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DUALITY THEOREMS FOR NONDIFFERENTIABLE MULTIOBJECTIVE PROGRAMS

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Duality for nonlinear nondifferentiable convex programs have been studied. Duality results for a multiobjective convex program with a square root term in each of the objectives have also been established.

Key Words : Duality — Weak Strong & Strict Converse; Programs — Nonlinear, Nondifferentiable & Multiobjective Convex

1. INTRODUCTION

Lai and Ho[1] presented a set of necessary and sufficient conditions, due to Wolkowicz, for the following scalar optimization problem :

(P) Minimize $f(x)$

subject to $g(x) \leqq 0$,

where f is a continuous convex functional defined on X, a locally convex real vector space and $g = (g_1, g_2, ..., g_m) : X \to \mathbb{R}^m$ is a continuous convex function. In this paper we are extending these results for the nondifferentiable multiobjective convex program :

(VP) Minimize $f(x) = (f_1(x), f_2(x), ..., f_k(x))^t$

subject to $g(x) \leqq 0$, $x \in C$ ... (1.1)

where $f=(f_1, f_2, ..., f_k)$ and $g=(g_1, g_2, ..., g_m)$ are continuous convex functions defined on a locally convex real vector space X and C is a convex subset of X. Duality results for the above program (VP) have also been established. Further, duality results for a program with a square root term in each of the objectives have also been proved.

2. PRELIMINARIES

The following convention for the inequalities will be used :

For $x=(x_1, x_2, ..., x_n)$, $y=(y_1, y_2, ..., y_n)$ in $\mathbb{R}^n$

(i) $x \geqq y \Leftrightarrow x_i \geqq y_i \quad \forall \quad i = 1, 2, ..., n$

(ii) $x \geq y \Leftrightarrow x \geqq y$ and $x \neq y$

(iii) $x > y \Leftrightarrow x_i > y_i \quad \forall \quad i = 1, 2, ..., n.$

Since the objectives in multiobjective problems generally conflict with one another, an optimal solution is chosen from the set of efficient solutions in the following sense.

Definition 2.1 — A feasible solution x^0 is said to be efficient for (VP) if there is no feasible x for (VP) such that

$$f(x) \leq f(x^0)$$

i.e., there is no feasible x for (VP) such that

$$f_i(x) \leqq f_i(x^0) \quad \forall \quad i = 1, 2, ..., k$$

and $f_r(x) < f_r(x^0)$ for at least one $r \in \{1, 2, ..., k\}$.

Geoffrion[2] defined a restricted concept of efficiency, called proper efficiency, as follows :

Definition 2.2 — An efficient solution x^0 of (VP) is said to be properly efficient if $\exists$ a scalar $M > 0$ such that for each $r \in \{1, 2, ..., k\}$ and x feasible for (VP) satisfying $f_r(x) < f_r(x^0)$ we have

$$\frac{f_r(x^0) - f_r(x)}{f_j(x) - f_j(x^0)} \leqq M$$

for at least one $j \in \{1, 2, ..., k\}$ satisfying $f_j(x^0) < f_j(x)$.

Definition 2.3 — A continuous linear map $\xi : X \to \mathbb{R}^n$ is said to be a subgradient of a convex map $f : X \to \mathbb{R}^n$ at a point $x^0 \in X$ if

$$\langle \xi, x - x^0 \rangle \leqq f(x) - f(x^0) \quad \forall \quad x \in X.$$

Definition 2.4 — A continuous linear map $\xi : X \to \mathbb{R}^n$ is said to be a subgradient of a strictly convex map $f : X \to \mathbb{R}^n$ at a point $x^0 \in X$ if

$$\langle \xi, x - x^0 \rangle < f(x) - f(x^0) \quad \forall \quad x \in X,\ x \neq x^0.$$

The set of all subgradients of f at x^0 is called the subdifferential of f at x^0 and is denoted by $\partial f(x^0)$.

Definition 2.5 — The positive polar cone of $K (\subseteq X)$, denoted by K^+ is defined as

$$K^+ = \{x^* \in X^* \mid \langle x^*, k \rangle \geqq 0 \quad \forall \quad k \in K\},$$

where X^* is the dual space of X.

Definition 2.6 — The normal cone to C, a convex subset of the locally convex space X, at $x^0 \in X$ is defined as

$$N_c(x^0) = \{x^* \in X^* : \langle x^*, x - x^0 \rangle \leqq 0 \quad \forall \quad x \in C\}.$$

Definition 2.7 — The relative interior of a convex set C in $\mathbb{R}^n$, denoted by ri C, is defined as the interior which results when C is regarded as a subset of its affine hull, aff C, where aff C is the smallest affine set containing C.

Thus ri $C = \{x \in \text{ aff } C \mid \exists\ \varepsilon > 0 : (x + \varepsilon B) \cap \text{aff } C \subseteq C\}$,

where B is the open unit ball.

Assumption (A1) (Rockafellar[3]) — The convex sets ri (dom f_i); $i = 1, 2, \ldots, k$ have a point in common, so that

$$\partial \left(\sum_{i=1}^{k} f_i \right)(x) = \sum_{i=1}^{k} \partial f_i(x) \quad \forall \quad x,$$

where f_i; $i = 1, 2, \ldots, k$ are proper convex functions.

Lemma 2.1 (Geoffrion[2]) — A feasible solution x^* for (VP) is properly efficient if and only if x^* is optimal to the Geoffrion's equivalent scalar minimization problem (P_α) for some $\alpha > 0$ where

$$(P_\alpha) \qquad \text{Minimize } \sum_{i=1}^{k} \alpha_i f_i(x)$$

$$\text{subject to } g(x) \leqq 0, \quad x \in C.$$

3. Necessary and Sufficient Conditions — Duality Theorems

Theorem 3.1 ***(Necessary and sufficient conditions)*** — ***A set of necessary and sufficient conditions for x^0 to be properly efficient for (VP) is that $\exists\ \alpha^0$ and λ^0 such that***

$$0 \in \sum_{i=1}^{k} \alpha_i^0 \partial f_i(x^0) + \sum_{j \in J(x^0)} \lambda_j^0 \partial g_j(x^0) - [D_j^{=}(x^0)]^+ + N_c(x^0) \quad \ldots (3.1)$$

and

$$\lambda_j^0 \geqq 0, \quad j \in J(x^0), \quad \alpha^0 = (\alpha_1^0, \alpha_2^0, \ldots, \alpha_k^0) > 0, \quad \ldots (3.2)$$

where $J(x^0) = \{j : g_j(x^0) = 0\}$,

$$D_{J^{=}}^{=}(x^0) = \bigcap_{j \in J^{=}} D_j^{=}(x^0),$$

$$J^{=} = \{j : g_j(x) = 0 \quad \forall \quad x \text{ feasible for (VP)}\},$$

and $$D_j^{=}(x^0) = \{d \in X : \exists\, \beta^0 > 0, \; g_j(x^0 + \beta d) = g_j(x^0), \quad \forall \quad \beta : 0 \leqq \beta \leqq \beta^0\}$$

$[D_{J^{=}}^{=}(x^0)]^+$ denotes the positive polar cone of $D_{J^{=}}^{=}(x^0)$ and $N_c(x^0)$ is the normal cone to C at x^0.

PROOF : The following conditions are necessary and sufficient for x^0 to be optimal for (P_{α^0}) for some $\alpha^0 > 0$, by Theorem 1 of Lai and Ho[1] and hence, by Lemma 2.1, for x^0 to be properly efficient for (VP) :

$$0 \in \partial\left(\sum_{i=1}^{k} \alpha_i^0 f_i\right)(x^0) + \sum_{j \in J(x^0)} \lambda_j^0 \partial g_j(x^0) - [D_{J^{=}}^{=}(x^0)]^+ + N_c(x^0)$$

$$\lambda_j^0 \geqq 0, \quad j \in J(x^0), \quad \alpha^0 = (\alpha_1^0, \alpha_2^0, \ldots, \alpha_k^0) > 0.$$

$N_c(x^0)$ is the subdifferential of the indicator function $\delta_c(x^0)$

i.e.,
$$0 \in \sum_{i=1}^{k} \alpha_i^0 \partial f_i(x^0) + \sum_{j \in J(x^0)} \lambda_j^0 \partial g_j(x^0) - [D_{J^{=}}^{=}(x^0)]^+ + N_c(x^0) \quad \text{(by (A1))}$$

$$\lambda_j^0 \geqq 0, \quad j \in J(x^0), \; \alpha^0 > 0.$$

Hence the theorem.

Lagrange type of dual for (VP) is

(VD) Maximize $(f_1(u) + \lambda^t g(u), f_2(u) + \lambda^t g(u), \ldots, f_k(u) + \lambda^t g(u))$

subject to

$$0 \in \sum_{i=1}^{k} \alpha_i \partial f_i(u) + \sum_{j=1}^{m} \lambda_j \partial g_j(u) - [D_{J^{=}}^{=}(u)]^+ + N_c(u) \quad \ldots (3.3)$$

$$u \in C, \quad g_{\mathcal{F}}(u) \leqq 0 \qquad \ldots (3.4)$$

$$\lambda \geqq 0, \ \lambda \in \mathbb{R}^m, \ \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k) > 0, \ \alpha^t e = 1. \qquad \ldots (3.5)$$

Theorem 3.2 *(Weak duality) — Let x be feasible for (VP) and (u, α, λ) be feasible for (VD) then*

$$f(x) \nleq f(u) + \lambda^t g(u)e.$$

PROOF : In view of (3.3) $\exists\ \xi_i \in \partial f_i(u)$, $i = 1, 2, \ldots, k$; $\eta_j \in \partial g_j(u)$, $j = 1, 2, \ldots, m$; $x^* \in [D^{=}_{\mathcal{F}}(u)]^+$ and $u^* \in N_c(u)$ such that

$$0 = \sum_{i=1}^{k} \alpha_i \xi_i + \sum_{j=1}^{m} \lambda_j \eta_j - x^* + u^*. \qquad \ldots (3.6)$$

In particular for x feasible for (VP), (3.4) gives

$$g_j(u) \leqq 0 \quad \forall \quad j : g_j(x) = 0. \qquad \ldots (3.7)$$

Now $x^* \in [D^{=}_{\mathcal{F}}(u)]^+$ which means

$\langle x^*, d \rangle \geqq 0$ where $d \in X$ and there is some

$$\beta^0 > 0 \ni g_j(u + \beta d) = g_j(u);$$

$$\forall \quad \beta : 0 \leqq \beta \leqq \beta^0, \quad \forall \quad j : g_j(x) = 0.$$

Since $\beta^0 > 0$, so one can set $\beta^0 = 1$.

Thus

$$\langle x^*, d \rangle \geqq 0, \ d \in X \text{ and } g_j(u + \beta d) \leqq 0;$$

$$\forall \quad \beta : 0 \leqq \beta \leqq 1, \quad \forall \quad j : g_j(x) = 0 \qquad \text{(using (3.7)).}$$

$$\ldots (3.8)$$

In particular for $d = x - u \in X$, we have

$$g_j(u + \beta d) = g_j(u + \beta(x - u))$$

$$= g_j(\beta x + (1 - \beta)u)$$

(*Equation Continued on* p.1036)

$$\leqq \beta g_j(x) + (1-\beta) g_j(u) \qquad \text{(by convexity of } g_j\text{)}$$

$$\leqq 0 \quad \forall \quad j : g_j(x) = 0 \qquad \text{(in view of (3.7)).}$$

Hence by (3.8)

$$\langle x^*, x-u \rangle \geqq 0. \qquad \ldots (3.9)$$

Consider

$$\alpha^t (f(x) - f(u)) \geqq \sum_{i=1}^{k} \alpha_i \quad \langle \xi_i, x-u \rangle$$

(by convexity of f and $\alpha > 0$)

$$= -\sum_{j=1}^{m} \lambda_j \langle \eta_j, x-u \rangle + \langle x^*, x-u \rangle - \langle u^*, x-u \rangle$$

(by (3.6))

$$\geqq -\lambda^t (g(x) - g(u))$$

(by convexity of g, $\lambda \geqq 0$, (3.9), and the fact that $u^* \in N_c(u)$)

$$\geqq \lambda^t g(u) = \lambda^t \alpha^t e\, g(u)$$

(by (1.1), $\lambda \geqq 0$ and (3.5)).

Hence

$$\alpha^t f(x) \geqq \alpha^t (f(u) + \lambda^t g(u) e)$$

which ensures

$$f(x) \nleq f(u) + \lambda^t g(u) e.$$

Theorem 3.3 (*Strong duality*) — *Let x^0 be properly efficient solution for (VP), then $\exists\ \alpha^0 \in \mathbb{R}^k$, $\lambda^0 \in \mathbb{R}^m$ such that $(x^0, \alpha^0, \lambda^0)$ is properly efficient for (VD). Further the two problems have the same extremal values.*

PROOF : Theorem 3.1 ensures the existence of $\alpha \in \mathbb{R}^k$, λ_j ; $j \in J(x^0)$ such that

$$0 \in \sum_{i=1}^{k} \alpha_i \partial f_i(x^0) + \sum_{j \in J(x^0)} \lambda_j \partial g_j(x^0) - [D_F^-(x^0)]^+ + N_c(x^0)$$

$$\lambda_j \geqq 0, \quad j \in J(x^0), \quad \alpha = (\alpha_1, \alpha_2, ..., \alpha_k) > 0$$

Set $\lambda_j = 0 \quad \forall \quad j \notin J(x^0)$.

Also $g_j(x^0) = 0 \quad \forall \quad j \in J(x^0)$

$\therefore \quad \lambda^t g(x^0) = 0.$

Dividing the above relations by $\sum_{i=1}^{k} \alpha_i > 0$ and setting

$$\alpha_i^0 = \frac{\alpha_i}{\sum_{i=1}^{k} \alpha_i}, \quad i = 1, 2, ..., k; \quad \lambda_j^0 = \frac{\lambda_i}{\sum_{i=1}^{k} \alpha_i}, \quad j = 1, 2, ..., m$$

and using the definitions of $[D_{f^=}(x^0)]^+$ and $N_c(x^0)$, we have

$$0 \in \sum_{i=1}^{k} \alpha_i^0 \partial f_i(x^0) + \sum_{j=1}^{m} \lambda_j^0 \partial g_j(x^0) - [D_{f^=}(x^0)]^+ + N_c(x^0)$$

$$\alpha^0 > 0, \ \lambda^0 \geqq 0; \ \alpha^{0t} e = 1$$

and $\lambda^{0t} g(x^0) = 0.$

Also feasibility of x^0 gives

$$g_j(x^0) \leqq 0 \quad \forall \quad j = 1, 2, ..., m.$$

In particular, $g_j(x^0) \leqq 0 \quad \forall \quad j \in J^=$

$$\Rightarrow \quad g_{J^=}(x^0) \leqq 0.$$

Thus $(x^0, \alpha^0, \lambda^0)$ is feasible for (VD).

Further since $\lambda^{0t} g(x^0) = 0$, the values of the two objectives are equal which, in turn, by Theorem 6 of Gulati and Talaat[4] makes $(x^0, \alpha^0, \lambda^0)$ properly efficient for (VD).

Theorem 3.4 *(Strict converse duality) — Let x, (u, α, λ) be feasible for (VP), (VD) respectively with*

$$f(x) = f(u) + \lambda^t g(u)e. \qquad \ldots (3.10)$$

If f_r is strictly convex at u for atleast one $r \in \{1, 2, ..., k\}$, then u is properly efficient for (VP).

PROOF : Properly efficiency of x and (u, α, λ) for (VP) and (VD) respectively follows from the given hypothesis and Theorem 6 of Gulati and Talaat[4].

Suppose, on the contrary, $u \neq x$. Feasibility of (u, α, λ) guarantees the existence of $\xi_i \in \partial f_i(u)$, $i = 1, 2, ..., k$; $\eta_j \in \partial g_j(u)$, $j = 1, 2, ..., m$; and $x^* \in [D_F^-(u)]^+$ and a $u^* \in N_c(u)$ such that

$$0 = \sum_{i=1}^{k} \alpha_i \xi_i + \sum_{j=1}^{m} \lambda_j \eta_j - x^* + u^*. \qquad \text{... (3.11)}$$

Convexity of f with one of its components as strictly convex and $\alpha > 0$ gives

$$\alpha^t (f(x) - f(u)) > \sum_{i=1}^{k} \alpha_i \langle \xi_i, x - u \rangle$$

$$= -\sum_{j=1}^{m} \lambda_j \langle \eta_j, x - u \rangle + \langle x^*, x - u \rangle - \langle u^*, x - u \rangle$$

(by (3.11))

$$\geqq -\lambda^t (g(x) - g(u))$$

(as g is convex, $\lambda \geqq 0$, $x^* \in [D_F^-(u)]^+$ and $u^* \in N_c(u)$)

$$\geqq \lambda^t g(u) = \lambda^t \alpha^t e\, g(u) \qquad \text{(by (1.1) and } \lambda \geqq 0).$$

Hence,

$$\alpha^t f(x) > \alpha^t (f(u) + \lambda^t g(u) e)$$

which contradicts (3.10).

Thus $u = x$ and so u is properly efficient for (VP).

4. Special Class

Now we consider a special case of (VP) where subgradients can be computed explicitly.

Consider the problem

$$\text{(SP)} \qquad v\text{-Min } F(x) = f(x) + (x^t B x)^{1/2} e$$

$$\text{subject to} \qquad g(x) \leqq 0, \quad x \in C. \qquad \text{... (4.1)}$$

Here f and g are convex and twice continuously differentiable functions on $\mathbb{R}^n$, B is an $n \times n$ positive semi-definite symmetric matrix and C is a convex subset of $\mathbb{R}^n$.

Lemma 4.1 (Craven and Mond[5]) — Let $\phi(x) = (x^t Bx)^{1/2}$. Then $\phi(x)$ is convex and $u \in \partial\phi(x)$ if and only if $u = Bz$, $z^t Bz \leqq 1$, $(x^t Bx)^{1/2} = x^t Bz$.

In view of this lemma, Kuhn-Tucker type necessary and sufficient conditions for x to be properly efficient for (SP) are

$$\sum_{i=1}^{k} \alpha_i (\nabla f_i(x) + Bz) + \sum_{j \in J(x)} \lambda_j \nabla g_j(x) - x^* + u^* = 0 \qquad \ldots (4.2)$$

$$\alpha_i > 0,\ i = 1, 2, \ldots, k;\ \lambda_j \geqq 0,\ j \in J(x),$$

$$x^* \in [D_F^=(x)]^+,\ u^* \in N_C(x), \qquad \ldots (4.3)$$

$$(x^t Bx)^{1/2} = x^t Bz \qquad \ldots (4.4)$$

and $$z^t Bz \leqq 1. \qquad \ldots (4.5)$$

A dual to the problem (SP) is

(SD) v-Max $G(u, \alpha, \lambda, z) = f(u) + [u^t Bz + \lambda^t g(u)]e$

subject to

$$\nabla\alpha^t f(u) + Bz + \nabla\lambda^t g(u) - x^* + u^* = 0, \qquad \ldots (4.6)$$

$$z^t Bz \leqq 1, \qquad \ldots (4.7)$$

$$g_J(u) \leqq 0, \qquad \ldots (4.8)$$

and $$\alpha > 0,\ \lambda \geqq 0,\ \alpha^t e = 1 \qquad \ldots (4.9)$$

with $$u \in C,\ x^* \in [D_F^=(u)]^+,\quad u^* \in N_C(u). \qquad \ldots (4.10)$$

Theorem 4.1 *(Weak duality) — Let x and (u, α, λ, z) be feasible solutions for (SP) and (SD) respectively.*

Then

$$F(x) \not\leqq G(u, \alpha, \lambda, z).$$

PROOF : Consider

$$\alpha^t (F(x) - G(u, \alpha, \lambda, z))$$

$$= \alpha^t (f(x) + (x^t Bz)^{1/2} e - f(u) - (u^t Bz + \lambda^t g(u))e$$

$$= \alpha^t (f(x) - f(u)) + \alpha^t ((x^t Bx)^{1/2} - u^t Bz - \lambda^t g(u))e$$

$$\geqq \alpha^t (f(x) - f(u)) + \alpha^t ((x^tBx)^{1/2} (z^tBz)^{1/2} - u^tBz - \lambda^t g(u))e$$

$$\geqq \alpha^t (f(x) - f(u)) + \alpha^t (x^tBz - u^tBz - \lambda^t g(u))e$$

(using Schwarz's inequality $x^tBz \leqq (x^tBx)^{1/2} (z^tBz)^{1/2}$)

$$\geqq \langle \nabla \alpha^t f(u), x - u \rangle + \alpha^t ((x-u)^t Bz - \lambda^t g(u))e$$

(by convexity of f and $\alpha > 0$)

$$= - \langle Bz + \nabla \lambda^t g(u) - x^* + u^*, x - u \rangle + (x-u)^t Bz - \lambda^t g(u) \quad \text{(by (4.6))}$$

$$= - \langle \nabla \lambda^t g(u), x - u \rangle + \langle x^*, x - u \rangle - \langle u^*, x - u \rangle - \lambda^t g(u)$$

$$\geqq - \lambda^t (g(x) - g(u)) - \lambda^t g(u)$$

(by convexity of g, $\lambda \geqq 0$ and (4.10))

$$= - \lambda^t g(x)$$

$$\geqq 0 \qquad \text{(by (4.1) and (4.9))}.$$

Hence,

$$\alpha^t F(x) \geqq \alpha^t G(u, \alpha, \lambda, z).$$

Thus $F(x) \not\leqq G(u, \alpha, \lambda, z)$.

Theorem 4.2 *(Strong duality) — Let x be properly efficient solution of (SP). Then there exists $\alpha \in \mathbb{R}^k$, $\lambda \in \mathbb{R}^m$, $z \in \mathbb{R}^n$ such that (x, α, λ, z) is properly efficient for (SD) and the two objectives are equal.*

PROOF : The theorem can be proved on the lines of proof of Theorem 3.3 and using (4.2) to (4.5).

5. REMARK

The above results can be easily extended for nondifferentiable multiobjective non-convex fractional program.

Consider the following nondifferentiable nonconvex multiobjective fractional programming problem :

$$\text{(FP)} \qquad v\text{-Min } \frac{f(x)}{h(x)} = \left(\frac{f_1(x)}{h_1(x)}, \frac{f_2(x)}{h_2(x)}, \ldots, \frac{f_k(x)}{h_k(x)} \right)^t$$

$$\text{subject to} \quad g(x) \leqq 0, \quad x \in C.$$

Here f, $-h$ and g are contiunuous convex functions defined on a locally convex real space X with $f_i(\cdot) \geqq 0$, $h_i(\cdot) > 0$; $i = 1, 2, ..., k$.

Assumption (A2) — The convex sets ri (dom f_i), ri (dom h_i); $i = 1, 2, ..., k$ have a point in common, so that

$$\partial\left(\sum_{i=1}^{k} f_i - \sum_{i=1}^{k} h_i\right)(x) = \sum_{i=1}^{k} \partial f_i(x) - \sum_{i=1}^{k} \partial h_i(x) \quad \forall \quad x$$

($f_i, -h_i$; $i = 1, 2, ..., k$ are proper convex functions).

Consider the following convex multiobjective program :

(VP′) Minimize $(f_1(x), f_2(x), ..., f_k(x), -h_1(x), -h_2(x), ..., -h_k(x))^t$

subject to $g(x) \leqq 0, \quad x \in C.$

Under Assumption (A2), the following Kuhn-Tucker type conditions are necessary and sufficient for x to be properly efficient for (VP′) and hence by equivalence of the fractional program (FP) with (VP′) in the sense of Bhatia and Datta[6], these conditions are necessary for x to be properly efficient for (FP) :

$$0 \in \sum_{i=1}^{k} \alpha_i \partial f_i(x) - \sum_{i=1}^{k} \beta_i \partial h_i(x) + \sum_{j \in J(x)} \lambda_j \partial g_j(x) - [D_F^{=}(x)]^+ + N_c(x)$$

$$\alpha > 0, \beta \geqq 0; \lambda_j \geqq 0 \quad \forall \quad j \in J(x).$$

Based on these conditions we have the following dual problem to (FP).

(FD) $$v\text{-Max } \frac{\beta}{\alpha} = \left(\frac{\beta_1}{\alpha_1}, \frac{\beta_2}{\alpha_2}, ..., \frac{\beta_k}{\alpha_k}\right)^t$$

subject to

$$0 \in \sum_{i=1}^{k} (\alpha_i \partial f_i(u) - \beta_i \partial h_i(u)) + \sum_{j=1}^{m} \lambda_j \partial g_j(u) - [D_F^{=}(u)]^+ + N_c(u)$$

and $$\sum_{i=1}^{k} (\alpha_i f_i(u) - \beta_i h_i(u)) + \sum_{j=1}^{m} \lambda_j g_j(u) \geqq 0,$$

where $\alpha_i > 0, \beta_i \geqq 0, i = 1, 2, ..., k; \lambda_j \geqq 0; j = 1, 2, ..., m.$

and $g_F(u) \leqq 0, \quad u \in C.$

Theorem 5.1 (*Weak duality*) — *Let* x *and* $(u, \alpha, \beta, \lambda)$ *be feasible for (FP) and (FD) respectively, then*

$$\frac{f(x)}{h(x)} \not< \frac{\beta}{\alpha}.$$

Theorem 5.2 (*Strong duality*) — *Let* x *be properly efficient solution for (FP). Then there exist* $\alpha \in \mathbb{R}^k$, $\beta \in \mathbb{R}^k$, $\lambda \in \mathbb{R}^m$ *such that* $(x, \alpha, \beta, \lambda)$ *is properly efficient for (FD).*

Theorem 5.3 (*Strict converse duality*) — *Let* x *and* $(u, \alpha, \beta, \lambda)$ *be feasible solutions for (FP) and (FD) respectively with*

$$\frac{f(x)}{h(x)} = \frac{\beta}{\alpha}.$$

If atleast one of the functions $f_i - h_i$; $i = 1, 2, \ldots, k$ *and* g_j; $j = 1, 2, \ldots, m$; *be strictly convex with the corresponding multiplier positive, then* u *is properly efficient for (FP).*

All the above theorems can be proved on the same lines as for the case of nondifferentiable multiobjective convex program.

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