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F-SEMINORMED SEQUENCE SPACES DEFINED BY A SEQUENCE OF MODULUS FUNCTIONS AND STRONG SUMMABILITY

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For a normal sequence space λ and a sequence of modulus functions $\mathcal{F} = (f_k)$ let $\lambda(\mathcal{F}) = \{(x_k) : (f_k(|x_k|)) \in \lambda\}$. We define an F-seminorm for $\lambda(\mathcal{F})$ and examine the completeness of $\lambda(\mathcal{F})$ according to this F-seminorm. As the concrete examples of the spaces of type $\lambda(\mathcal{F})$ we consider the spaces of strongly summable and strongly bounded with respect to $\mathcal{F}$ sequences.

Key Words : Sequence Space; *F*-seminorm; Modulus Function; Strong Summability

1. INTRODUCTION*

An F-space is usually understood as a complete metrizable topological vector space over the field $\mathbb{K}$ of real numbers $\mathbb{R}$ or complex numbers $\mathbb{C}$. It is known [Kothe[1], p.167; Kamthan and Gupta[2], p.13] that the topology of an F-space X can be given by an F-norm, i.e., by a functional $g : X \to \mathbb{R}$ with the conditions

(N_1) $g(0) = 0$,

(N_2) $g(x+y) \le g(x) + g(y) \qquad (x, y \in X)$,

(N_3) $|\alpha| \le 1 \quad (\alpha \in \mathbb{K}), \quad x \in X \Rightarrow g(\alpha x) \le g(x)$,

(N_4) $\lim_n \alpha_n = 0 \quad (\alpha_n \in \mathbb{K}), \quad x \in X \Rightarrow \lim_n g(\alpha_n x) = 0$,

(N_5) $g(x) = 0 \Rightarrow x = 0$.

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A functional g with the conditions (N_1) – (N_4) is called an F-seminorm [Kamthan and Gupta[2], p. Wilansky[4]]. Wilansky[3, 4] and some other authors used here a paranorm instead of an F-seminorm. A paranorm in X is defined as a functional $g : X \to \mathbb{R}$ which satisfies the conditions (N_1), (N_2) and

(N_6) $g(-x) = g(x)$ $(x \in X)$,

(N_7) $\lim_n \alpha_n = \alpha$ $(\alpha_n, \alpha \in \mathbb{K})$, $\lim_n g(x_n - x) = 0$ $(x_n, x \in X)$

$\Rightarrow$ $\lim_n g(\alpha_n x_n - \alpha x) = 0$.

A seminorm is a functional $g : X \to \mathbb{R}$ with the conditions (N_1), (N_2) and

(N_8) $g(\alpha x) = |\alpha| g(x)$ $(\alpha \in \mathbb{K}, \quad x \in X)$.

An F-seminorm (paranorm, seminorm) g is called total in X if (N_5) holds. An F-norm (norm) is a total F-seminorm (seminorm).

It should be noted that since (N_3) yields (N_6), and from (N_2) and (N_3) it follows

$$g(\alpha x) \leq 2([|\alpha|] + 1)\, g(x) \qquad (\alpha \in \mathbb{K}, \quad x \in X), \qquad \ldots (1)$$

where $[|\alpha|]$ denotes the integer part of $|\alpha|$, any F-seminorm is a paranorm with the condition (N_3).

By a sequence space we mean any linear subspace of the vector space s of all real or complex sequences $x = (x_k) = (x_k)_{k \in \mathbb{N}}$, where $\mathbb{N} = \{1, 2, \ldots\}$. The topological sequence space in which all coordinate functionals π_k, $\pi_k(x) = x_k$, are continuous is called a K-space. An FK-space is defined as a K-space which is also an F-space, an FK-space with normable topology is called a BK-space.

The theory of usual summability is essentially based on the notion of FK-space. So for every infinite scalar matrix $A = (a_{nk})$ the space of all A-summable sequences

$$c_A = \left\{ x = (x_k) : \exists \lim_n \sum_k a_{nk} x_k \right\}$$

is a locally convex FK-space, topology of which is determined by a certain countable system of seminorms [[5], Theorem 4.3.13; [2], Theorem 4.3.7]. If we consider strong summability, then in place of FK-spaces there appear F-seminormed (or paranormed) spaces. For example, if $p \geq 1$ and $A = (a_{nk})$ is a matrix with non-negative elements, then the equality

$$g_A^p(x) = \sup_n \left(\sum_k a_{nk} |x_k|^p \right)^{1/p}$$

defines clearly a seminorm in the space of strongly summable to zero sequences[6]

$$w_0^p(A) = \left\{ x = (x_k) : \lim_n \sum_k a_{nk} |x_k|^p = 0 \right\}.$$

More generally, if $p = (p_k)$ with $0 < p_k \le H < \infty$, then, analogously, the equality[7]

$$g_A^p(x) = \sup_n \left(\sum_k a_{nk} |x_k|^{p_k} \right)^{1/M},$$

where $M = \max\{1, H\}$ defines an F-seminorm in the space

$$w_0^p(A) = \left\{ x = (x_k) : \lim_n \sum_k a_{nk} |x_k|^{p_k} = 0 \right\}.$$

Seminorm g_A^p and F-seminorm g_A^p are not total in general, they are total only if A has no zero columns. We remark that $w_0^1(A)$ is precisely the space nor-$A^{-1}(c_0)$ of Johnson and Mohapatra[8] where c_0 denotes the space of all convergent to zero sequences.

The modulus function f is a function $f : [0, \infty) \to [0, \infty)$ such that

(M_1) $f(t) = 0 \Leftrightarrow t = 0$,

(M_2) $f(t + u) \le f(t) + f(u)$ $\quad (t, u \ge 0)$,

(M_3) f is increasing,

(M_4) f is continuous from the right at 0.

Ruckle[9], Maddox[10] and some others used a modulus function f to construct new sequence spaces for certain sequence spaces λ in the form

$$\cdot\lambda(f) = \{x = (x_k) : (f(|x_k|)) \in \lambda\}.$$

An extension of this definition was given[11] by means of a sequence of modulus functions. For a sequence space λ and a sequence of modulus functions $\mathcal{F} = (f_k)$ we define

$$\lambda(\mathcal{F}) = \{x = (x_k) : \mathcal{F}(|x|) \in \lambda\},$$

where $\mathcal{F}(|x|) = (f_k(|x_k|))$. It is not difficult to see that $\lambda(\mathcal{F})$ is a normal sequence space whenever λ is normal, i.e. $(x_k) \in \lambda$ and $|y_k| \le |x_k|$ $(k \in \mathbb{N})$ yield $(y_k) \in \lambda$.

Some sequence spaces of type $\lambda(\mathcal{F})$ were considered by Kolk[12-14] and Pehlivan and Fisher[15]. A natural paranorm topology for $\lambda(\mathcal{F})$ was given by Soomer[16, 17].

The purpose of this paper is to examine $\lambda(\mathcal{F})$ as an F-seminormed space assuming that λ is an F-seminormed normal sequence space. We give an F-seminorm for $\lambda(\mathcal{F})$ and investigate the completeness of $\lambda(\mathcal{F})$ according to this F-seminorm. As the concrete examples of the spaces of type $\lambda(\mathcal{F})$ we consider the space of strongly summable and strongly bounded with respect to $\mathcal{F}$ sequences. Our spaces contain as particular cases the most known spaces of strongly summable and strongly bounded sequences.

2. Definitions and Preliminary Results

Let g be an F-seminorm in X. From the axioms (N_1)-(N_3) it follows clearly that $g(x) \geq 0$, $g(-x) = g(x)$ and

$$|g(x) - g(y)| \leq g(x - y) \qquad \ldots (2)$$

for all $x, y \in X$.

The following lemma gives a similar to (2) inequality for modulus functions.

Lemma 1 — Any modulus function f satisfies the condition

$$|f(t) - f(u)| \leq f(|t - u|) \quad (t, u \geq 0). \qquad \ldots (3)$$

Proof : If $t \geq u$, then $t - u \geq 0$ and by (M_2) we have $f(t) = f((t-u)+u) \leq f(|t-u|) + f(u)$ which gives

$$f(t) - f(u) \leq f(|t - u|).$$

Further, by (M_3), $f(t) \geq f(u)$ and so $f(u) \leq f(t) \leq f(t) + f(|t-u|)$, i.e.,

$$-f(|t - u|) \leq f(t) - f(u).$$

Consequently, (3) holds for $t \geq u$.

If $t < u$, then $u > t$ and by above-proved we get

$$|f(u) - f(t)| \leq f(|u - t|)$$

which is equivalent to (3). Lemma is proved.

Now we formulate some auxiliary results concerning the normal sequence spaces. We recall that a sequence space λ is said to be normal (or solid) if $(y_k) \in \lambda$ whenever $(x_k) \in \lambda$ and $|y_k| \leq |x_k|$ $(k \in \mathbb{N})$.

Let λ be a normal sequence space and $x = (x_k) \in \lambda$. Let e^k $(k \in \mathbb{N})$ denotes the sequence $(\delta_{ki})_{i \in \mathbb{N}}$, where δ is the Kronecker delta. If there is an index k_0 such that $x_{k_0} \neq 0$, then $z = (z_k) = x_{k_0} e^{k_0}$ satisfies the condition $|z_k| \leq |x_k|$ $(k \in \mathbb{N})$ and so $z \in \lambda$. Thus $e^{k_0} = (1/x_{k_0})z$ also belongs to λ. Hence, we have proved

Lemma 2 — Let λ be a normal sequence space. If $x = (x_k) \in \lambda$ with $x_j \neq 0$, then $e^j \in \lambda$. If $e^j \notin \lambda$, then $x_j = 0$ for all $x \in \lambda$.

An F-seminorm g in a normal sequence space λ is said to be absolutely monotone [2, p. 64] if $g(y) \leq g(x)$ for all $x, y \in \lambda$ with $|y_k| \leq |x_k|$ $(k \in \mathbb{N})$. Let $g^{\perp} = \{x \in \lambda : g(x) = 0\}$.

If g is an absolutely monotone F-seminorm in a normal sequence space λ, then

$$(x_k) \in g^{\perp}, \quad |y_k| \leq |x_k| \quad (k \in \mathbb{N}) \Rightarrow (y_k) \in g^{\perp}. \qquad \ldots (4)$$

If, in addition, g is not total, then there exists $x = (x_k) \neq 0$ in $g^{\perp}$. Let k_0 be an index with $x_{k_0} \neq 0$. By Lemma 2 we have $e^{k_0} \in \lambda$. We may write $e^{k_0} = (1/x_{k_0})(x - y)$ where $y = x - x_{k_0} e^{k_0} \in g^{\perp}$ by (4). Thus also $e^{k_0} \in g^{\perp}$ because

$$g(e^{k_0}) \leq 2([|\, 1/x_{k_0} \,|] + 1)\,(g(x) + g(y))$$

by (1) and (N_2). Hence, the index set

$$I_0 = I_0(g) = \{i \in \mathbb{N} : e^i \in g^{\perp}\}$$

is non-empty. Moreover, by (1) we have

$$i \in I_0(g) \Leftrightarrow \alpha e^i \in g^{\perp} \qquad \ldots (5)$$

for all $\alpha \in \mathbb{K}$ with $\alpha \neq 0$.

Further, if $x = (x_k)$ and $y = (y_k)$ are such that $x_k = y_k$ for $k \notin I_0$ and $x_k \neq y_k$ for $k \in I_0$, then by (2) and (N_2) we get

$$|\, g(x) - g(y) \,| \leq \sum_{i \in I_0} g((x_i - y_i)\, e^i)$$

which yields $g(x) = g(y)$ by (5). This in turn implies the discontinuity of coordinate functionals π_k with $k \in I_0$.

Let

$$I_1 = I_1(\lambda) = \{i \in \mathbb{N} : e^i \notin \lambda\}.$$

By Lemma 2, $x_i = 0$ for all $x = (x_k) \in \lambda$ if $i \in I_1$. So the coordinate functionals π_i with $i \in I_1$ are always continuous.

Thus we have proved the following statements.

Proposition 1 — Let g be an absolutely monotone F-seminorm in a normal sequence space λ. Then :

(i) $g(x)$ $(x \in \lambda)$ does not depend on the values of coordinates x_k with $k \in I_0$;

(ii) The coordinate functionals π_k $(k \in I_0)$ are discontinuous; and

(iii) The coordinate functionals π_k $(k \in I_1)$ are continuous.

Subsequently, by an F-seminormed space (seminormed space, F-space) we mean a pair (λ, g) where λ is a sequence space and g is an F-seminorm (seminorm, F-norm) in λ. Proposition 1 suggests the following definition.

Definition 1 — An F-seminorm g in a normal sequence space λ is said to have the property (K) if the coordinate functionals π_k with $k \notin I_0$ are continuous.

We remark that if in Definition 1 g is total (i.e. $g^{\perp}$ is an F-norm), then I_0 is empty and (λ, g) is a K-space.

A distinguished subset of the set of all FK-spaces form the AK-spaces. An FK-space λ is called an AK-space [5, p. 59; 2, p. 59] if $\varphi \subset \lambda$ and $\lim_n x^{[n]} = x$ for all $x \in \lambda$, where $x^{[n]} = \sum_{k=1}^{n} x_k e^k$. Here the condition $\varphi \subset \lambda$ guarantees that λ is section-closed, i.e. λ contians all sections $x^{[n]}$ $(n \in \mathbb{N})$ of every element $x \in \lambda$. But $\varphi \subset \lambda$ is not necessary for the section-closedness of λ. For example, $\varphi \subset w_0^p(A)$ only

if $\lim_n a_{nk} = 0$ $(k \in \mathbb{N})$, but the space $w_0^p(A)$ is section-closed for every non-negative A. More generally, every normal sequence space λ is section-closed.

The last remarks justify the following extension [8, p.180] of the notion of AK-space.

Definition 2 — An F-seminormed sequence space (λ, g) is called an AK-space if λ is section-closed and $\lim_n x^{[n]} = x$ for all $x \in \lambda$.

The concept of strong summability was generalized[18, 19] by the use of a sequence of matrices. For a sequence $\mathcal{B} = (B_i)$ of infinite scalar matrices $B_i = (b_{nk}(i))$ with $b_{nk}(i)) \geq 0$ $(n, k, i \in \mathbb{N})$ we define

$$w_0^p(\mathcal{B}) = \left\{ x = (x_k) : \lim_n \sum_k b_{nk}(i) \mid x_k \mid^p = 0 \text{ uniformly in } i \right\},$$

$$w^p(\mathcal{B}) = \{x = (x_k) : (x - le) \in w_0^p(\mathcal{B}) \text{ for some } l \in \mathbb{K}\},$$

$$w_\infty^p(\mathcal{B}) = \left\{ x = (x_k) : \sup_{n,i} \sum_k b_{nk}(i) \mid x_k \mid^p < \infty \right\},$$

where $p \geq 1$ and $e = (1, 1, \ldots)$. All three sets are obviously sequence spaces, $w_0^p(\mathcal{B})$ and $w_\infty^p(\mathcal{B})$ of which are normal.

The functional $g_{\mathcal{B}}^p$ defined by

$$g_{\mathcal{B}}^p(x) = \sup_{n,i} \left(\sum_k b_{nk}(i) \mid x_k \mid^p \right)^{1/p},$$

is a seminorm in $w_0^p(\mathcal{B})$ and $w_\infty^p(\mathcal{B})$ since the axiom (N_1) is evident, (N_2) follows from Minkowski's inequality and

$$g_{\mathcal{B}}^p(\alpha x) = \mid \alpha \mid g_{\mathcal{B}}^p(x) \qquad (\alpha \in \mathbb{K})$$

is (N_8). It is easy to see that $g_{\mathcal{B}}^p$ is absolutely monotone. Further, for $x \in w^p(\mathcal{B})$ we have

$$\left(\sum_k b_{nk}(i) \mid x_k \mid^p \right)^{1/p} \leq \left(\sum_k b_{nk}(i) \mid x_k - l \mid^p \right)^{1/p} + \mid l \mid \left(\sum_k b_{nk}(i) \right)^{1/p}$$

if the series $\Sigma_k b_{nk}(i)$ $(n, i \in \mathbb{N})$ converge. So $w^p(\mathcal{B}) \subset w_\infty^p(\mathcal{B})$ and $g_{\mathcal{B}}^p$ is a seminorm in $w^p(\mathcal{B})$ when

$$(R_1) \quad \| \mathcal{B} \| = \sup_{n,i} \sum_k b_{nk}(i) < \infty.$$

A sequence $x = (x_k)$ is said to be positive [2, p.74] if $x_k \geq 0$ $(k \in \mathbb{N})$ and $x_k \neq 0$ for at least one $k \geq 1$. We say that a matrix $A = (a_{nk})$ is column-positive if

all its columns are positive. The sequence of matrices $\mathcal{B}$ is called column-positive if $b_{nk}(i) \geq 0$ $(n, k, i \in \mathbb{N})$ and for every $k \in \mathbb{N}$ there exists an index i_k such that the column $(b_{nk}(i_k))_{n \in \mathbb{N}}$ is positive. It is easy to see that the seminorm $g^p_{\mathcal{B}}$ is total if and only if $\mathcal{B}$ is column-positive.

Since

$$I_0(g^p_{\mathcal{B}}) = \{k \in \mathbb{N} : b_{nk}(i) = 0 \quad (n, i \in \mathbb{N})\},$$

for every $j \notin I_0(g^p_{\mathcal{B}})$ there exist indices i_j and n_j such that $b_{n_j, j}(i_j) > 0$. Thus we can write

$$b_{n_j, j}(i_j) \mid x_k \mid^p \geq \sum_k b_{n_j, k}(i_j) \mid x_k \mid^p \leq (g^p_{\mathcal{B}}(x))^p$$

which implies the continuity of coordinate functionals π_j with $j \notin I_0(g^p_{\mathcal{B}})$. So, by Definition 1, the seminorm $g^p_{\mathcal{B}}$ has the property (K). A routine verification shows that the seminormed spaces $(w^p_0(\mathcal{B})\ g^p_{\mathcal{B}})$ and $(w^p_\infty(\mathcal{B})\ g^p_{\mathcal{B}})$ are complete.

The sequence space $w^p_0(\mathcal{B})$ is normal and hence section-closed. If $x = (x_k) \in w^p_0(\mathcal{B})$, then

$$\lim_n \sum_k b_{nk}(i) \mid x_k \mid^p = 0 \text{ uniformly in } i$$

which clearly yields

$$\lim_m \sum_{k=m+1}^{\infty} b_{nk}(i) \mid x_k \mid^p = 0 \text{ uniformly in } n \text{ and } i.$$

So, using also the equality

$$g^p_{\mathcal{B}}(x^{[m]} - x) = \sup_{n, i} \sum_{k=m+1}^{\infty} b_{nk}(i) \mid x_k \mid^p,$$

we see that $\lim_m x^{[m]} = x$ in seminormed space $(w^p_0(\mathcal{B})\ g^p_{\mathcal{B}})$.

As a result we can formulate the following proposition.

Proposition 2 — Let $\mathcal{B}$ be a sequence of non-negative scalar matrices $B_i = (b_{nk}(i))$. Then $(w^p_0(\mathcal{B})\ g^p_{\mathcal{B}})$ is a complete seminormed AK-space, $(w^p_\infty(\mathcal{B})\ g^p_{\mathcal{B}})$ is a complete seminormed space and $(w^p(\mathcal{B})\ g^p_{\mathcal{B}})$ is a seminormed space when (R_1) holds. The seminorm $g^p_{\mathcal{B}}$ is absolutely monotone and has the property (K). If $\mathcal{B}$ is column-positive, then $(w^p_0(\mathcal{B}), g^p_{\mathcal{B}})$ is a BK-AK-space and $(w^p_\infty(\mathcal{B}), g^p_{\mathcal{B}})$ is a BK- space.

3. Topologization of $\lambda(\mathcal{F})$

For a normal sequence space λ, a sequence of modulus functions $\mathcal{F} = (f_k)$ and a sequence $x = (x_k)$, let $\mathcal{F}(|x|) = (f_k(|x_k|))$ and

$$\lambda(\mathcal{F}) = \{x = (x_k) : \mathcal{F}(|x|) \in \lambda\}.$$

If $\alpha, \beta \in \mathbb{K}$ and $x = (x_k)$, $y = (y_k)$ belong to $\lambda(\mathcal{F})$, then by (M_2) and (M_3),

$$f_k(|\alpha x_k + \beta y_k|) \leq f_k(|\alpha x_k|) + f_k(|\beta y_k|) \leq ([|\alpha|] + 1) f_k(|x_k|) + ([|\beta|] + 1) f_k(|y_k|)$$

for all $k \in \mathbb{N}$. This implies $\alpha x + \beta y \in \lambda(\mathcal{F})$ because λ is a normal sequence space. So $\lambda(\mathcal{F})$ is the sequence space and since $|y_k| \leq |x_k|$ $(k \in \mathbb{N})$ yields $f_k(|y_k|) \leq f_k(|x_k|)$ $(k \in \mathbb{N})$, then $\lambda(\mathcal{F})$ is also normal.

Soomer[16, 17] showed that the functional $g_{\mathcal{F}}$,

$$g_{\mathcal{F}}(x) = g(\mathcal{F}(|x|)),$$

defines a paranorm in $\lambda(\mathcal{F})$ if g is an absolutely monotone paranorm in λ and the sequence $\mathcal{F} = (f_k)$ satisfies the condition :

(M_5) There exists a function v such that $f_k(ut) \leq v(u) f_k(t)$ $(0 \leq u < 1,\ t \geq 0)$ and $\lim_{u \to 0+} v(u) = 0$.

Here we prove an analogous result for F-seminormed spaces.

Theorem 1 — *Let (λ, g) be an F-seminormed space. If g is absolutely monotone and the sequence of modulus functions $\mathcal{F} = (f_k)$ is such that*

$$(M_5) \quad \lim_{u \to 0+} \sup_{t > 0} \sup_k \frac{f_k(ut)}{f_k(t)} = 0,$$

then $g_{\mathcal{F}}$ is an absolutely monotone F-seminorm in $\lambda(\mathcal{F})$.

PROOF : First we prove that $g_{\mathcal{F}}$ is an F-seminorm, i.e. $g_{\mathcal{F}}$ satisfies (N_1)-(N_4). Since g is an F-seminorm, (N_1) holds by (M_1). If $|\alpha| \leq 1$, then

$$f_k(|\alpha x_k|) \leq f_k(|x_k|) \qquad (k \in \mathbb{N})$$

by (M_3). So, since g is absolutely monotone, we get

$$g_{\mathcal{F}}(\alpha x) = g((f_k(|\alpha x_k|))) \leq g((f_k(|x_k|))) = g_{\mathcal{F}}(x),$$

i.e., (N_3) is valid. The axiom (N_2) follows immediately from the subadditivity of g and f_k $(k \in \mathbb{N})$.

To prove (N_4) for $g_{\mathcal{F}}$, let $\lim_n \alpha_n = 0$ $(\alpha_n \in \mathbb{K})$ and $x = (x_k) \in \lambda(\mathcal{F})$. Since $f_k(t) > 0$ for $t > 0$ and $f_k(|\alpha_n x_k|) = 0$ for $k \in K_0 = \{k : x_k = 0\}$, we have

$$f_k(|\alpha_n x_k|) \leq v_n f_k(|x_k|) \quad (k \in \mathbb{N}), \qquad \ldots (6)$$

where by (M_5),

$$v_n = \sup_{k \in K_0} \frac{f_k(|\alpha_n x_k|)}{f_k(|x_k|)} \to 0 \quad (n \to \infty).$$

So, using that g satisfies (N_4), $\lim_n g(v_n \mathcal{F}(|x|)) = 0$. This implies $\lim_n g_{\mathcal{F}}(\alpha_n x) = 0$ because

$$g_{\mathcal{F}}(\alpha_n x) = g((f_k(|\alpha_n x_k|))) \leq g((v_n f_k(|x_k|))) = g(v_n \mathcal{F}(|x|))$$

by (6). Thus $g_{\mathcal{F}}$ is F-seminorm in $\lambda(\mathcal{F})$.

Now let $x = (x_k)$ and $y = (y_k)$ be in $\lambda(\mathcal{F})$ and $|y_k| \leq |x_k| \quad (k \in \mathbb{N})$. Then, by the increase of f_k, $f_k(|y_k|) \leq f_k(|x_k|)$ $(k \in \mathbb{N})$ and since g is absolutely monotone,

$$g_{\mathcal{F}}(y) = g(\mathcal{F}(|y|)) = g((f_k(|y_k|))) \leq g((f_k(|x_k|))) = g(\mathcal{F}(|x|)) = g_{\mathcal{F}}(x).$$

Hence $g_{\mathcal{F}}$ is absolutely monotone and the proof is completed.

By the investigation of the spaces of type $\lambda(\mathcal{F})$, the sequences $\mathcal{F} = (f_k)$ with the property[13-15]

$$(M_6) \quad \lim_{t \to 0+} \sup_k f_k(t) = 0$$

are important. If, in addition, $\mathcal{F}$ satisfies

$$(M_7) \quad f_k(ut) \leq f_k(u) f_k(t) \quad (u, t \geq 0;\ k \in \mathbb{N}),$$

then (M_5) clearly holds and we can formulate

Corollary 1 — If an F-seminorm g is absolutely monotone in λ and $\mathcal{F} = (f_k)$ satisfies (M_6) and (M_7), then $g_{\mathcal{F}}$ is an absolutely monotone F-seminorm in $\lambda(\mathcal{F})$.

Remark 1 : The conditions (M_5') and (M_5) are equivalent.

In fact, if (M_5') holds, then (M_5) follows from

$$v(u) \geq \frac{f_k(ut)}{f_k(t)} \geq 0 \quad (k \in \mathbb{N}\ ;\ t > 0).$$

Conversely, if (M_5) is true, then by defining

$$v(u) = \sup_{t>0} \sup_k \frac{f_k(ut)}{f_k(t)}$$

we get (M_5').

The next theorem says that (M_5) is superfluous in Theorem 1 if (λ, g) is an AK-space.

Theorem 2 — *Let (λ, g) be an F-seminormed AK-space. If g is absolutely*

monotone, then $g_{\mathcal{F}}$ is an absolutely monotone F-seminorm in $\lambda(\mathcal{B})$ for an arbitrary sequence of modulus functions $\mathcal{F} = (f_k)$. Moreover, $(\lambda(\mathcal{F}), g_{\mathcal{F}})$ is an AK-space.

PROOF : The functional $g_{\mathcal{F}}$ satisfies the axioms (N_1)-(N_3) by Theorem 1. To prove (N_4), let $\lim_n \alpha_n = 0$ $(\alpha_n \in \mathbb{N})$ and $x = (x_k) \in \lambda(\mathcal{F})$. Then $\mathcal{F}(|x|) \in \lambda$ and since (λ, g) has *AK*,

$$\mathcal{F}(|x|) = \lim_m \mathcal{F}(|x|)^{[m]} \qquad \ldots (7)$$

in λ by (2). So, using also the equality

$$\mathcal{F}(|x - x^{[m]}|) = \mathcal{F}(|x|) - \mathcal{F}(|x|)^{[m]}, \qquad \ldots (8)$$

for given $\varepsilon > 0$ we can find an index i such that

$$g_{\mathcal{F}}(|x - x^{[i]}|) < \varepsilon/2. \qquad \ldots (9)$$

Further, since g satisfies (N_4) and $\lim_n f_k(|\alpha_n x_k|) = 0$ $(k \in \mathbb{N})$ by the continuity of modulus functions f_k, we get

$$\lim_n g(f_k(|\alpha_n x_k|)\, e^k) = 0$$

for all $k \in \mathbb{N}$. Thus the inequality

$$g_{\mathcal{F}}(\alpha_n x^{[i]}) \le \sum_{k=1}^{i} g(f_k(|\alpha_n x_k|)\, e^k)$$

implies $\lim_n g_{\mathcal{F}}(\alpha_n x^{[i]}) = 0$ and so there exists an index n_0 such that

$$g_{\mathcal{F}}(\alpha_n x^{[i]}) < \varepsilon/2 \quad (n \ge n_0) \qquad \ldots (10)$$

and $|\alpha_n| \le 1$ $(n \ge n_0)$ because $\lim_n \alpha_n = 0$. Since $g_{\mathcal{F}}$ satisfies (N_3), then

$$g_{\mathcal{F}}(\alpha_n x - \alpha_n x^{[i]}) \le g_{\mathcal{F}}(x - x^{[i]}),$$

and from (9), (10) by (N_2) we deduce

$$g_{\mathcal{F}}(\alpha_n x) \le g_{\mathcal{F}}(\alpha_n x^{[i]}) + g_{\mathcal{F}}(x - x^{[i]}) < \varepsilon/2 + \varepsilon/2 = \varepsilon$$

for $n \ge n_0$. This implies $\lim_n g_{\mathcal{F}}(\alpha_n x) = 0$. So (N_4) is true for $g_{\mathcal{F}}$.

In Theorem 1 it was already proved that $g_{\mathcal{F}}$ is absolutely monotone.

Finally, using (8), by (7) we have $\lim_m g_{\mathcal{F}}(x - x^{[m]}) = 0$ for all $x \in \lambda(\mathcal{F})$. Thus $\lambda(\mathcal{F})$ is an *AK*-space and the proof is completed.

4. COMPLETENESS OF $(\lambda(\mathcal{F}),\ g_{\mathcal{F}})$

By the assumptions of Theorems 1 and 2, $(\lambda(\mathcal{F}),\ g_{\mathcal{F}})$ is an F-seminormed space. For the investigation of the completeness of $(\lambda(\mathcal{F}),\ g_{\mathcal{F}})$ we need some auxiliary results.

Proposition 3 — If an F-seminorm g with the property (K), a normal sequence space λ and the sequence of modulus functions $\mathcal{F} = (f_k)$ are such that $\lambda(\mathcal{F})$ is F-seminormed by $g_{\mathcal{F}}$, then $g_{\mathcal{F}}$ has the property (K).

PROOF : First we observe that $g_{\mathcal{F}}(e^i) = g(f_i(1)\, e^i)$ implies

$$I_0\,(g_{\mathcal{F}}) \;=\; I_0(g)$$

by (5) because $f_i(1) > 0$. Now, if $i \notin I_0\,(g_{\mathcal{F}})$, then $i \notin I_0(g)$ and since g has the property (K), the coordinate functional π_i is continuous on λ. So, for given $\varepsilon > 0$ by $f_i\,(\varepsilon) > 0$ we can find a number $\delta > 0$ such that

$$f_i\,(|\,x_i\,|) < f_i(\varepsilon)$$

when $g_{\mathcal{F}}(x) = g(\mathcal{F}(|\,x\,|)) < \delta$. This yields $|\,x_i\,| < \varepsilon$ by the increase of f_i. Consequenlty, the ith coordinate functional π_i must be continuous on $(\lambda(\mathcal{F}),\ g_{\mathcal{F}})$. The proof is completed.

An F-seminorm g in a section-closed sequence space λ is called sequential (or monotone) [2, p.70; 5, p.104] if for all $x \in \lambda$,

$$g(x^{[n]}) \le g(x^{[n+1]}) \quad (n \in \mathbb{N}) \qquad \ldots (11)$$

and

$$g(x) = \sup_n\ g(x^{[n]}). \qquad \ldots (12)$$

We note that (11) obviously holds whenever g is absolutely monotone. If, in addition, (λ, g) is an AK-space, then (12) is also true because the sequence $(g(x^{[n]}))$ is non-decreasing by (11) and $g(x) = \lim_n g(x^{[n]})$ by (2). Thus we can formulate

Proposition 4 — Let λ be a normal sequence space and let g be an absolutely monotone F-seminorm in λ. If (λ, g) is an AK-space, then the F-seminorm g is sequential.

Suppose that $g_{\mathcal{F}}$ is F-seminorm in $\lambda(\mathcal{F})$. If the F-seminorm g is sequential in λ, then by $\mathcal{F}(|\,x\,|)^{[n]} = \mathcal{F}(|\,x\,|^{[n]})$ $(n \in \mathbb{N})$ we have

$$g_{\mathcal{F}}\ (x) \;=\; g(\mathcal{F}(|\,x\,|)) \;=\; \sup_n\ g(\mathcal{F}(|\,x\,|)^{[n]}) \;=\; \sup_n\ g(\mathcal{F}(|\,x\,|^{[n]}))$$

$$=\; \sup_n\ g_{\mathcal{F}}\ (x^{[n]}).$$

So the following proposition holds.

Proposition 5 — Let (λ, g) be an F-seminormed space and $\mathcal{F} = (f_k)$ be a sequence of modulus functions such that $(\lambda(\mathcal{F}), g_{\mathcal{F}})$ is an F-seminormed space. If g is sequential in λ, then $g_{\mathcal{F}}$ is sequential in $\lambda(\mathcal{F})$.

Now we shall prove the main theorem of this section.

Theorem 3 — *Let (λ, g) be a complete F-seminormed space according to an absolutely monotone F-seminorm g with the property (K). If the sequence of modulus functions $\mathcal{F} = (f_k)$ is such that $g_{\mathcal{F}}$ is a sequential F-seminorm in $\lambda(\mathcal{F})$, then $(\lambda(\mathcal{F}), g_{\mathcal{F}})$ is complete.*

PROOF : Let (x^i), $x^i = (x_k^i)$, be a Cauchy sequence in $(\lambda(\mathcal{F}), g_{\mathcal{F}})$, i.e.

$$g_{\mathcal{F}}(x^i - x^j) \to 0 \quad (i, j \to \infty). \qquad \ldots (13)$$

Since by Proposition 3 the coordinate functionals π_k with $k \notin I_0$ are continuous on $\lambda(\mathcal{F})$, we have

$$|x_k^i - x_k^j| \to 0 \quad (i, j \to \infty; \ k \in \mathbb{N} \setminus I_0).$$

So $(x_k^i)_{i \in \mathbb{N}}$ are Cauchy sequences for $k \in \mathbb{N} \setminus I_0$, and by the completeness of $\mathbb{K}$ there exist limits

$$\lim_i x_k^i = x_k \quad (k \in \mathbb{N} \setminus I_0). \qquad \ldots (14)$$

For the indices $k \in I_0$ the sequences $(x_k^i)_{i \in \mathbb{N}}$ must not converge in general. By Proposition 1(i), the "limits" x_k of $(x_k^i)_{i \in \mathbb{N}}$ for $k \in I_0$ can be arbitrarily defined.

Let $x = (x_k)$. Since

$$|f_k(|x_k^i|) - f_k(|x_k^j|)| \le f_k(|x_k^i - x_k^j|)$$

by Lemma 1, we have

$$g((f_k(|x_k^i|) - f_k(|x_k^j|))) \le g_{\mathcal{F}}(x^i - x^j)$$

because g is absolutely monotone. Thus by (13), $(f_k(|x_k^i|))_{i \in \mathbb{N}}$ is a Cauchy sequence in the complete space (λ, g) for any $k \in \mathbb{N}$. So, since by (14) and the continuity of f_k, $\lim_i f_k(|x_k^i|) = f_k(|x_k|)$ for $k \in \mathbb{N} \setminus I_0$, by Proposition 1(i), using also the property (K) of g, we have

$$\lim_i (f_k(|x_k^i|))_{k \in \mathbb{N}} = (f_k(|x_k|)) \in \lambda,$$

i.e. $x \in \lambda(\mathcal{F})$.

To prove that $\lim_i x^i = x$ in $(\lambda(\mathcal{F}), g_{\mathcal{F}})$, let $\varepsilon > 0$. Then, since $g_{\mathcal{F}}$ is sequential, by (13) there exists an index r such that

$$g_{\mathcal{F}}((x^i - x^j)^{[n]}) < \varepsilon \quad (i, j \ge r; \ n \in \mathbb{N}). \qquad \ldots (15)$$

Using that

$$(x^i - x^j)^{[n]} = \sum_{k=1}^{n} (x_k^i - x_k^j)\, e^k$$

and $\lim_j (x_k^i - x_k^j) = x_k^i - x_k$ $(k \in \mathbb{N} \setminus I_0)$ by (14), we have

$$\lim_j g_{\mathcal{F}}((x^i - x^j)^{[n]}) = g_{\mathcal{F}}((x^i - x)^{[n]})$$

by Proposition 1(*i*). So (15) gives

$$g_{\mathcal{F}}((x^i - x)^{[n]}) \leq \varepsilon \quad (i \geq r;\ n \in \mathbb{N})$$

which implies

$$g_{\mathcal{F}}(x^i - x) \leq \varepsilon \quad (i \geq r)$$

because $g_{\mathcal{F}}$ is sequential. Hence $\lim_i x^i = x$ in $\lambda(\mathcal{F})$. Theorem is proved.

Theorem 3 together with Theorem 2 and Proposition 3-5 gives the following result.

Corollary 2 — Let $\mathcal{F} = (f_k)$ be a sequence of modulus functions. If (λ, g) is a complete *AK*-space according to an absolutely monotone *F*-seminorm g with the property (K), then $(\lambda(\mathcal{F}), g_{\mathcal{F}})$ is also a complete *F*-seminormed *AK*-space, where $g_{\mathcal{F}}$ is absolutely monotone and has the property (K).

5. Strong Summability and Boundedness with Respect to a Sequence of Modulus Functions

Let $\mathcal{F} = (f_k)$ be a sequence of modulus functions, $\mathcal{B} = B_i$ a sequence of non-negative matrices $B_i = (b_{nk}(i))$ and $p \geq 1$. As the generalizations of the sequence spaces $w_0^p(\mathcal{B})$, $w^p(\mathcal{B})$ and $w_\infty^p(\mathcal{B})$ we define

$$w_0^p(\mathcal{B}, \mathcal{F}) = \{x = (x_k) : \lim_n t_{n,i}(|x|) = 0 \text{ uniformly in } i\},$$

and

$$w^p(\mathcal{B}, \mathcal{F}) = \{x = (x_k) : x - le \in w_0^p(\mathcal{B}, \mathcal{F}) \text{ for some } l \in \mathbb{K}\}$$

$$w_\infty^p(\mathcal{B}, \mathcal{F}) = \{x = (x_k) : \sup_{n,i} t_{n,i}(|x|) < \infty\},$$

where $t_{n,i}(|x|) = \Sigma_k b_{nk}(i)(f_k(|x_k|))^p$. A sequence x with $x - le \in w_0^p(\mathcal{B}, \mathcal{F})$ is called strongly $\mathcal{B}$-summable to l with respect to the sequence of modulus functions $\mathcal{F}$ and a sequence $x \in w_\infty^p(\mathcal{B}, \mathcal{F})$ is called strongly $\mathcal{B}$-bounded with respect to $\mathcal{F}$.

For the constant sequences $\mathcal{B} = (B)$ and $\mathcal{F} = (f)$ we write B and f instead of $\mathcal{B}$ and $\mathcal{F}$, respectively. If, moreover, $\mathcal{B} = C_1$ the Cesàro matrix of order 1 (or $f(t) = t$), then we omit B (respectively, f) in the notations. So, for example, $w_0^p(A, \mathcal{F})$

$= w_0^p(\mathcal{B}, \mathcal{F})$ with $\mathcal{B} = (A)$, $w^p(f) = w^p(A, \mathcal{F})$ with $A = C_1$ and $\mathcal{F} = (f)$ etc.

Analogously to the functional $g_{\mathcal{B}}^p$ we introduce the functional $g_{\mathcal{B},\mathcal{F}}^p$, where

$$g_{\mathcal{B},\mathcal{F}}^p(x) = \sup_{n,i} (t_{n,i}(|x|))^{1/p}.$$

It is easy to see that $w_0^p(\mathcal{B}, \mathcal{F})$ and $w_\infty^p(\mathcal{B}, \mathcal{F})$ are the spaces of type $\lambda(\mathcal{F})$ with $\lambda = w_0^p(\mathcal{B})$ and $\lambda = w_\infty^p(\mathcal{B})$, respectively. In addition, $g_{\mathcal{B},\mathcal{F}}^p = (g_{\mathcal{B}}^p)_{\mathcal{F}}$. Thus by the investigation of these spaces we can use the results of sections 2-4.

For example, using Proposition 2, from Corollary 2 we deduce

Proposition 6 — For any sequence $\mathcal{F}$ of modulus functions $(w_0^p(\mathcal{B}, \mathcal{F}), g_{\mathcal{B},\mathcal{F}}^p)$ is a complete F-seminormed AK-space, where $g_{\mathcal{B},\mathcal{F}}^p$ is absolutely monotone and has the property (K). If $\mathcal{B}$ is column-positive, then $(w_0^p(\mathcal{B}, \mathcal{F}), g_{\mathcal{B},\mathcal{F}}^p)$ is an FK-AK-space.

Since the seminorm $g_{\mathcal{B}}^p$ is sequential, by Proposition 5 from Theorems 2 and 3 it follows the characterization of $w_\infty^p(\mathcal{B}, \mathcal{F})$.

Proposition 7 — If the sequence of modulus functions $\mathcal{F} = (f_k)$ satisfies (M_5), then $(w_\infty^p(\mathcal{B}, \mathcal{F}), g_{\mathcal{B},\mathcal{F}}^p)$ is a complete F-seminormed space, where $g_{\mathcal{B},\mathcal{F}}^p$ is absolutely monotone and has the property (K). If, moreover, $\mathcal{B}$ is column-positive, then $(w_\infty^p(\mathcal{B}, \mathcal{F}), g_{\mathcal{B},\mathcal{F}}^p)$ is an FK-space.

By the investigation of $w^p(\mathcal{B}, \mathcal{F})$ we suppose that (M_6) holds and

(R_2) $\quad \exists \lim_n \sum_k b_{nk}(i) = b$ uniformly in i.

Then for any $x = (x_k) \in w^p(\mathcal{B}, \mathcal{F})$ by Minkovski's inequality we have

$$(t_{n,i}(|x|))^{1/p} \le (t_{n,i}(|x - le|))^{1/p} + H(|l|)\left(\sum_k b_{nk}(i)\right)^{1/p} \quad (n, i \in \mathbb{N}), \qquad \ldots (16)$$

where $H(t) = \sup_k f_k(t)$. Thus if $b = 0$ in (R_2), then $w^p((\mathcal{B}, \mathcal{F}) = w_0^p(\mathcal{B}, \mathcal{F})$ and $(w^p(\mathcal{B}, \mathcal{F}), g_{\mathcal{B},\mathcal{F}}^p)$ is a complete F-seminormed space by Proposition 6. But if $b \neq 0$, then from (16) it follows that $w^p(\mathcal{B}, \mathcal{F}) \subset w_\infty^p(\mathcal{B}, \mathcal{F})$ if (R_1) holds. As in Theorem 1 we may prove that $g_{\mathcal{B},\mathcal{F}}^p$ satisfies $(N_1)-(N_3)$. To prove (N_4) let $\lim_m \alpha_m = 0$. Then $\lim_m g_{\mathcal{B},\mathcal{F}}^p(\alpha_m, (l - le)) = 0$ by Proposition 6, Moreover, $\lim_m H(|\alpha_m l|) = 0$ by (M_6). So by (16) (with $\alpha_m x$ and $\alpha_m l$ instead of x and l), $\lim_m g_{\mathcal{B},\mathcal{F}}^p(\alpha_m x) = 0$, i.e. $g_{\mathcal{B},\mathcal{F}}^p$ is F-seminorm in $w^p(\mathcal{B}, \mathcal{F})$.

To prove in this case the completeness of $(w^p(\mathcal{B}, \mathcal{F}), g^p_{\mathcal{B},\mathcal{F}})$ let

(M_8) $\quad \inf_k f_k(t) = h(t) > 0 \qquad (t > 0)$

and let (x^m), $x^m = (x^m_k)$, be a Cauchy sequence in $w^p(\mathcal{B}, \mathcal{F})$, i.e.

$$g^p_{\mathcal{B},\mathcal{F}}\ (x^m - x^r) \to 0 \quad (m, r \to \infty). \qquad \ldots (17)$$

Since $g^p_{\mathcal{B},\mathcal{F}}$ is sequential, as in the proof of Theorem 3, we can find the sequence $x = (x_k)$ such that

$$g^p_{\mathcal{B},\mathcal{F}}\ (x^m - x) \to 0 \quad (m \to \infty). \qquad \ldots (18)$$

It remains to prove that $x \in w^p(\mathcal{B}, \mathcal{F})$.

By $x^m \in w^p(\mathcal{B}, \mathcal{F})$ there exist numbers l_m $(m \in \mathbb{N})$ such that

$$\lim_m t_{n,i}(|\, x^m - l_m e\,|) = 0 \text{ uniformly in } i. \qquad \ldots (19)$$

Since by (M_8), $h(|\, l_m - l_r\,|) \le f_k(|\, l_m - l_r\,|)$ $(k \in \mathbb{N})$, by Minkowski's inequality we get

$$h(|\, l_m - l_r\,|)\left(\sum_k b_{nk}(i)\right)^{1/p} \le (t_{n,i}(|\, x^m - l_m e\,|))^{1/p} + (t_{n,i}(|\, x^r - l_r e\,|))^{1/p} + g^p_{\mathcal{B},\mathcal{F}}\ (x^m - x^r)$$

which by (R_2) and (19) gives

$$h(|\, l_m - l_r\,|) \le b^{-1/p}\ g^p_{\mathcal{B},\mathcal{F}}\ (x^m - x^r) \qquad (m, r \in \mathbb{N}),$$

as $n \to \infty$. By (17), $h(|\, l_m - l_r\,|) \to 0$ $(m, r \to \infty)$ which in turn implies $|\, l_m - l_r\,| \to 0$ $(m, r \to \infty)$ because h increases and $\lim_{t \to 0+} h(t) = 0$. Hence, there exists $\lim_m l_m = l$.

Again by Minkowski's inequality we have

$$(t_{n,i}(|\, x - le\,|))^{1/p} \le g^p_{\mathcal{B},\mathcal{F}}\ (x^m - x) + (t_{n,i}(|\, x^m - l_m e\,|))^{1/p} + g^p_{\mathcal{B},\mathcal{F}}\ ((l_m - l)e)$$

for every $m \in \mathbb{N}$. Since $e \in w^p_\infty(\mathcal{B}, \mathcal{F})$ by (R_1) and (M_6), and $g^p_{\mathcal{B},\mathcal{F}}$ satisfies (N_4), $\lim_m g^p_{\mathcal{B},\mathcal{F}}\ ((l_m - l)e) = 0$. So by (18) and $x^m - l_m e \in w^p_0(\mathcal{B}, \mathcal{F})$ $(m \in \mathbb{N})$ we get

$$\lim_n t_{n,i}(|\, x - le\,|) = 0 \text{ uniformly in } i,$$

i.e. $x = (x_k) \in w^p(\mathcal{B}, \mathcal{F})$. Thus the following proposition holds for $w^p(\mathcal{B}, \mathcal{F})$.

Proposition 8 — Suppose that the sequence $\mathcal{B}$ satisfies (R_2) and the sequence of modulus functions $\mathcal{F}$ satisfies (M_6). If $b = 0$ in (R_2), then $(w^p(\mathcal{B}, \mathcal{F}), g^p_{\mathcal{B},\mathcal{F}})$ is a complete F-seminormed space where $g^p_{\mathcal{B},\mathcal{F}}$ has the property (K), and an FK-space when $\mathcal{B}$ is column-positive. In the case of $b \neq 0$ the same is true whenever (R_1) and (M_8) hold.

In the following we consider some essential special cases of sequences $\mathcal{B}$ and $\mathcal{F}$ which lead to some known and new spaces of strongly summable and strongly bounded sequences.

For a modulus function f and a bounded sequence $\boldsymbol{p} = (p_k)$ with $0 < p_k < \sup_k p_k = H < \infty$ Bilgin[2] defined

$$w_0^{\boldsymbol{p}}(\mathcal{B}, f) = \left\{ x=(x_k) : \lim_n \sum_k b_{nk}(i)\,(f(|x_k|))^{p_k} = 0 \text{ uniformly in } i \right\},$$

$$w^{\boldsymbol{p}}(\mathcal{B}, f) = \{x = (x_k) : x - le \in w_0^{\boldsymbol{p}}(\mathcal{B}, f) \text{ for some } l \in \mathbb{K}\},$$

$$w_0^{\boldsymbol{p}}(\mathcal{B}, f) = \left\{ x=(x_k) : \sup_{n,i} \sum_k b_{nk}(i)\,(f(|x_k|))^{p_k} < \infty \right\}.$$

Since $0 < p_k/M \leq 1$ with $M = \max\{1, H\}$, the function g_k, $g_k(t) = t^{p_k/M}$, is a modulus function for every $k \in \mathbb{N}$. Using that the superposition of two modulus functions is also a modulus function, by $(f(|x_k|))^{p_k} = ((f(|x_k|))^{p_k/M})^M$ we have $w_0^{\boldsymbol{p}}(\mathcal{B}, f) = w_0^M(\mathcal{B}, \mathcal{F}_1)$, $w^{\boldsymbol{p}}(\mathcal{B}, f) = w^M(\mathcal{B}, \mathcal{F}_1)$ and $w_\infty^{\boldsymbol{p}}(\mathcal{B}, f) = w_\infty^M(\mathcal{B}, \mathcal{F}_1)$ where $\mathcal{F}_1 = (g_k \circ f)$. Thus from Proposition 16 we get [cf. [2], Theorem 2]

Corollary 3 — For any modulus function f the space $w_0^{\boldsymbol{p}}(\mathcal{B}, f)$ is a complete AK-space according to the absolutely monotone F-seminorm $g^{\boldsymbol{p}}_{\mathcal{B},f}$ with the property (K), where

$$g^{\boldsymbol{p}}_{\mathcal{B},f}(x) = \sup_{n,i} \left(\sum_k b_{nk}(i)\,(f(|x_k|))^{p_k} \right)^{1/M}.$$

If $\mathcal{B}$ is column-positive, then $(w_0^{\boldsymbol{p}}(\mathcal{B}, f), g^{\boldsymbol{p}}_{\mathcal{B},f})$ is an FK-AK-space.

For the constant sequence $\mathcal{F} = (f)$ the condition (M_5) reduces to

$$(M_9) \qquad \lim_{u \to 0+} \sup_{t>0} \frac{f(ut)}{f(t)} = 0.$$

Since $\lim_{t \to 0+} t^{p_k} = 0$ is equivalent to $\inf_k p_k > 0$, the sequence $\mathcal{F}_1 = (g_k \circ f)$ satisfies (M_5) whenever $\inf_k p_k > 0$ and (M_9) holds. Thus by Proposition 7 we can formulate

Corollary 4 — If $\inf_k p_k > 0$ and the modulus function f satisfies (M_9), then $(w_0^{\boldsymbol{p}}(\mathcal{B}, f), g_{\mathcal{B},f}^{\boldsymbol{p}})$ is a complete F-seminormed space, where $g_{\mathcal{B},f}^{\boldsymbol{p}}$ is absolutely monotone and has the property (K). If, moreover, $\mathcal{B}$ is column-positive, then $w_0^{\boldsymbol{p}}(\mathcal{B}, f), g_{\mathcal{B},f}^{\boldsymbol{p}})$ is an FK-space.

Proposition 8 gives the following characterization of $w^{\boldsymbol{p}}(\mathcal{B}, f)$ if we note that for $\mathcal{F}_1$ the condition (M_6) reduces to $\inf_k p_k > 0$ and (M_8) is immediately fulfilled by the boundedness of $p = (p_k)$.

Corollary 5 — Suppose that $\inf_k p_k > 0$ and $\mathcal{B}$ satisfies (R_2). If $b = 0$ in (R_2), then $(w^{\boldsymbol{p}}(\mathcal{B}, f), g_{\mathcal{B},f}^{\boldsymbol{p}})$ is a complete F-seminormed space where $g_{\mathcal{B},f}^{\boldsymbol{p}}$ has the property (K), it is an FK-space when $\mathcal{B}$ is column-positive. For $b \neq 0$ the same holds whenever (R_1) is satisfied.

For $f(t) = t$ our results contain Theorem 2.1 of Mursaleen[19] about the spaces $w_0^{\boldsymbol{p}}(\mathcal{B})$, $w^{\boldsymbol{p}}(\mathcal{B})$ and $w_\infty^{\boldsymbol{p}}(\mathcal{B})$.

We get further examples by specifying $\mathcal{B}$. For a constant sequence $\mathcal{B} = (A)$, where $A = (a_{nk})$ is a non-negative matrix, we have the spaces $w_0^{\boldsymbol{p}}(A, f)$, $w^{\boldsymbol{p}}(A, f)$ and $w_\infty^{\boldsymbol{p}}(A, f)$ introduced by Bilgin[21] and Öztürk and Bilgin[22] $(A = C_1)$. The spaces $w_0^1(A, f)$ and $w^1(A, f)$ have been considered by Connor[23] and Maddox[10] $(A = C_1)$. It should be noted that by Theorem 8 of Maddox[10] the space $w_\infty^1(f)$ is not a paranormed space in general and thus Theorem 1 of Bilgin[21] is untrue in part. The space $w_\infty^{\boldsymbol{p}}(A, f)$ cannot be paranormed by $g_{A,f}^{\boldsymbol{p}}$ for all f and A. Corollary 4 completes Maddox's result showing that $w_\infty^1(f)$ (and also $w_\infty^{\boldsymbol{p}}(A, f)$ with $\inf_k p_k > 0$) are paranormed whenever f satisfies (M_9).

The spaces $w_0^{\boldsymbol{p}}(A, f)$, $w^{\boldsymbol{p}}(A, f)$ and $w_\infty^{\boldsymbol{p}}(A, f)$ in the case $f(t) = t$ have been considered by Maddox[7, 24]. If here $A = E$ the unit matrix, then Maddox's spaces $w_0^{\boldsymbol{p}}(A)$, $w^{\boldsymbol{p}}(A)$ and $w_\infty^{\boldsymbol{p}}(A)$ reduce to

$$c_p(\boldsymbol{p}) = \left\{ x = (x_k) : \lim_k |x_k|^{p_k} = 0 \right\},$$

$$c(\boldsymbol{p}) = \{x = (x_k) : x - le \in c_0(\boldsymbol{p}) \text{ for some } l \in \mathbb{K}\},$$

$$l_\infty(\boldsymbol{p}) = \left\{ x = (x_k) : \sup_k |x_k|^{p_k} < \infty \right\},$$

respectively. So Corollaries 3-5 imply the following version of the known results[25] concerning $c_0(\boldsymbol{p})$, $c(\boldsymbol{p})$ and $l_\infty(\boldsymbol{p})$.

Corollary 6 — $c_p(\boldsymbol{p})$ is an FK-AK-space according to the absolutely monotone F-norm $g^{\boldsymbol{p}}$, where $g^{\boldsymbol{p}}(x) = \sup_x (|x_k|^{p_k})^{1/M}$. If $\inf_k p_k > 0$, then $(c(\boldsymbol{p}), g^{\boldsymbol{p}})$ and $(l_\infty(\boldsymbol{p}), g^{\boldsymbol{p}})$ are FK-spaces.

Lacunary strong convergence is determined by a special matrix. A strictly increasing seqeunce of non-negative integers $\Theta = (k_r)$ is called a lacunary sequence if $k_1 = 0$ and $h_r = k_{r+1} - k_r \to \infty$, as $r \to \infty$. The intervals determined by Θ are denoted by $I_r = (k_r, k_{r+1}]$. If $A_\Theta = (a_{rk}^\Theta)$ where $a_{rk}^\Theta = 1/h_r$ for $k \in I_r$ and $a_{rk}^\Theta = 0$ otherwise, then $w^1(A_\Theta)$ is precisely the space of lacunary strongly convergent sequences[26]

$$N_\Theta = \left\{ x = (x_k) : \lim_r \frac{1}{h_r} \sum_{k \in I_r} |x_k - l| = 0, \text{ for some } l \in \mathbb{K} \right\}.$$

Generalizing the ideas of Pehlivan and Fisher[15], we may consider the spaces $w_0^p(A_\Theta, \mathcal{F})$, $w^p(A_\Theta, \mathcal{F})$ and $w_\infty^p(A_\Theta, \mathcal{F})$ with $p \geq 1$. Propositions 6-8 give the following result concerning these spaces if we note that A_Θ is column-positive and $\sum_k a_{rk}^\Theta = 1 \quad (r \in \mathbb{N})$.

Corollary 7 — For $p \geq 1$ and any sequence $\mathcal{F}$ of modulus functions $w_0^p(A_\Theta, \mathcal{F})$ is an *FK-AK*-space according to the absolutely monotone *F*-norm $g_{\Theta,\mathcal{F}}^p$, where

$$g_{\Theta,\mathcal{F}}^p(x) = \sup_r \left(1/h_r \sum_{k \in I_r} (f_k(|x_k|))^p \right)^{1/p}.$$

If $\mathcal{F}$ satisfies (M_5), then $(w_\infty^p(A_\Theta, \mathcal{F}), g_{\Theta,\mathcal{F}}^p)$ is an *FK*-space. $(w^p(A_\Theta, \mathcal{F}), g_{\Theta,\mathcal{F}}^p)$ is an *FK*-space when (M_6) and (M_8) holds.

The known examples of non-constant sequences $\mathcal{B}$ are connected with strong σ-convergence and strong almost convergence. Let σ be a mapping of $\mathbb{N}$ into itself. The mapping σ is assumed to be one-to-one and such that $\sigma^k(n) \neq n$ for all $n, k \in \mathbb{N}$, where $\sigma^k(n)$ denotes the iterate of order k of the mapping σ at n. For every $i \in \mathbb{N}$ let $a_{nk}^\sigma(i) = a_{nj}$ if $k = \sigma^j(i)$ and $a_{nk}^\sigma(i) = 0$ otherwise. The sequence $\mathcal{B}_A^\sigma = (A_i^\sigma)$, $A_i^\sigma = (a_{nk}^\sigma(i))$, defines the spaces $w_0^p(\mathcal{B}_A^\sigma, \mathcal{F})$, $w^p(\mathcal{B}_A^\sigma, \mathcal{F})$ and $w_\infty^p(\mathcal{B}_A^\sigma, \mathcal{F})$ for $p \geq 1$. By the use of these spaces we can define as above the spaces $w_0^p(\mathcal{B}_A^\sigma, \mathcal{F})$, $w^p(\mathcal{B}_A^\sigma, \mathcal{F})$ and $w_\infty^p(\mathcal{B}_A^\sigma, \mathcal{F})$ which generalize the spaces of Nuray and Savas[27], Savas[28, 29]. Savas and Mursaleen[30] considered the space $w^1(\mathcal{B}_{A_\Theta}^\sigma)$.

The case, where $\sigma = \tau$ with $\tau(n) = n + 1$, is related to strong almost convergence and summability. Spaces of type $w_0^p(\mathcal{B}_A^\tau, f)$, $w^p(\mathcal{B}_A^\tau, f)$ and $w_\infty^p(\mathcal{B}_A^\tau, f)$ have been considered by Pehlivan[32, 33] ($A = C_1$ and $p_k = 1$ $(k \in \mathbb{N})$) and Nanda[23] ($f(t) = t$). For $f(t) = t$, $A = C_1$ and $p_k = 1$ $(k \in \mathbb{N})$, $w^p(\mathcal{B}_A^\tau, f)$ is the space of strongly almost convergent sequences[26, 35, 36, 37]. It should be noted that the notion of strong almost convergence was given already by Nanda[37] in a more general form. Das and Patel[38] studied the space $w^1(\mathcal{B}_{A_\Theta}^\tau)$.

Combining the given non-negative matrix scalar $A = (a_{nk})$ and the sequence of matrices $\mathcal{B}$, Mursaleen[19] defined a new sequence of matrices $\mathcal{B}A = (A_i)$, $A_i = (a_{nk}(i))$, where

$$a_{nk}(i) = \sum_j b_{nj}(i)\, a_{jk},$$

and considered the spaces $w_0^p(\mathcal{B}\ A)$, $w^p(\mathcal{B}\ A)$ and $w_\infty^p(\mathcal{B}\ A)$. If here $\mathcal{B} = \mathcal{B}_A^r$ with $A = C_1$, then these are the spaces of Nanda[37]. Savas[39] generalized the spaces of Nanda by means of a modulus function f.

Topologies for $w_0^p(\mathcal{B}_A^\sigma, \mathcal{F})$, $w^p(\mathcal{B}_A^\sigma, \mathcal{F})$ and $w_\infty^p(\mathcal{B}_A^\sigma, \mathcal{F})$ and for $w_0^p(\mathcal{B}_A^\sigma, f)$, $w^p(\mathcal{B}_A^\sigma, f)$ and $w_\infty^p(\mathcal{B}_A^\sigma, f)$ give Propositions 6-8 and Corollaries 3-5, respectively. For example, if we note that for $\mathcal{B}_A^\sigma$ the conditions (R_1) and (R_2) reduce to

$$(R_3) \qquad \exists \lim_n \sum_k a_{nk} = a,$$

then from Corollaries 3-5 we get the following extension of Theorem 3 of Nuray and Savas[27].

Corollary 8 — For any modulus function f the set $w_0^p(\mathcal{B}_A^\sigma, f)$ is a complete F-seminormed AK-space according to the F-seminormed $g_{\sigma,f}^p$, where

$$g_{\sigma,f}^p(x) = \sup_{n,i} \left(\sum_k a_{nk} (f(|\, x_{\sigma^i}(i)\,|))^{p_k} \right)^{1/M}.$$

If $\inf_k p_k > 0$ and f satisfies (M_9), then $(w_0^p(\mathcal{B}_A^\sigma, f),\ g_{\sigma,f}^p)$ is a complete F-seminormed space. $(w^p(\mathcal{B}_A^\sigma, f),\ g_{\sigma,f}^p)$ is a complete F-seminormed space when $\inf_k p_k > 0$ and A satisfies (R_3). The F-seminorm $g_{\sigma,f}^p$ is absolutely monotone and has the property (K).

Finally we remark that all our results remain valid if we consider for a Banach space $(X, \|\cdot\|)$ the X-valued sequences instead of number sequences. Let $s(X)$ be the set of all X-valued sequences $x = (x_k)$. Defining $\mathcal{F}(|\, x\,|) = (f_k(\|x_k\|))$ for $x \in s(X)$, we have

$$\lambda(\mathcal{F}, X) = \{x = (x_k) \in s(X) : \mathcal{F}(|\, x\,|) \in \lambda\}$$

instead of $\lambda(\mathcal{F})$. Analogously,

$$w_0^p(\mathcal{B}, \mathcal{F}, X) = \{x = (x_k) \in s(X) : \lim_n t_{n,i}(|\, x\,|) = 0 \text{ uniformly in } i\},$$

$$w^p(\mathcal{B}, \mathcal{F}, X) = \{x = (x_k) \in s(X) : x - e(l) \in w_0^p(\mathcal{B}, \mathcal{F}, X) \text{ for some } l \in X\}$$

and

$$w_\infty^p(\mathcal{B}, \mathcal{F}, X) = \left\{ x = (x_k) \in s(X) : \sup_{n,i} t_{n,i}(|\, x\,|) < \infty \right\},$$

where $t_{n,i} = \sum_k b_{nk}(i)\,(f_k(\|x_k\|))^p$ and $e(l) = (l, l, \ldots)$, occur instead of $w_0^p(\mathcal{B}, \mathcal{F})$, $w^p(\mathcal{B}, \mathcal{F})$ and $w_\infty^p(\mathcal{B}, \mathcal{F})$, respectively.

Some sequence spaces of this type have been considered by Kolk[12, 13], Pehlivan[33] and Pehlivan and Fisher[15].

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