x)\pi(x) f \circ T(x)) \leq 1.$$

In particular, by taking $x = x_0$, we have

$$v(x_0)\, p_G(\pi(x_0)t_0) \leq \frac{\alpha}{3} = \frac{1}{3} v(x_0)\, p_G(\pi(x_0)t_0)$$

which is absurd. This prove our claim and the proposition.

Theorem 3.2 — *Let $\mathcal{N}$ be a base of neighbourhoods of 0 in a Hausdorff topological vector space E. Then for mappings $\pi : X \to B(E)$ and $T : X \to X$, the following are equivalent :*

(a) $W_{\pi,T}$ is a weighted composition operator on $FV_b(X, E)$;

(b) for every $v \in V$ and $G \in \mathcal{N}$, there exists $u \in V$ and $H \in \mathcal{N}$ such that $v(x)\, p_G(\pi(x)t) < u \circ T(x)\, p_H(t)$ for all $x \in X$ and $t \in E$.

PROOF : We may assume that $\mathcal{N}$ consists of closed, balanced and shrinkable sets.

(a) $\Rightarrow$ (b) : This implication follows from above Proposition.

(b) $\Rightarrow$ (a) : We first show that $W_{\pi,T}$ takes $FV_b(X, E)$ into itself. Let $f \in FV_b(X, E)$, and let $v \in V$ and $G \in \mathcal{N}$. Choose $u \in V$ and $H \in \mathcal{N}$ such that

(*) $v(x)\, p_G(\pi(x)t) < u \circ T(x)\, p_H(t)$ for all $x \in X$ and $t \in E$. Since $f \in FV_b(X, E)$, there exists a scalar $m > 0$ such that $p_H(u(x)f(x)) \leq m$ for all $x \in X$. Now for any $x \in X$,

$$p_G(v(x)\pi(x) f \circ T(x)) \leq u \circ T(x)\, p_H(f \circ T(x)) \leq m.$$

This implies that $v \,.\, W_{\pi,T} f(X) = \subseteq mG$, and so $W_{\pi,T} f \in FV_b(X, E)$.

We next establish the continuity of $W_{\pi,T}$. For this, let $\{f_\alpha : \alpha \in \Delta\}$ be a net in $FV_b(X, E)$ with $f_\alpha \to 0$, and let $v \in V$ and $G \in \mathcal{N}$. Choose $u \in V$ and $H \in \mathcal{N}$ as above satisfying (*). There exists an $\alpha_0 \in \Delta$ such that $f_\alpha \in S(u, H)$ for all $\alpha \geq \alpha_0$. Then, for any $x \in X$ and $\alpha \geq \alpha_0$,

$$v(x)\, p_G(\pi(x) f_\alpha \circ T(x)) < u \circ T(x)\, p_H(f_\alpha \circ T(x)) < 1,$$

or equivalently, $v(x)W_{\pi,T}f_a(x) \in G$. Thus $W_{\pi,T}$ is continuous at 0 and hence, by its linearity, on $FV_b(X, E)$.

With a slight modification in the proof of Theorem 3.2 above, we have the following result which extend Theorem 2.2 of Singh and Summers[1] to the present setting :

Proposition 3.3 — Let E be a Hausdorff topological vector space and let T be a selfmap on X. Then C_T is a composition operator on $FV_b(X, E)$ if and only if $V \leq V \circ T$, that is, for given $v \in V$, there exists $u \in V$ such that $v(x) \leq u(T(x))$ for all $x \in X$.

In the following propositions we shall deduce that, under some additional conditions, M_π is automatically a multiplication operator on $FV_b(X, E)$.

Proposition 3.4 — Let E be a Hausdorff topological vector space and let $\pi : X \to B(E)$ be a constant mapping. Then M_π is a multiplication operator on $FV_b(X, E)$.

PROOF : We need only to verify that condition (b) of Theorem 3.2 holds in this setting. Let $v \in V$ and $G \in \mathcal{N}$. Suppose $A \in B(E)$ such that $\pi(x) = A$ for all $x \in X$.. Since $A : E \to E$ is continuous, there exists a closed shrinkable set $H \in \mathcal{N}$ and $m > 0$ such that $p_G(A(t)) \leq mp_H(t)$ for all $t \in E$. Let $u = (m + 1)v$. Then, for any $x \in X$. and $t \in E$,

$$v(x)\, p_G(\pi(x)t) \leq mv(x)\, p_H(t) < u(x)\, p_H(t),$$

as required.

Proposition 3.5 — Let E be a locally bounded space and let $\pi : X \to B(E)$ be a bounded mapping. Then M_π is a multiplication operator on $FV_b(X, E)$.

PROOF : Let $v \in E$ and $G \in \mathcal{N}$. We may assume that G is bounded. Choose a closed and shrinkable $H \in \mathcal{N}$ with $H \subseteq G$. Since $\pi(X)$ is bounded in $B(E)$, there exists $m > 0$ such that $\pi(X) \subseteq mU(H, H)$, that is, $p_H(\pi(x)(t)) \leq m$ for all $x \in X$ and $t \in H$. Let $u = (m + 1)v$. Then for any $x \in X$ and $t \in E$,

$$v(x)\, p_G(\pi(x)(t)) \leq v(x)\, p_H(\pi(x)(t)) < u(x)\, p_H(t), \text{ as required.}$$

Remark 3.6 : (i) Theorem 3.2, Proposition 3.4 and Proposition 3.5 above respectively generalize Theorem 2.1, Remark 2.1 (i) and Remark 2.1 (ii) of Singh and Manhas[2] to a non-locally convex setting of weighted spaces of vector-valued functions.

(ii) The condition (b) of Theorem 3.2 is not sufficient for the pair (π, T) to induce a weighted composition operator on $FV_0(X, E)$ as can be seen from the following example :

Example 3.7 — Let $x = \mathbb{N}$, with the discrete topology, $E = \mathbb{C}$ with the usual topology and let $V = K^+(X)$. Then $FV_0(X, E) = c_0$, the Banach space of all null sequences of complex numbers. Define $\pi(x) = I$ and $T(x) = 3$ for all $x \in X$. Then the pair (π, T) satisfies the condition (b) of Theorem 3.2, but $W_{\pi,T}$ is not an operator

on c_0. In fact, it is not an even into map. For, if $f(x)=\frac{1}{x}$, then $f\in c_0$ but $W_{\pi,T}f\notin c_0$.

Therefore, we require some more conditions on the pair (π, T) so that $W_{\pi,T}$ is a weighted composition operator on $FV_0(X\,E)$. In the following theorem, we shall present a necessary and sufficient condition for $W_{\pi,T}$ to be a weighted composition operator on $FV_0(X, E)$, which also answers Remark 4.4.12 of Singh and Manhas[15] in the present setting.

Theorem 3.8 — *Let $\mathcal{N}$ be a base of neighbourhoods of 0 in a Hausdorff topological vector space E. Then the following conditions are sufficient for the mapping $\pi: X\to B(E)$ and $T: X\to X$ to induce a weighted composition operator on $FV_0(X, E)$:*

(a) for every $v\in V$ and $G\in\mathcal{N}$ *there exists* $u\in V$ *and* $H\in\mathcal{N}$ *such that* $v(x)\,p_G(\pi(x)t) < u\circ T(x)\,p_H(t)$ *for all* $x\in X$ *and* $t\in E$.

(b) for every $v\in V, G\in\mathcal{N}$, $\varepsilon>0$ *and compact subset K of X, the set* $T^{-1}(K)\bigcap X_t$ *is compact for all* $t\in E$, *where*

$$X_t = \{x\in X : v(x)\,p_G(\pi(x)t)\geq\varepsilon\}.$$

Furthermore, the above conditions (a) and (b) are necessary for $W_{\pi,T}$ to be an operator on $FV_0(X, E)$ if T is continuous.

PROOF : We may assume that $\mathcal{N}$ consists of closed balanced, and shrinkable sets.

Sufficient Part — Assume that conditions (a) and (b) hold. Then, by Theorem 3.2, $W_{\pi,T}$ is a weighted composition operator on $FV_b(X, E)$. We show that $FV_0(X\,E)$ is invariant under $W_{\pi,T}$. For this, let $f\in FV_0(X, E)$, $v\in V$, and $G\in\mathcal{N}$. By (a), we choose $u\in V$ and $H\in\mathcal{N}$ such that (*) $v(x)\,p_G(\pi(x)t) < u\circ T(x)\,p_H(t)$ for all $x\in X$ and $t\in E$. Now given any $\varepsilon>0$, there exists a compact subset K of X such that

$$u(x)\,p_H(f(x)) < \varepsilon \text{ for all } x\notin K.$$

For any $t\in E$, the set $K_t = T^{-1}(K)\bigcap X_t$ is compact in view of (b). If $x\notin K_t$, then either $x\notin T^{-1}(K)$ or $x\notin X_t$. Suppose first that $x\notin T^{-1}(K)$. Then $T(x)\notin K$, and so $u\circ T(x)\,p_H(f\circ T(x)) < \varepsilon$. By (*), this implies that $v(x)\,p_G(\pi(x)f\circ T(x)) < \varepsilon$. If $x\notin K_t$, then $v(x)\,p_H(\pi(x)t) < \varepsilon$. Thus $v\circ W_{\pi,T}\in F_0(X, E)$. Since $v\in V$ was arbitrary, we conclude that $W_{\pi,T}f\in FV_0(X, E)$.

Necessary Part — Assume that T is a continuous self-map on X, and let $W_{\pi,T}$ be a weighted composition operator on $FV_0(X, E)$. Then condition (a) follows in view of Theorem 3.2. To see that condition (b) holds, let $v\in V, G\in\mathcal{N}$, $\varepsilon>0$ and take K to be a compact subset of X. For any $t\in E$, define $f_t: X\to E$ as

$$f_t(x) = \begin{cases} t & \text{if } x \in K \\ 0 & \text{if } x \notin K. \end{cases}$$

Then clearly $f_t \in FV_0(X, E)$, and so the set

$$K' = \{x \in X : v(x)\, p_G(\pi(x) f_t \circ T(x)) \geq \varepsilon\}$$

is compact. We note that $K_t = T^{-1}(K) \cap X_t$ is a closed subset of K'. Hence K_t is compact.

Similarly, we can prove the following proposition which extends Proposition 1 of Prolla[10] as well as Theorem 2.5 of Singh and Manhas[3] to a non-locally convex setting.

Proposition 3.9 — Let E be a Hausdorff topological vector space. Then the following conditions are sufficient for a selfmap T on X to induce a composition operator C_T on $FV_0(X, E)$:

(*a*) $V \leq VoT$,

(*b*) For every $v \in V, \varepsilon > 0$ and compact subset K of X, the set $T^{-1}(K) \cap \{x \in X : v(x) \geq \varepsilon\}$ is compact.

Furthermore,the above conditions are also necessary for C_T to be a composition operator on $FV_0(X, E)$ if T is continuous.

If E is a locally bounded space and $\pi : X \to B(E)$ is a bounded mapping, then it follows from Proposition 3.5 that M_π is a multiplication operator on $F(X, E), k)$. Further, if T is a continuous selfmap on X, then it can seen that C_T is a composition operator on $(F(X, E), k)$ (for a proof see Singh and Kour[7], Proposition 3.13)). Since $W_{\pi, T} = M_\pi C_T$, we obtain

Proposition 3.11 — Assume that E is a locally bounded space. Let $\pi : X \to B(E)$ be a bounded mapping and let T be a continuous selfmap on X. Then $W_{\pi, T}$ is a weighted composition on $(F(X, E), k)$.

We now consider the cases when $V = S_0^+(X)$ and $V = K^+(X)$, and see that Proposition 3.11 above does not hold here.

Example 3.12 — Let $X = \mathbb{R}^+ \setminus \{0\}$, $E = \mathbb{C}$ and $V = \{\alpha v : \alpha > 0\}$, where $v(x) = \frac{1}{x}$ for all $x \in X$. Let $\pi : X \to B(E) = \mathbb{C}$ and $T : X \to X$ be respectively defined as $\pi(x) = x^2$ and $T(x) = \frac{1}{x^2}$ for all $x \in X$. Then the pair (π, T) does not satisfy the condition (b) of Theorem 3.2, and so $W_{\pi, T}$ is not a weighted composition operator on $(F_b(X, E), \beta_0)$.

Example 3.13 — Let X, V and E be the same as taken in Example 3.7. Define $\pi(x) = x$ and $T(x) = 3$ for all $x \in X$. Then it is easily seen that $W_{\pi, T}$ is not weighted composition operator on $F_0(X, E), \sigma) = c_0$.

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