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EIGENVALUE APPROACH TO GENERALIZED THERMOELASTICITY

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The fundamental equations of the problems of generalized thermoelasticity with one relaxation time parameter including heat sources have been written in the form of a vector-matrix differential equation in the (*i*) Laplace transform domain for a one dimensional problem, and in the (*ii*) Laplace-Fourier transform domain for a two dimensional problem and they are solved by the eigenvalue approach. The results are compared with the existing literature.

Key Words : Eigenvalue; Generalized Thermoelasticity; Laplace & Laplace-Fourier Transform; One-Dimensional Problem; Vector-Matrix Differential Equation; Numerical Solution

NOMENCLATURE

λ, μ	=	Lame constants
α_t	=	Coefficient of linear thermal expansion
β	=	$[(\lambda + 2\mu)/\mu]^{1/2}$
γ	=	$(3\lambda + 2\mu)\,\alpha_t$
ρ	=	Density
v	=	Speed of propagation of longitudinal waves
	=	$[(\lambda + 2\mu)/\rho]^{1/2}$
σ'	=	σ'_{xx} component of the stress tensor in x'-direction
u'	=	Component of displacement in x'-direction
C_E	=	Specific heat at constant stain
K	=	Thermal conductivity
K_1	=	Thermal diffusivity
δ	=	$\rho C_E/K$

t'	=	Time
T	=	Absolute temperature
T_0	=	Reference temperature chosen that $\lvert (T-T_0)/T_0 \rvert < 1$
b	=	$\gamma T_0/\mu$
g	=	$\gamma/\rho C_E$
τ_0	=	Relaxation time
q	=	Heat flux
Q	=	Intensity of the applied heat source per unit volume
τ_{ij}	=	Stress tensor
A_{ij}	=	Elastic moduli of the material
β_j	=	Stress-temperature coefficients
K_y	=	Coefficient of thermal conductivity in Y-direction
K_z	=	Coefficient of thermal conductivity in Z-direction
C	=	Specific heat per unit mass
α, α_0	=	Material constants
τ	=	Thermal relaxation parameter.

INTRODUCTION

Apart from the constitutive relations, the governing equations for displacement and the temperature fields, as in the linear dynamical theory of classical thermoelasticity consist of the coupled partial differential equation of motion and the Fourier heat conduction equation. The former equation, in terms of displacement field, is governed by a wave type hyperbolic equation, whereas, the latter equation for the temperature field is a parabolic diffusion type equation. This amounts to saying that classical thermoelasticity predicts a finite speed for predominantly elastic disturbances but as infinite speed for predominantly thermal disturbances, which are coupled together. This means that a part of every solution of the equations extends to infinity, vide, Lord and Shulman[1]. Experimental investigations by Ackerman *et al.*[2-4], Von Gutfeld and Nethercot[5], Taylor *et al.*[6], Jackson and Walker[7] etc. conducted on different solids have shown that heat pulses do not propagate with infinite speeds. In order to overcome this paradox, efforts were made to modify classical thermoelasticity, on different grounds, for obtaining a wave type heat conduction equation ; vide Kaliski[8], Norwood and Warren[9], Green and Lindsay[10] ; Suhubi[11], Lebon[12] etc. A resume on this generalization for the last twenty years can be found in Chandrasekharaiah[13].

At present there are two different generalizations of the classical theory of thermoelasticity. The first proposed by Green and Lindsay[10] (G-L theory) involves two relaxations times for a thermoelastic process, the second by Lord and Shulman[1] (L-S theory) takes into account only one parameter on relaxation time. Owing to the mathematical difficulties encountered in coupled thermoelasticity, mainly due to inertia and coupling terms in the governing equations, this kind of problems are mostly confined in one dimensional problems, vide, Furukawa *et al.*[14], Anwar *et al.*[15] and Chandrasekharaiah *et al.*[16] etc. However, in the present paper following one parameter L-S theory, the authors have considered two problems, one in each part, of generalized thermoelasticity by taking into account both the dynamic effect and the influence of coupling terms. In part I, a one dimensional problem with heat sources distributed over a plane area in an infinite isotropic elastic solids, and in part II, a two dimensional problem with instantaneous heat sources in an infinite, transversely isotropic, elastic medium have been considered. The new eigenvalue approach of Das *et al.*[17, 18] has been applied for the solution of the problem.

PART I : ONE DIMENSIONAL PROBLEM

1. BASIC EQUATIONS AND FORMULATIONS

We consider a homogeneous isotropic thermoelastic solid occupying the whole space; $-\alpha \le x' \le \alpha$, whose state depends only on the space variables x' and the time variable t' in dimensional form.

The nondimensional equation of motion and the generalized equation of heat conduction with one relaxation time parameter are given by[19].

$$\beta^2 \frac{\partial^2 u}{\partial x^2} - b \frac{\partial \theta}{\partial x} = \beta^2 \frac{\partial^2 u}{\partial t^2} \qquad \text{... (1.1)}$$

$$\frac{\partial^2 \theta}{\partial x^2} = \frac{\partial \theta}{\partial t} + \tau_0 \frac{\partial^2 \theta}{\partial t^2} + g\left(\frac{\partial^2 u}{\partial x\, \partial t} + \tau_0 \frac{\partial^3 u}{\partial x\, \partial t^2} \right) - \left(1 + \tau_0 \frac{\partial}{\partial t} \right) Q. \qquad \text{... (1.2)}$$

The constitutive relation takes the form

$$\sigma = \beta^2 \frac{\partial u}{\partial x} - b\theta. \qquad \text{... (1.3)}$$

In the above equations, we have used the following nondimensional variables

$$\theta = T/T_0$$

$$x = V\,\delta x',\quad u = V\delta u',\quad t = V^2\,\delta t',\quad \tau_0 = V^2\,\delta \tau_0',\quad \sigma = \sigma'/\mu,$$

$$Q = \rho Q'/K T_0\, \delta^2 V^2. \qquad \text{... (1.4)}$$

We assume that the heat source acts on the plane $x = 0$, and is of the form

$$Q = Q_0\, \delta(x)\, H(t), \qquad \text{... (1.5)}$$

where Q_0 is a constant, $\delta(x)$ and $H(t)$ are Dirac delta function of x and the Heaviside unit step function of t respectively.

For the solution of these equations, we now apply the Laplace transform of the parameter P defined by

$$\bar{u}(x,p) = \int_0^{\alpha} u(x,t)\exp(-pt)\,dt$$

and
$$\bar{\theta}(x,p) = \int_0^{\alpha} \theta(x,t)\exp(-pt)\,dt. \qquad \text{... (1.6)}$$

Taking the Laplace transform on both sides of eq. (1.1) (1.2), and (1.3) we get,

$$\left[\frac{d^2}{dx^2} - p^2\right]\bar{u} = a\frac{d\bar{\theta}}{dx}, \qquad \text{... (1.7)}$$

$$\left[\frac{d^2}{dx^2} - p - \tau_0 p^2\right]\bar{\theta} = gp\,(1+\tau_0 p)\frac{d\bar{u}}{dx} - Q_0\frac{(1+\tau_0 p)}{p}\delta(x) \qquad \text{... (1.8)}$$

and

$$\bar{\sigma} = \beta^2\frac{d\bar{u}}{dx} - b\bar{\theta} \qquad \text{... (1.9)}$$

where
$$a = b/\beta^2,\ \int_0^{\alpha}\delta(t)\exp(-pt)\,dt = 1 \text{ and } \int_0^{\alpha} H(t)\exp(-pt)\,dt = \frac{1}{p}. \qquad \text{... (1.10)}$$

In the above expressions, we have used the conditions that $u, \frac{\partial u}{\partial t}$, and θ are initially zero, while $\bar{\theta}, \frac{\partial\bar{\theta}}{\partial x}$, and $\frac{\partial\bar{u}}{\partial x}$ are zero at infinity.

As in Das *et al.*[17], eqs. (1.7) and (1.8) can be written in the form of a vector matrix differential equation as follows :

$$\frac{d}{dx}\begin{bmatrix}\bar{\theta}(x,p)\\ \bar{u}(x,p)\\ \bar{\theta}'(x,p)\\ \bar{u}'(x,p)\end{bmatrix} = \begin{bmatrix}0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ C_{31} & 0 & 0 & C_{34}\\ 0 & C_{42} & C_{43} & 0\end{bmatrix}\begin{bmatrix}\bar{\theta}(x,p)\\ \bar{u}(x,p)\\ \bar{\theta}'(x,p)\\ \bar{u}'(x,p)\end{bmatrix} + \begin{bmatrix}0\\ 0\\ \dfrac{-Q_0(1+\tau_0 p)\delta(x)}{p}\\ 0\end{bmatrix}, \qquad \text{... (1.11)}$$

where,
$$C_{31} = p(1+\tau_0 p),\ C_{34} = gp(1+\tau_0 p),\ C_{42} = p^2,\ C_{43} = a. \qquad \text{... (1.12)}$$

The primes indicate differentiation with respect to x.

Eq. (1.11) can be compactly written as :

$$\frac{d}{dx}\tilde{V}(x,p)=\tilde{A}(p)\,\tilde{V}(x,p)+\tilde{B}(x,p), \quad \dots (1.13)$$

where

$$\tilde{V}(x,p)=[\,\bar{\theta}(x,p),\ \bar{u}(x,p),\ \bar{\theta}'(x,p),\ \bar{u}'(x,p)]^T,$$

$$\tilde{B}(x,p) = \left[\,0, 0, -\frac{Q_0(1+\tau_0 p)\,\delta(x)}{p},\ 0\right]^T$$

and

$$\tilde{A}(p) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ C_{31} & 0 & 0 & C_{34} \\ 0 & C_{42} & C_{43} & 0 \end{bmatrix}.$$

Eq. (1.13) is to be solved subject to the conditions

(i) $\theta=u=\frac{\partial u}{\partial t}=0$ at $t=0,\ \forall\, x$

and (ii) $\bar{\theta}=\bar{u}=\frac{\partial\bar{\theta}}{\partial x}=\frac{\partial\bar{u}}{\partial x}=0$ at $x=\pm\alpha,\ \forall\, t>0.$ ... (1.14)

2. Solution of the Vector Matrix Differential Equation

The rudiments of the solution methodology through eigenvalue approach, as in Das *et al.*[17] have been presented in the appendix. We now proceed to solve eq. (1.13) by eigenvalue approach.

The characteristic equation of the matrix $\tilde{A}(p)$ takes of the form

$$\lambda^4-\lambda^2(C_{31}+C_{42}+C_{34}C_{43})+C_{31}C_{42}=0. \quad \dots (2.1)$$

The roots of the characteristic equation which are also the eigenvalues of the matrix $\tilde{A}(p)$ are the form

$$\lambda=\pm\lambda_1,\quad \lambda=\pm\lambda_2, \quad \dots (2.2)$$

where, $\lambda_1^2+\lambda_2^2=C_{31}+C_{42}+C_{34}C_{43}=(1+\varepsilon)(p+\tau_0 p^2)+p^2$

$$\lambda_1^2\lambda_2^2=C_{31}C_{42}=p^3(1+\tau_0 p) \quad \dots (2.3)$$

where, $\varepsilon=ga.$

The right eigenvector $X=[X_1, X_2, X_3, X_4]^T$ corresponding to the eigenvalue λ can be written as :

$$X=[(C_{42}-\lambda^2),\ -\lambda C_{43},\ \lambda(C_{42}-\lambda^2),\ -\lambda^2 C_{43}]^T. \quad \dots (2.4)$$

From (2.3) we can easily calculate the eigenvector X_i, corresponding to the eigenvalue $\lambda = \lambda_i$, $i = 1, 2, 3, 4$. For our further reference we shall use the following notations :

$$X_1 = [X]_{\lambda=\lambda_1}, \quad X_2 = [X]_{\lambda=-\lambda_1}, \quad X_3 = [X]_{\lambda=\lambda_2}, \quad X_4 = [X]_{\lambda=-\lambda_2}. \qquad \text{... (2.5)}$$

The left eigenvector $Y = [y_1, y_2, y_3, y_4]$ corresponding to the eigenvalue λ can be calculated as :

$$Y = \left[(\lambda^2 - C_{42}),\ \lambda C_{34} C_{42},\ \frac{\lambda(\lambda^2 - C_{42})}{C_{31}},\ \frac{\lambda^2 C_{34}}{C_{31}} \right]. \qquad \text{... (2.6)}$$

As early the left eigenvector Y_i, corresponding to the eigenvalue λ_i, $i = 1, 2, 3, 4$ be denoted as

$$Y_1 = [Y]_{\lambda=\lambda_1}, \quad Y_2 = [Y]_{\lambda=-\lambda_1}, \quad Y_3 = [Y]_{\lambda=\lambda_2}, \quad Y_4 = [Y]_{\lambda=-\lambda_2}. \qquad \text{... (2.7)}$$

Considering other physical conditions of the problem as in Das and Bhakta[18] the solution of the eq. (1.13) can be written as :

$$\tilde{V}(x,p) = a_2(x) X_2 \exp(-\lambda_1 x) + a_4(x) X_4 \exp(-\lambda_2 x), \quad x > 0 \qquad \text{... (2.8)}$$

where,
$$a_2(x) = \frac{1}{Y_2 X_2} \int_{s(=-\alpha)}^{x} Y_2\, \tilde{B}(s,p) \exp(-\lambda_1 s)\, ds, \quad x > 0$$

$$= \frac{1}{Y_2 X_2} \int_{s(=-\alpha)}^{x} \frac{\lambda_1(\lambda_1^2 - C_{42})}{C_{31}} \frac{Q_0(1+\tau_0 p)}{p} \delta(s) \exp(-\lambda_1 s)\, ds \quad x > 0.$$

Integrating, these expression we get

$$a_2(x) = \frac{1}{Y_2 X_2} \frac{\lambda_1(\lambda_1^2 - C_{42})}{C_{31}} \frac{Q_0(1+\tau_0 p)}{p} = \frac{Q_0}{Y_2 X_2} \frac{\lambda_1(\lambda_1^2 - C_{42})}{p^2}. \qquad \text{... (2.9)}$$

Similarly,

$$a_4(x) = \frac{1}{Y_4 X_4} \frac{\lambda_2(\lambda_2^2 - C_{42})}{C_{31}} \frac{Q_0(1+\tau_0 p)}{p} = \frac{Q_0}{Y_4 X_4} \frac{\lambda_2(\lambda_2^2 - C_{42})}{p^2}. \qquad \text{... (2.10)}$$

Thus the displacement and temperature field can be written from (2.8) as

$$\bar{u}(x,p) = \frac{Q_0 a}{p^2} \left[\frac{\lambda_1^2(\lambda_1^2 - C_{42})}{V_2} \exp(-\lambda_1 x) + \frac{\lambda_2^2(\lambda_2^2 - C_{42})}{V_4} \exp(-\lambda_2 x) \right] \qquad \text{... (2.11)}$$

and

$$\bar{\theta}(x,p) = \frac{-Q_0}{p^2} \left[\frac{\lambda_1(\lambda_1^2 - C_{42})^2}{V_2} \exp(-\lambda_1 x) + \frac{\lambda_2(\lambda_2^2 - C_{42})^2}{V_4} \exp(-\lambda_2 x) \right], \qquad \text{... (2.12)}$$

where $V_2 = Y_2X_2$ and $V_4 = Y_4X_4$.

Using eqs. (2.11) and (2.12) in (1.9), we write the stress component $\bar{\sigma}(x,p)$ as :

$$\bar{\sigma}(x,p) = \frac{-\beta^2 Q_0 a}{p^2}\left[\frac{\lambda_1^3(\lambda_1^2 - C_{42})}{V_2}\exp(-\lambda_1 x) + \frac{\lambda_2^3(\lambda_2^2 - C_{42})}{V_4}\exp(-\lambda_2 x)\right]$$

$$+\frac{bQ_0}{p^2}\left[\frac{\lambda_1(\lambda_1^2 - C_{42})^2}{V_2}\exp(-\lambda_1 x) + \frac{\lambda_2(\lambda_2^2 - C_{42})^2}{V_4}\exp(-\lambda_2 x)\right]. \quad \text{... (2.13)}$$

Eqs. (2.11), (2.12) and (2.13) determine completely the state of the solid for $x > 0$. The solution for the whole space (when the space $x \le 0$ is also included) is obtained from (2.11), (2.12) and (2.13) by taking the symmetrics under consideration. Thus, considering the heat source to act at the location $x = c$, instead of $x = 0$, we may write down the field variables as follows.

$$\bar{u}(x,p) = \frac{Q_0 a}{p^2}\left[\frac{\lambda_1^2(\lambda_1^2 - C_{42})}{V_2}\exp(\pm(-\lambda_1(x-c)))\right.$$

$$\left.+\frac{\lambda_2^2(\lambda_2^2 - C_{42})}{V_4}\exp(\pm(-\lambda_2(x-c)))\right] \quad \text{... (2.14)}$$

$$\bar{\theta}(x,p) = \frac{-Q_0}{p^2}\left[\frac{\lambda_1(\lambda_1^2 - C_{42})^2}{V_2}\exp(\pm(-\lambda_1(x-c)))\right.$$

$$\left.+\frac{\lambda_2(\lambda_2^2 - C_{42})^2}{V_4}\exp(\pm(-\lambda_2(x-c)))\right] \quad \text{... (2.15)}$$

$$\bar{\sigma}(x,p) = \frac{-\beta^2 Q_0 a}{p^2}\left[\frac{\lambda_1^3(\lambda_1^2 - C_{42})}{V_2}\exp(\pm(-\lambda_1(x-c)))\right.$$

$$\left.+\frac{\lambda_2^3(\lambda_2^2 - C_{42})}{V_4}\exp(\pm(-\lambda_2(x-c)))\right]$$

$$+\frac{bQ_0}{p^2}\left[\frac{\lambda_1(\lambda_1^2 - C_{42})^2}{V_2}\exp(\pm(-\lambda_1(x-c)))\right.$$

$$\left.+\frac{\lambda_2(\lambda_2^2 - C_{42})^2}{V_4}\exp(\pm(-\lambda_2(x-c)))\right], \quad \text{... (2.16)}$$

where the upper (plus) sign denotes the solution in the region $x \le c$, while the lower (minus) sign denotes the solution in the region $x > c$. We now compare our results (2.14), (2.15) and (2.16) with those of the existing literature (Sherief[19]) obtained through state space methodology, for determining the state of the solid. If we write $p = s$ in the eqs. [(2.14)-(2.16)] and use the characteristic eq. (2.1), then these equations representing displacement, temperature, and stress, are in complete agreement with those of Sherief[19] as in his eqs. (31), (32) and (33).

3. Numerical Solution

Sherief[19] presented the solutions for two fixed values of time with varying space variable x. The present author prefer to determine the state in the space time domain numerically by Bellman[20] method for a fixed value of the space variable and for varying time. The computations for the state variables are carried our for a fixed value of $x = 1$ and for values of time $t = 0.0257750, 0.138382, 0.352509, 0.693147, 1.21376, 2.04612, 3.67119$ which are also the roots of the Legendre polynomial of degree seven, *vide* Bellman[20]. The Copper material is chosen for purposes of numerical evaluations.

The values of the nondimensional constants ε and β^2 as well as the values of τ_0 and C are taken as

$$\varepsilon = 0.0168, \ \beta^2 = 3.5, \quad \tau_0 = 0.02, \ (\text{sec}) \ c = 0 \ (\text{meter}).$$

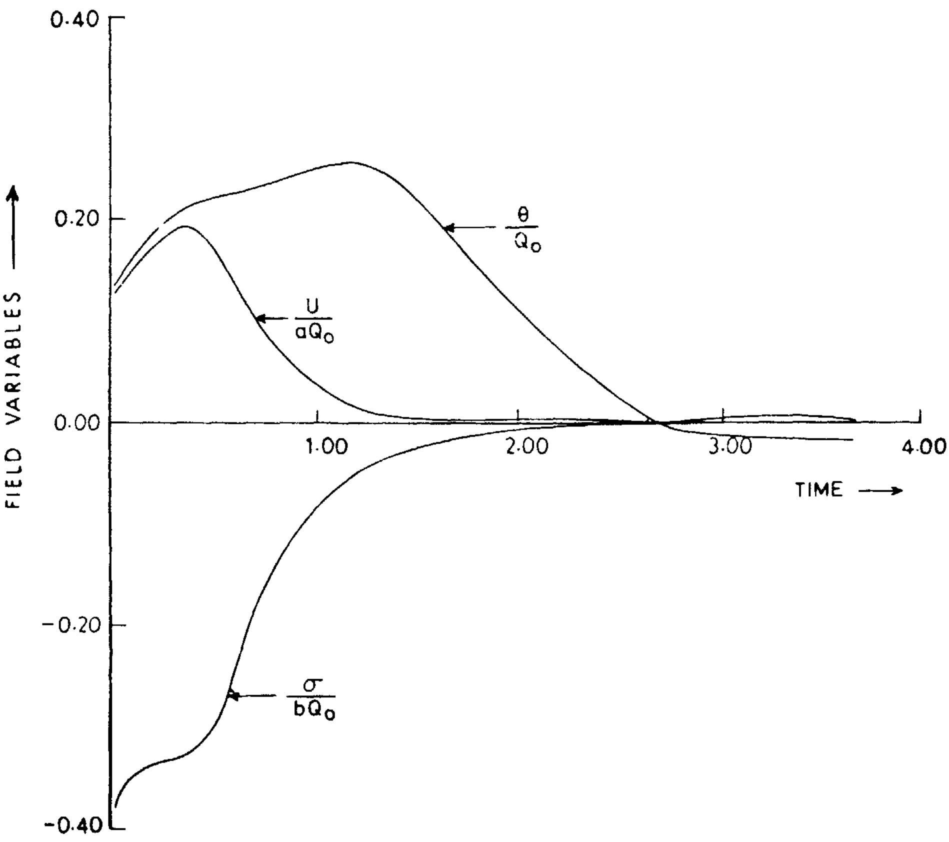

Fig. 1. Distribution of displacement temperature and stress (for $x = 1$) versus time.

Using Cubic spline formalism the computed values of the state variables are then plotted in a graph. It is clear from the graph (Fig. 1) that in the time range considered; the temperature, displacement and stress fields rise from the initial value of the time upto a certain value and then decrease to vanish at $t = 2.7$ (approximately). It may be noted that this value for $t = 2.7$ when $x = 1$, is the location of the wave front, and this distance x from the heat source depends on t.

PART II : TWO DIMENSIONAL PROBLEM

4. BASIC EQUATIONS AND FORMULATIONS

We consider a transversely isotropic infinite elastic medium subject to plane strain parallel to yz-plane. As such

$$U = 0,\ V = V(y, z, t),\ W = W(y, z, t), \qquad \dots (4.1)$$

where U, V and W are displacement components in the x, y, z directions respectively, *vide* Tauchert and Akoz[21], as follows :

$$\tau_{xx} = A_{12}\frac{\partial V}{\partial y} + A_{13}\frac{\partial W}{\partial z} - \beta_1\left(1 + \alpha\frac{\partial}{\partial t}\right)T,$$

$$\tau_{yy} = A_{22}\frac{\partial V}{\partial y} + A_{23}\frac{\partial W}{\partial z} - \beta_2\left(1 + \alpha\frac{\partial}{\partial t}\right)T,$$

$$\tau_{zz} = A_{23}\frac{\partial V}{\partial y} + A_{33}\frac{\partial W}{\partial z} - \beta_3\left(1 + \alpha\frac{\partial}{\partial t}\right)T$$

and

$$\tau_{yz} = A_{44}\left[\frac{\partial V}{\partial y} + \frac{\partial W}{\partial z}\right]. \qquad \dots (4.2)$$

The corresponding displacement equations of motion are

$$A_{22}\frac{\partial^2 V}{\partial y^2} + A_{44}\frac{\partial^2 V}{\partial z^2} + [A_{23} + A_{44}]\frac{\partial^2 W}{\partial y\,\partial z} = \rho\frac{\partial^2 V}{\partial t^2} + \beta_2\frac{\partial}{\partial y}\left[1 + \alpha\frac{\partial}{\partial t}\right]T$$

and

$$A_{44}\frac{\partial^2 W}{\partial y^2} + A_{33}\frac{\partial^2 W}{\partial z^2} + [A_{23} + A_{44}]\frac{\partial^2 V}{\partial y\,\partial z} = \rho\frac{\partial^2 W}{\partial t^2} + \beta_3\frac{\partial}{\partial z}\left[1 + \alpha\frac{\partial}{\partial t}\right]T. \qquad \dots (4.3)$$

The temperature field $T(y, z, t)$ is assumed to satisfy the general heat conduction equation

$$K_y\frac{\partial^2 T}{\partial y^2} + K_z\frac{\partial^2 T}{\partial z^2} = \rho C\left[\frac{\partial T}{\partial t} + \alpha_0\frac{\partial^2 T}{\partial t^2}\right] + T_0\left[\frac{\partial}{\partial t} + \tau\frac{\partial^2}{\partial t^2}\right]\left[\beta_2\frac{\partial V}{\partial y} + \beta_3\frac{\partial W}{\partial z}\right] - \left[1 + \tau\frac{\partial}{\partial t}\right]Q(y, z, t). \qquad \dots (4.4)$$

From these general eqs. (4.2)-(4.4) we now classify the problem into three classes, *vide* Chandrasekharaiah[13], for our further reference and for the comparison of our numerical computation of the results.

(i) If $\alpha_0 = 0$, $\alpha = 0$ and $\tau = 0$, then the problem reduces the problem of Classical thermoelasticity (CTE).

(ii) In the case when $\alpha = 0$, but $\alpha_0 = \tau \neq 0$ we call the problem as a problem of Extended thermoelasticity (ETE).

(iii) When $\alpha \neq 0$, $\alpha_0 \neq 0$ but $\tau = 0$, then equations reduces to a problem of temperature Rate Dependent thermoelasticity (TRDTE).

We now assume that the heat sources are instantaneous and acts on the line $y = 0$, $z = 0$; then we can write

$$Q(y, z, t) = q_0\, \delta(y)\, \delta(z)\, \delta(t), \quad \text{... (4.5)}$$

where q_0 is the strength of the heat source and $\delta(t)$ is a Dirac delta function of t.

5. Method of Solution : Formulation of a Vector Matrix Differential Equation

We apply the Laplace-Fourier double integral transform of the form

$$\overline{T}(y, z, p) = \int_0^{\alpha} T(y, z, t) \exp(-pt)\, dt \quad \text{... (5.1)}$$

and

$$\overline{T}_1(\xi, z, p) = \frac{1}{\sqrt{2\pi}} \int_{-\alpha}^{\alpha} \overline{T}(y, z, p) \exp(i\xi y)\, dy, \quad \text{... (5.2)}$$

(where p and ξ are transform parameters) to eqs. (4.3)-(4.5).

$$-K_y\, \xi^2\, \overline{T}_1 + K_z \frac{d^2 \overline{T}_1}{dz^2} = \rho C' p\, \overline{T}_1 + T_0' p \left[-i\xi \beta_2' \overline{V}_1 + \beta_3' \frac{d\overline{W}_1}{dz} \right] - \frac{q_0' \delta(z)}{\sqrt{2\pi}} \quad \text{... (5.3)}$$

$$-(\xi^2 A_{22} + \rho p^2)\, \overline{V}_1 + A_{44} \frac{d^2 \overline{V}_1}{dz^2} - i\xi [A_{23} + A_{44}] \frac{d\overline{W}_1}{dz} + i\xi \beta_2' \overline{T}_1 = 0 \quad \text{... (5.4)}$$

$$-(\xi^2 A_{44} + \rho p^2)\, \overline{W}_1 + A_{33} \frac{d^2 \overline{W}_1}{dz^2} - i\xi [A_{23} + A_{44}] \frac{d\overline{V}_1}{dz} - \beta_3' \frac{d\overline{T}_1}{dz} = 0, \quad \text{... (5.5)}$$

where we have used

$$\int_0^{\alpha} \delta(t) \exp(-pt)\, dt = 1 \ \text{ and } \ \frac{1}{\sqrt{2\pi}} \int_{-\alpha}^{\alpha} \delta(y) \exp(i\xi y)\, dy = \frac{1}{\sqrt{2\pi}}, \quad \text{... (5.6)}$$

and

$$C' = C(1 + \alpha_0 p),\ T_0' = T_0(1 + \tau p),\ q_0' = q_0(1 + \tau p),\ \beta_1' = \beta_1 (1 + \alpha p)$$

$$\beta_2' = \beta_2\,(1+\alpha p), \quad \beta_3' = \beta_3\,(1+\alpha p). \qquad \ldots (5.7)$$

We also assume that at time $t = 0$; the body is at rest; in an undeformed and unstressed state, and is maintained at the reference temperature. Then the following initial conditions hold.

$$V(y,\ z,\ 0) = \frac{\partial V(y,z,0)}{\partial t} = 0,$$

$$W(y,\ z,\ 0) = \frac{\partial W(y,z,0)}{\partial t} = 0$$

and

$$T(y,\ z,\ 0) = \frac{\partial T(y,z,0)}{\partial t} = 0. \qquad \ldots (5.8)$$

We have further assumed that $\overline{T}$, $\overline{V}$ and $\overline{W}$ as well as their derivatives with respect to y vanish at infinity.

Eqs. (5.3), (5.4) and (5.5) can be written in the form of a vector matrix differential equation as follows : *vide* Das *et al.*[17] and Das and Bhakta[18]

$$\frac{d\widetilde{V}}{dz} = \widetilde{A}\,\widetilde{V} + f(z), \qquad \ldots (5.9)$$

where

$$\widetilde{V} = [\,\overline{V}_1, \overline{W}_1, \overline{T}_1, \overline{V}_1', \overline{W}_1', \overline{T}_1'\,]^T$$

and

$$f(z) = \left[\,0, 0, 0, 0, 0, -\frac{q_0'\,\delta(z)}{\sqrt{2\pi}\,K_z}\,\right]^T. \qquad \ldots (5.10)$$

We now solve eq. (5.9) subject to the condition

(i) as given in (5.8)

and (ii) $\overline{V}$, $\overline{W}$, $\overline{T}$, $\dfrac{\partial \overline{V}}{\partial y}$, $\dfrac{\partial \overline{W}}{\partial y}$, $\dfrac{\partial \overline{T}}{\partial y}$, are zero at $y = \pm\infty$,

where the primes indicate differentiation with respect to z. The matrix $\widetilde{A}$ is

$$\widetilde{A} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ C_{41} & 0 & C_{43} & 0 & C_{45} & 0 \\ 0 & C_{52} & 0 & C_{54} & 0 & C_{56} \\ C_{61} & 0 & C_{63} & 0 & C_{65} & 0 \end{bmatrix} \qquad \ldots (5.11)$$

$$C_{41}=\frac{\xi^2 A_{22}+\rho p^2}{A_{44}},\quad C_{43}=\frac{-i\xi\beta_2'}{A_{44}},\quad C_{45}=\frac{i\xi(A_{23}+A_{44})}{A_{44}}$$

$$C_{52}=\frac{\xi^2 A_{44}+\rho p^2}{A_{33}},\quad C_{54}=\frac{i\xi(A_{23}+A_{44})}{A_{33}},\quad C_{56}=\frac{\beta_3'}{A_{33}}$$

$$C_{61}=\frac{-i\xi\beta_2 T_0' p}{K_z},\quad C_{63}=\frac{K_y\xi^2+\rho pC'}{K_z},\quad C_{65}=\frac{T_0'\beta_3 p}{K_z}. \quad \ldots (5.12)$$

6. Solution of the Vector Matrix Equations

The characteristic equation of the matrix $\tilde{A}$ takes the form

$$\lambda^6-\lambda^4[C_{41}+C_{52}+C_{63}+C_{45}C_{54}+C_{56}C_{65}]+\lambda^2[C_{52}C_{63}$$
$$+C_{41}C_{52}+C_{41}C_{63}-C_{43}C_{61}+C_{63}C_{45}C_{54}-C_{43}C_{54}C_{65}$$
$$+C_{41}C_{56}C_{65}-C_{45}C_{61}C_{56}]-[C_{41}C_{52}C_{63}-C_{43}C_{61}C_{52}]=0. \quad \ldots (6.1)$$

The roots of characteristic eq. (6.1) which are also the eigenvalues of the matrix $\tilde{A}$ are of the form

$$\lambda=\pm\lambda_1,\ \lambda=\pm\lambda_2,\ \lambda=\pm\lambda_3. \quad \ldots (6.2)$$

The right eigenvector $X = [X_1, X_2, X_3, X_4, X_5, X_6]^T$ corresponding to eigenvalue λ can be calculated as :

$$X=\begin{bmatrix}\lambda^2[C_{45}C_{56}+C_{43}]-C_{43}C_{52}\\ \lambda[\lambda^2C_{56}+(C_{54}C_{43}-C_{41}C_{56})]\\ \lambda^4-\lambda^2[C_{41}+C_{52}+C_{45}C_{54}]+C_{41}C_{52}\\ \lambda[\lambda^2[C_{45}C_{56}+C_{43}]-C_{43}C_{52}]\\ \lambda^2[\lambda^2C_{56}+(C_{54}C_{43}-C_{41}C_{56})]\\ \lambda[\lambda^4-\lambda^2[C_{41}+C_{52}+C_{45}C_{54}]+C_{41}C_{52}]\end{bmatrix}. \quad \ldots (6.9)$$

From (6.3) we can easily calculate the eigenvector X corresponding to the eigenvalue $\lambda=\lambda_i$. For our further reference we shall use the following notations :

$$X_1=[X]_{\lambda=\lambda_1},\quad X_2=[X]_{\lambda=-\lambda_1},\quad X_3=[X]_{\lambda=\lambda_2},\quad X_4=[X]_{\lambda=-\lambda_2}$$
$$X_5=[X]_{\lambda=\lambda_3},\quad X_6=[X]_{\lambda=-\lambda_3}. \quad \ldots (6.4)$$

The left eigenvector $\boldsymbol{Y} = [y_1, y_2, y_3, y_4, y_5, y_6]$ corresponding to the eigenvalue λ can be calculated as :

$$\boldsymbol{Y} = \begin{bmatrix} \lambda^3 C_{61} + \lambda\,[-C_{41} C_{61} - C_{61} C_{52} - C_{61} C_{45} C_{54} + C_{41} C_{61} + C_{41} C_{54} C_{65}] \\ C_{52}\,[\lambda^2 C_{65} + C_{45} C_{61} - C_{65} C_{41}] \\ \lambda^5 - \lambda^3 [C_{41} + C_{52} + C_{45} C_{54} + C_{56} C_{65}] \\ \qquad + \lambda\,[C_{41} C_{52} - C_{45} C_{61} C_{65} + C_{41} C_{56} C_{65}] \\ \lambda^2 [C_{61} + C_{54} C_{65}] - C_{61} C_{52} \\ \lambda\,[\lambda^2 C_{65} + C_{45} C_{61} - C_{65} C_{41})] \\ \lambda^4 - \lambda^2 [C_{41} + C_{52} + C_{45} C_{54}] + C_{41} C_{52} \end{bmatrix}^T \quad \ldots (6.5)$$

For simplicity, henceforth, we shall denote them as

$$Y_1 = [Y]_{\lambda = \lambda_1}, \quad Y_2 = [Y]_{\lambda = -\lambda_1}, \quad Y_3 = [Y]_{\lambda = \lambda_2}, \quad Y_4 = [Y]_{\lambda = -\lambda_2},$$

$$Y_5 = [Y]_{\lambda = \lambda_3}, \quad Y_6 = [Y]_{\lambda = -\lambda_3}. \quad \ldots (6.6)$$

Assuming the regularity condition at infinity as in Das *et al.*[17, 18] the solution of the equation (5.9) can be written as :

$$\tilde{V}\,(x, p) = a_2(x)\,X_2 \exp(-\lambda_1 x) + a_4(x)\,X_4 \exp(-\lambda_2 x) + a_6(x)\,X_6 \exp(-\lambda_3 x) \quad \ldots (6.7)$$

where

$$a_2(z) = -\frac{1}{Y_2 X_2} \int_{s(=-\alpha)}^{z} [\lambda_1^4 - \lambda_1^2 [C_{41} + C_{52} + C_{45} C_{54}] + C_{41} C_{52}]\, \frac{q_0' \,\delta(z)\, e^{-\lambda_1 s}}{\sqrt{2\pi}\, K_z}\, ds, \quad z > 0.$$

Integrating, this expression

$$a_2(z) = \frac{-q_0'}{Y_2 X_2 \sqrt{2\pi}\, K_z} [\lambda_1^4 - \lambda_1^2 .[C_{41} + C_{52} + C_{45} C_{54}] + C_{41} C_{52}], \quad z > 0. \quad \ldots (6.8)$$

Similarly, $a_4(z)$ and $a_6(z)$ become

$$a_4(z) = \frac{-q_0'}{Y_4 X_4 \sqrt{2\pi}\, K_z} [\lambda_2^4 - \lambda_2^2 [C_{41} + C_{52} + C_{45} C_{54}] + C_{41} C_{52}], \quad z > 0. \quad \ldots (6.9)$$

and

$$a_6(z) = \frac{-q_0'}{Y_6 X_6 \sqrt{2\pi}\, K_z} [\lambda_3^4 - \lambda_3^2 [C_{41} + C_{52} + C_{45} C_{54}] + C_{41} C_{52}], \quad z > 0. \quad \text{... (6.10)}$$

Writing (a_2, a_4, a_6) as (A_1, A_2, A_3) the deformations $\overline{V}_1 (\xi, z, p)$, $\overline{W}_1 (\xi, z, p)$ and temperature $\overline{T}_1 (\xi, z, p)$ can be compactly written from eq. (6.7) as

$$\overline{V}_1 (\xi, z, p) = \sum_{i=1}^{3} A_i [\lambda_i^2 [C_{45} C_{56} + C_{43}] - C_{43} C_{52}] \exp(-\lambda_i z), \quad \text{... (6.11)}$$

$$\overline{W}_1 (\xi, z, p) = \sum_{i=1}^{3} A_i [-\lambda_i \{\lambda_i^2 C_{56} + (C_{54} C_{43} - C_{41} C_{56})] \exp(-\lambda_i z), \quad \text{... (6.12)}$$

and

$$\overline{T}_1 (\xi, z, p) = \sum_{i=1}^{3} A_i [\lambda_i^4 - \lambda_i^2 \{C_{41} + C_{52} + C_{45} C_{54}\} + C_{41} C_{54}] \exp(-\lambda_i z). \quad \text{... (6.13)}$$

Using eqs. (4.2), (6.11), (6.12) and (6.13) and stresses in the Laplace-Fourier transform domain can be written as

$$(\overline{\tau}_{xx})_1 = -i\xi A_{12} \sum_{i=1}^{3} A_i [\lambda_i^2 \{C_{45} C_{56} + C_{43}\} - C_{43} C_{52}] \exp(-\lambda_i z)$$
$$+ A_{13} \sum_{i=1}^{3} A_i [\lambda_i^2 \{\lambda_i^2 C_{56} + (C_{54} C_{43} - C_{41} C_{56})\}] \exp(-\lambda_i z)$$
$$-\beta_1' \sum_{i=1}^{3} A_i [\lambda_i^4 - \lambda_i^2 \{C_{41} + C_{52} + C_{45} C_{54}\} + C_{41} C_{52}] \exp(-\lambda_i z) \quad \text{... (6.14)}$$

$$(\overline{\tau}_{yy})_1 = -i\xi A_{22} \sum_{i=1}^{3} A_i [\lambda_i^2 \{C_{45} C_{56} + C_{43}\} - C_{43} C_{52}] \exp(-\lambda_i z)$$

$$+ A_{23} \sum_{i=1}^{3} A_i [\lambda_i^2 \{\lambda_i^2 C_{56} + (C_{54} C_{43} - C_{41} C_{56})\}] \exp(-\lambda_i z)$$

$$-\beta_2' \sum_{i=1}^{3} A_i [\lambda_i^4 - \lambda_i^2 \ \{C_{41} + C_{52} + C_{45} C_{54}\} + C_{41} C_{52}] \exp(-\lambda_i z)$$

$$\dots (6.15)$$

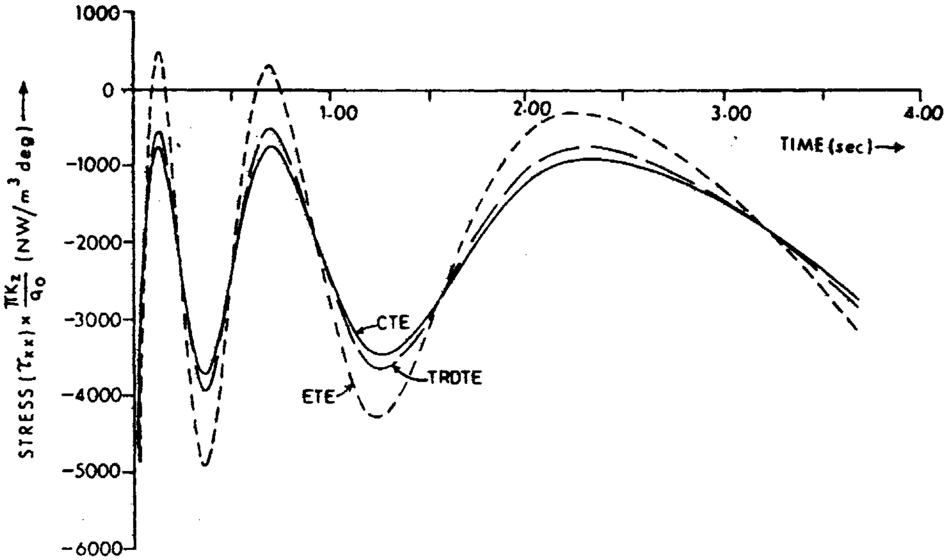

FIG. 2. Distribution of stress (τ_{xx}) (for y = 20 and z = 1) versus time.

$$(\bar{\tau}_{zz})_1 = - i\xi A_{23} \sum_{i=1}^{3} A_i [\lambda_i^2 \{C_{45} C_{56} + C_{43}\} - C_{43} C_{52}] \exp(-\lambda_i z)$$

$$+ A_{33} \sum_{i=1}^{3} A_i [\lambda_i^2 \{\lambda_i^2 C_{56} + (C_{54} C_{43} - C_{41} C_{56})\}] \exp(-\lambda_i z)$$

$$-\beta_3' \sum_{i=1}^{3} A_i [\lambda_i^4 - \lambda_i^2 \ \{C_{41} + C_{52} + C_{45} C_{54}\} + C_{41} C_{52}] \exp(-\lambda_i z)$$

$$\dots (6.16)$$

$$(\bar{\tau}_{yz})_1 = A_{44} \sum_{i=1}^{3} A_i [-\lambda_i^3 \{C_{45} C_{56} + C_{43}\} + \lambda_1 C_{43} C_{52}] \exp(-\lambda_i z)$$

$$- i\xi A_{44} \sum_{i=1}^{3} A_i \left[-\lambda_i\{\lambda_i^2 C_{56} + (C_{54} C_{43} - C_{41} C_{56})\}\right] \exp(-\lambda_i z) \qquad \ldots (6.17)$$

We now write down from eqs. (6.13), (6.14), (6.15), (6.16) and (6.17), the expressions of the temperature and the stress fields from the Laplace-Fourier transform domain to the Laplace transform domain as

$$[\bar{T}, \bar{\tau}_{xx}, \bar{\tau}_{yy}, \bar{\tau}_{zz}](y, z, p) = \sqrt{2/\pi} \int_0^{\alpha} [\bar{T}_1, (\bar{\tau}_{xx})_1, (\bar{\tau}_{yy})_1, (\bar{\tau}_{zz})_1] \cos(\xi y)\, d\xi \qquad \ldots (6.18)$$

and

$$\bar{\tau}_{yz}(y, z, p) = \sqrt{2/\pi} \int_0^{\alpha} (\bar{\tau}_{yz})_1 \sin(\xi y)\, d\xi. \qquad \ldots (6.19)$$

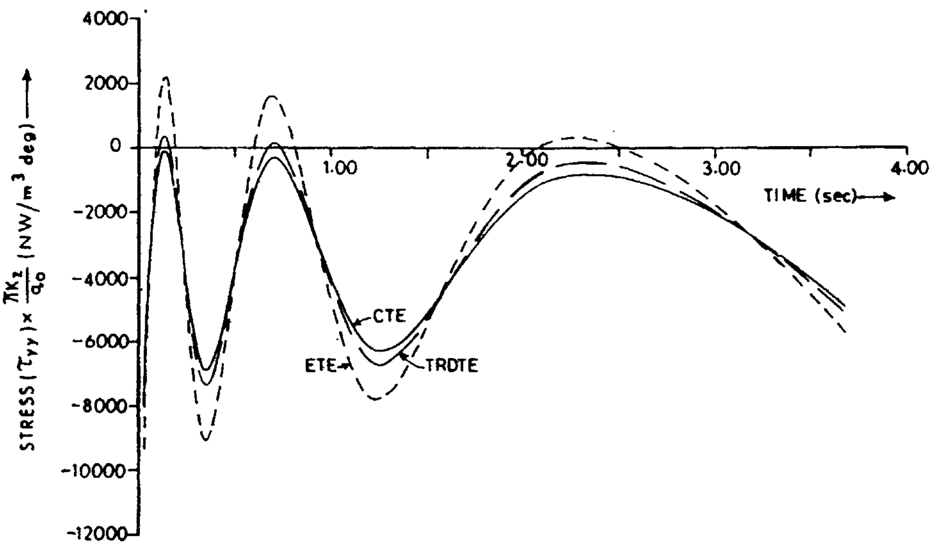

FIG. 3. Distribution of stress (τ_{yy}) (for y = 20 and z = 1) versus time.

7. NUMERICAL SOLUTION

The Fourier-Laplace inversion of the expressions for temperature and stresses in the space time domain are very complex and we prefer to develop an efficient computer software for the purpose of inversion of these double integral transforms. As such, as in the previous case, for the inversion of Laplace transform we follow the method of Bellman[20] and choose seven values of the time $t = t_i$, $i = 1, 2, 3, 4, 5, 6, 7$, at which the stresses are to be determined, where, t_i are the roots of the Legendre polynomial of degree seven, *vide* Bellman[20]. Simultaneous calculations for the inversion of the Fourier-transform were done by evaluating the infinite integrals (6.18), (6.19) numerically by five point Gaussian quadrature formula for several prescribed values of y and z.

The following data S.I. units of Cobalt (considered as transversely isotropic) have been used, *vide* Dhaliwal and Singh[22].

MATERIAL CONSTANTS

A_{12}	A_{13}	A_{22}	A_{23}	A_{33}	A_{44}	$\times 10^{11}$ Nm^{-2}
1.650	1.027	3.071	1.027	3.581	1.510	
β_1	β_2	β_3	$\times 10^6$ Nm^{-2} deg^{-1}			
7.04	7.04	6.90				
K_y	K_z	Wm^{-1} deg^{-1}				
69	69					

$\rho = 8.836 \times 10^3$ kgm^{-3}, $c = 4.27 \times 10^2$ J (kg)$^{-1}$ deg^{-1}

$T_0 = 298^0$ K.

We now present our results in the form of graphs (Figs. 2-5) to compare with the cases CTE, ETE, TRDTE for the stresses $[\tau_{xx}, \tau_{yy}, \tau_{zz}, \tau_{yz}]$ (y, z, t) when time variable t = 0.025775, 0.138382, 0.352509, 0.693147, 1.21376, 2.04612, and 3.67119 are labelled in the abscissa and for particular values of the space variables $y = 20$ and $z = 1$. The material constants α, α_0 and thermal relaxation parameter τ were taken as for different cases (i) *CTE*, (ii) *ETE* and (iii) *TRDTE* as follows :

(i) *CTE* $\alpha=0$, $\alpha = 0$, $\tau = 0$

(ii) *ETE* $\alpha=0$, $\alpha = 10^{-5}$, $\tau = 10^{-5}$

(iii) *TRDTE* $\alpha=10^{-5}$, $\alpha = 10^{-7}$, $\tau = 0$.

The behaviour of the stresses etc. for $t \to 0$ can be estimated from the initial value theorem

$$\lim_{t \to 0} \phi(t) = \lim_{p \to \infty} p\bar{\phi}(p).$$

As such the stresses all assume very large value, *vide* Paria[23] and Hetnarski[24].

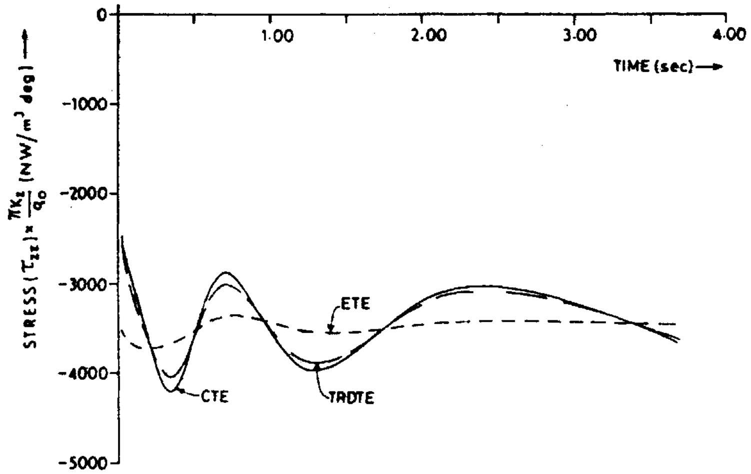

FIG. 4. Distribution of stress (τ_{zz}) (for $y = 20$ and $z = 1$) versus time.

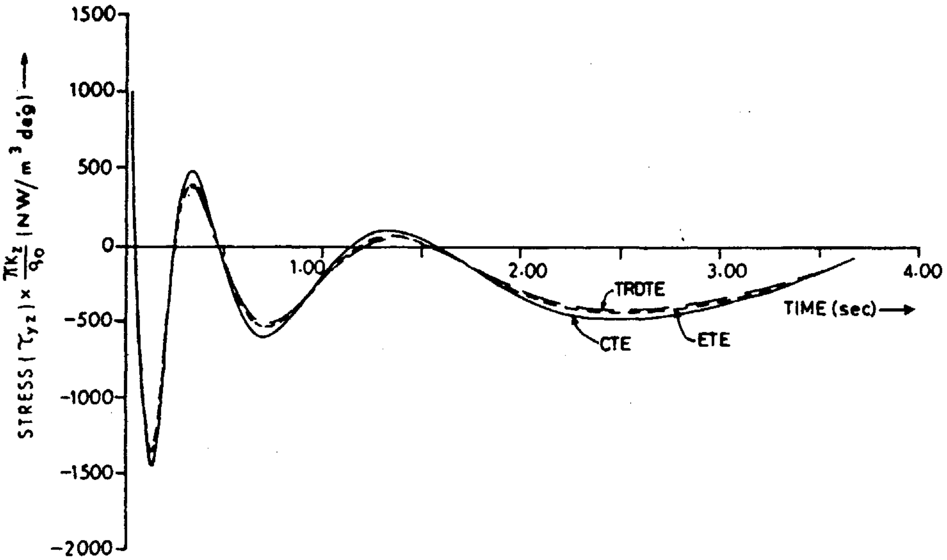

FIG. 5. Distribution of stress (τ_{yz}) (for $y = 20$ and $z = 1$) versus time.

Further, to determine the location of the wave front, a separate graph for the stresses in the case of TRDTE has been made for values of the variables $y = 20$, $z = .001$, 0.01, 0.1, 1 for $t = 3.67119$. It is found from Fig. 6 that all the stresses tend to zero when $z = 1$. Thus $(y, z, t) = (20, 1, 3.67119)$ is a location of the wave front. As in the earlier case, the graphs have been drawn by using the cubic spline formalisms.

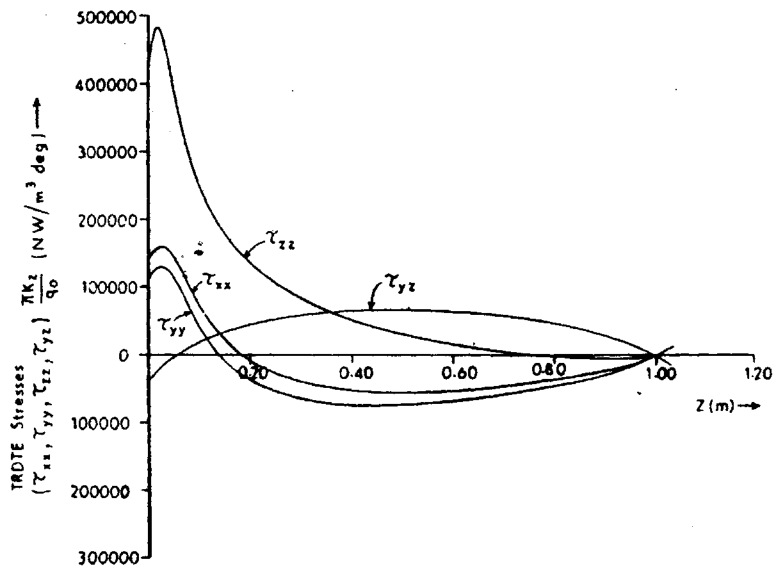

FIG. 6. Distribution of TRDTE stresses (τ_{xx}, τ_{yy}, τ_{zz}, τ_{yz}) (for time 3.67119 and $y = 20$) versus z.

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APPENDIX

8. Solution of the Vector-Matrix Differential Equation

Consider a vector-matrix differential equation

$$\frac{dV}{dx} = A\,V + f(x) \qquad \text{... (8.1)}$$

with the condition

$$V(x_0) = C, \qquad \text{... (8.2)}$$

where A is an $n \times n$ constant real matrix, C is a given constant real n vector and f is a real n vector function.

Let,

$$V = X \exp(\lambda x) \qquad \text{... (8.3)}$$

be a solution of the homogeneous equation

$$\frac{dV}{dx} = A\,V \qquad \text{... (8.4)}$$

where λ is a scalar and X is an n-vector independent of x. Substituting (8.3), (8.4) we get,

$$(AX - \lambda X)\, e^{\lambda x} = 0 \;\Rightarrow\; AX - \lambda X = 0 \;\Rightarrow\; AX = \lambda X.$$

This may be interpreted that λ is an eigenvalues of the matrix A and X the corresponding right eigenvector.

Let $\lambda_1, \lambda_2, \lambda_3, ..., \lambda_n$, be n distinct eigenvalues of the matrix A and $X_1, X_2, X_3, ..., X_n$ be the corresponding right eigenvectors of the matrix A. Then the vectors $X_1, X_2, ..., X_n$ are linearly independent and so they form a basis of the space Γ^n, where Γ denotes the field of complex numbers. We can find scalars $b_1, b_2, ..., b_n$ such that

$$C = b_1 X_1 + b_2 X_2 + \ldots b_n X_n$$

choose,

$$c_i = b_i e^{-\lambda_i x_0} \quad (i = 1, 2, \ldots n).$$

Let,

$$u(x) = \sum_{i=1}^{n} c_i X_i e^{\lambda_i x}. \qquad \ldots (8.5)$$

Thus $u(x)$ is a solution of the differential equation (8.4) and

$$u(x_0) = \sum_{i=1}^{n} c_i X_i e^{\lambda_i x_0}, \; X_i = \sum_{i=1}^{n} b_i X_i = C.$$

Now, let

$$w(x) = \sum_{i=1}^{n} a_i(x) X_i e^{\lambda_i x} \qquad \ldots (8.6)$$

be a solution of eq. (8.1), where $a_1(x), a_2(x), \ldots, a_n(x)$ are scalar function of x such that $a_i(x_0) = 0$.

Differentiating (8.6) with respect to x we get,

$$w'(x) = \sum_{i=1}^{n} a_i'(x) X_i e^{\lambda_i x} + \sum_{i=1}^{n} a_i(x) \lambda_i X_i e^{\lambda_i x} \qquad \ldots (8.7)$$

Substituting (8.6) and (8.7) in (8.1) we have

$$\sum_{i=1}^{n} a_i'(x) X_i e^{\lambda_i x} + \sum_{i=1}^{n} a_i(x) \lambda_i X_i e^{\lambda_i x} = \sum_{i=1}^{n} a_i(x) A X_i e^{\lambda_i x} + f(x)$$

or,

$$\sum_{i=1}^{n} a_i'(x) X_i e^{\lambda_i x} = \sum_{i=1}^{n} a_i(x) [A X_i - \lambda_i X_i] e^{\lambda_i x} + f(x) = f(x) \qquad \ldots (8.8)$$

multiplying (8.8) by $Y_j e^{-\lambda_j x}$ (where $Y_1, Y_2, \ldots, Y_n$ are left eigenvector of A corresponding to the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$).

We get,

$$\sum_{i=1}^{n} a_i'(x) Y_j X_i e^{(\lambda_i - \lambda_j)x} = Y_j f(x) e^{-\lambda_j x}$$

or,

$$a_i'(x)\, Y_j X_j = Y_j f(x)\, e^{-\lambda_j x}\, [Y_j X_i = 0 \quad \text{for} \quad i \neq j]$$

$$a_j'(x) = \frac{1}{Y_j X_j} Y_j f(x)\, e^{-\lambda_j x}$$

or,

$$a_j(x) = \int_{x_0}^{x} (Y_j X_j)^{-1}\, Y_j f(s)\, e^{\lambda_j s}\, ds \qquad \text{... (8.9)}$$

$$[a_j(x_0) = 0 \text{ for } j = 1, 2, \ldots, n]$$

Not take

$$V(x) = u(x) + w(x) \qquad \text{... (8.10)}$$

Differentiating we get,

$$\begin{aligned} V'(x) &= u'(x) + w'(x) \\ &= Au(x) + Aw(x) + f(x) \\ &= A[u(x) + w(x)] + f(x) \\ &= AV(x) + f(x) \end{aligned}$$

and

$$V(x_0) = u(x_0) + w(x_0) = C.$$

Hence, $V(x) = u(x) + w(x)$ is the unique solution of the differential eq. (8.1) satisfying the condition (8.2).