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THE REGRESSION PERIOD OF A SATELLITE MOVING UNDER THE GRAVITATIONAL FORCES OF THE SUN, THE MOON, THE EARTH AND THE SOLAR RADIATION PRESSURE

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The regression period of a satellite moving under the gravitational forces of the sun, the moon, the earth (including ellipticity of the earth's equator) and the solar radiation pressure has been studied. It is seen that due to the earth's equatorial ellipticity the mean value about which the regression period oscillates is slightly reduced, due to solar radiation pressure the regression period always decreases and due to the combined effect of both these phenomena it varies between 0.0978 years and 411.16 years for different orbital inclinations and altitudes.

Key Words : Gravitational Forces; Solar Radiation; Regression Period

NOTATIONS

A_i, B_i, C_i, D_i	-	amplitudes of *i*th oscillatory components in the equations of motion,
q	-	solar radiation pressure parameter,
r	-	vector from the centre of the earth to the satellite,
t	-	time,
α	-	inclination of the satellite orbital plane to the reference plane,
α_1	-	inclination of the reference plane to the ecliptic,
α_m	-	inclination of the moon's orbital plane to the ecliptic,
β	-	coefficient due to the solar radiation pressure,
Γ	-	angle measured from the minor axis of the earth's equatorial ellipse to the satellite,
ϕ	-	orbital angle of the earth around the sun,

θ	-	orbital angle of the satellite around the erath,
ψ	-	satellite orbital regression angle,
ω_i	-	oscillatory frequency of the *i*th component in the equations of motion,
$r_0\ \alpha_0\ \dot{\theta}_0\ \dot{\psi}_0$	-	steady state values of the corresponding variables,
T_R	-	Regression rate of the orbital plane,
G	-	Universal gravitational constant = 6.668×10^{-8} dyne cm^2,
J_2	-	coefficient due to the oblateness of the earth = 1.08219×10^{-3},
$J_2^{(2)}$	-	coefficient due to the earth's equatorial ellipticity = 2.32×10^{-6},
M_s	-	mass of the sun = $(332, 946.8 \times M_E)$gm,
M_E	-	mass of the earth = 600.06780×10^{25} gm,
M_m	-	mass of the moon = 7.380×10^{25} gm,
R_0	-	mean earth radius = 3963 miles,
R	-	distance between the centre of sun to the centre of mass of the earth-moon system = 1.4959965×10^8 gm,
ε	-	Obliquity = 23°27′,
$\dot{\theta}_m$	-	moon's orbital rate = 0.22998 rad/solar day,
$\dot{\psi}_m$	-	regression rate of the moon's orbital plane, = $-\ 9.249 \times 10^{-4}$ rad/solar day
$\dot{\phi}$	-	earth's angular rate = 0.0172 rad/solar day,
$\dot{\theta}_E$	-	earth's rotation rate = 6.3004 rad/solar day.

INTRODUCTION

It was shown by Bhatnagar and Mehra[1, 2] that for an orbit of given radius an orbital orientation can be found which remains invariant relative to inertial space, taking this plane as the reference plane, the equations of motion of a satellite moving under the gravitational forces of the sun, the moon, the earth (including ellipticity of the earth's equator) and the solar, radiation pressure in a rotating frame of reference were determined in the form :

$$\dot{\alpha} = \dot{\alpha}_0 + \sum_{i=1}^{182} A_i \sin \omega_i t \qquad \text{... (1)}$$

$$\dot{\psi} = \dot{\psi}_0 + \frac{1}{\sin \alpha_0} \sum_{i=1}^{182} B_i \cos \omega_i t \qquad \text{... (2)}$$

$$(\ddot{r} - r\dot{\theta}^2) + \frac{GM_E}{r^2} = \xi + \sum_{i=1}^{179} C_i \cos \omega_i t \qquad \text{... (3)}$$

$$\frac{d}{dt}(r^2 \dot{\theta}) = \eta + \sum_{i=1}^{179} D_i \sin \omega_i t \qquad \text{... (4)}$$

where

$$\dot{\alpha}_0 = -\frac{3\,\dot{\theta}_0 J_2^{(2)} R_0^2}{r_0^2} \cos(\varepsilon - \alpha_1) \sin \alpha_0 \sin 2\Gamma_0$$

$$\dot{\psi}_0 = -\frac{3\,\dot{\phi}^2}{8\,\dot{\theta}_0}\left[\left\{1 + \frac{\dot{\theta}_m^2}{2\mu\,\dot{\phi}^2}(2 - 3\sin^2\alpha_m)\right\}(2 - 3\sin^2\alpha_1) + \frac{3\beta}{\dot{\phi}^2 R^3}\sin^2\alpha_1\right.$$

$$\left. + \frac{2J_2\,\dot{\theta}_0^2 R_0^2}{\dot{\phi}^2 r_0^2}\{2 - 3\sin^2(\varepsilon - \alpha_1)\}\right]\cos\alpha_0$$

$$+ \frac{3\,\dot{\theta}_0 J_2^{(2)} R_0^2}{2r_0^2}\{2 - 3\sin^2(\varepsilon - \alpha_1)\} \cos\alpha_0 \cos 2\Gamma_0 \qquad \text{... (5)}$$

$$\xi = -\frac{3GM_E J_2 R_0^2}{8r_0^4}\{2 - 3\sin^2(\varepsilon - \alpha_1)\}(2 - 3\sin^2\alpha_0)$$

$$+ \frac{r_0\,\dot{\phi}^2}{8}(2 - 3\sin^2\alpha_0)(2 - 3\sin^2\alpha_1)$$

$$+ \frac{r_0\,\dot{\theta}_m^2}{16\mu}(2 - 3\sin^2\alpha_0)(2 - 3\sin^2\alpha_1)(2 - 3\sin^2\alpha_m) + \frac{r_0\beta}{R^3}$$

$$+ \frac{9GM_E J_2^{(2)} R_0^2}{4r_0^4}\left[(1 + \cos^2\alpha_0)\{1 + \cos^2(\varepsilon - \alpha_1)\}\right.$$

$$\left. + 2\sin^2(\varepsilon - \alpha_1)\sin^2\alpha_0\right] \cos 2\Gamma_0$$

$$\eta = 6\,\dot{\theta}_0^2 J_2^{(2)} R_0^2 \cos(\varepsilon - \alpha_1)\cos\alpha_0 \sin 2\Gamma_0$$

α_1 is defined by

$$\tan 2\alpha_1 = \frac{2\left(K - \dfrac{2\,\dot{\theta}_0^2 J_2^{(2)} R_0^2}{\dot{\phi}^2 r_0^2}\cos 2\Gamma_0\right)\sin 2\varepsilon}{\left[1 + \dfrac{\dot{\theta}_m^2}{2\mu\dot{\phi}^2}(2 - 3\sin^2\alpha_m) - \dfrac{\beta}{\dot{\phi}^2 R^3}\right] + 2\left(K - \dfrac{2\dot{\theta}_0^2 J_2^{(2)} R_0^2}{\dot{\phi}^2 r_0^2}\cos 2\Gamma_0\right)\cos 2\varepsilon} \qquad \text{... (6)}$$

we may note that Γ_0 during the course of study has been used in two ways (i) Γ_0, the value of Γ at a particular instant (ii) Γ_0 the value of Γ at synchronous altitude, that is, the steady state value of Γ

$$\mu = \frac{M_E + M_m}{M_m}, \quad \dot{\theta}_0^2 = \frac{GM_E}{r_0^3}$$

$$\beta = GM_s(1-q), \quad K = \frac{GM_E J_2 R_0^2}{\dot{\phi}^2 r_0^5}$$

where α, ψ determine the orbital plane and r, θ the position of the satellite in the orbital plane. The amplitudes A_i, B_i and C_i, D_i are functions of any or all the quantities α_0, α_1, α_m, r_0, J_2, $J_2^{(2)}$ and β and the frequencies ω_i are linear combinations of $\dot{\theta}_0$, $\dot{\theta}_m$, $\dot{\phi}$, $\dot{\psi}_m$ and $\dot{\psi}_0$. All symbols have been defined in the list of notations.

The orientation of the orbital plane was studied with the help of eqs. (1) and (2). However, the regression period of the orbital plane is given by :

$$T_R = \frac{2\pi}{|\dot{\psi}_0|} \qquad \dots (7)$$

where $\dot{\psi}_0$ is given by (5). In section 2, the orbital regression rate in some special cases has been discussed and in section 3, the effects of the earth's equatorial ellipticity and the solar radiation pressure on the regression period of the orbital plane are discussed in the following four cases :

Case 1 : $J_2^{(2)} = 0$, $q = 1$ (Frick[3]).

Case 2 : $J_2^{(2)} \neq 0$, $q = 1$.

Case 3 : $J_2^{(2)} = 0$, $q \neq 1$.

Case 4 : $J_2^{(2)} \neq 0$, $q \neq 1$.

2. Orbital Regression Rate in Special Cases

Case I (a) : *When only earth's oblateness (excluding ellipticity of its equator) is taken into account* — In this case the effect is given by setting $\dot{\phi}$, $\dot{\theta}_m$, β and $J_2^{(2)}$ equal to zero in (5). In this case α_1 is equal to ε [eq. (6)]. This means that the reference plane coincides with the earth's equatorial plane and the regression takes place about the axis of the earth and the regression rate is given by

$$\dot{\psi}_0 = \frac{-3J_2 R_0^2 \dot{\theta}_0}{2r_0^2} \cos \alpha_0$$

which is the well known result for the regression due to oblateness of the earth.

Case I (b) : *When the effect of earth's oblateness including ellipticity of its equator is taken into account* — This effect is given by setting $\dot{\phi}$, $\dot{\theta}_m$ and β equal to zero in (5). In this case α_1 is again equal to ε [eq. (6)] and the reference plane coincides with the earth's equatorial plane and the regression takes place about the axis of the earth. The regression rate is given by

$$\dot{\psi}_0 = \frac{3R_0^2 \dot{\theta}_0}{r_0^2}\left[-\frac{1}{2} J_2 + J_2^{(2)} \cos 2\Gamma_0 \right] \cos \alpha_0.$$

Case II : *Effect of Sun and Moon* — For extremely high altitude orbits, the effect of earth's oblateness is negligible. Setting J_2 and $J_2^{(2)}$ equal to zero in Eq. (6), we have α_1 equal to zero. This means the reference plane coincides with the plane of the ecliptic and the regression takes about the normal to the ecliptic. The regression rate is here given by

$$\dot{\psi}_0 = -\frac{3\dot{\phi}^2}{4\dot{\theta}_0} \cos \alpha_0 - \frac{3\dot{\theta}_m^2}{8\mu\, \dot{\theta}_0} (2 - 3 \sin^2 \alpha_m) \cos \alpha_0 - \frac{3\beta}{\dot{\phi}^2 R^3} \sin^2 \alpha_1 \cos \alpha_0.$$

Here the first term is due to the effect of the sun and the second is due to the moon. Numerical computation shows that the regression rate due to the moon is approximately twice that due to the sun.

Case III (a) : *Lunar regression under the effect of the earth's oblateness and the sun's gravitational force* — The results determined above can be used to determine the regression of the moon itself under the effect of the earth's oblateness (excluding ellipticity) and the sun's gravitational force. In this case eq. (5) becomes

$$\dot{\psi}_m = -\frac{3\dot{\phi}^2}{4\dot{\theta}_m} \cos \alpha_m \left[1 + \frac{\dot{\theta}_m^2 J_2 R_0^2}{\dot{\phi}^2 \rho_0^2} (2 - 3 \sin^2 \varepsilon) \right].$$

In $\dot{\psi}_m$, the first term in the bracket is due to the sun's gravitational force and the second is due to the earth's oblateness. In fact, the oblateness term is very small and therefore

$$\dot{\psi}_m \simeq -\frac{3\dot{\phi}^2}{4\dot{\theta}_m} \cos \alpha_m$$

$$\simeq -\frac{3\dot{\phi}^2}{4\dot{\theta}_m}, \quad \alpha_m \text{ is taken small.} \qquad \text{... (8)}$$

From this relation it is found that the regression period is nearly 17.9 years whereas the actual period is 18.6 years. This difference is due to the fact that eq. (8) does not include higher order terms which are essential for calculating the orbital regression rate of the moon. If higher order terms are included then

$$\dot{\psi}_m = -\frac{3\dot{\phi}^2}{4\dot{\theta}_m}\left[1 - \frac{3}{8}\left(\frac{\dot{\phi}}{\dot{\theta}_m}\right) - \frac{91}{32}\left(\frac{\dot{\phi}}{\dot{\theta}_m}\right)^2 \right]$$

which gives regression rate as 18.6 years. In the case of artificial satellites for calculating regression rate higher order terms are not necessary as then $\dot{\phi}/\dot{\theta}_m$ is of the order of 1/365 or less.

3. Regression Period 'T_R' of the Orbital Plane

The regression period T_R, that is the time required by the normal to the orbital plane to make one complete rotation about the normal to the reference plane is given by eq. (7). This equation contains $\dot{\psi}_0$ given by eq. (5). It is observed that $\dot{\psi}_0$ contains α_1 given by eq. (6). From eqns. (5), (6) and (7), T_R can be determined as a function of constant orbital inclination α_0 relative to the reference plane, the orbital radius r_0, the solar radiation pressure β and the ellipticity angle Γ_0.

The study has been done in the four cases mentioned earlier. In all these cases α_0 is measured in terms of degrees, r_0 in terms of earth's radii and T_R in years.

Case 1 : $J_2^{(2)} = 0$, $q = 1$ (Figs. 1 and 2) — We observe that the regression period increases as both the orbital inclination α_0 and the altitude Γ_0 increase. It is very large for α_0 close to 90° for a fixed r_0 and attains a maximum in the vicinity of nine times the earth's radii for a fixed α_0. Also, the minimum value of T_R is 0.0988 years when $r_0 = R_0$, $\alpha_0 = 0°$ and the maximum value is 409.9800 years when $r_0 = 9R_0$, $\alpha_0 = 80°$. However, at zero inclination at synchronous altitude T_R is 52.84 years. Condition for synchronism is given by $\dot{\theta}_0 = \dot{\theta}_E - \dot{\psi}_0$. On substituting the values of $\dot{\theta}_0$, $\dot{\theta}_E$ and $\dot{\psi}_0$ we can determine synchronous altitude for different values of α_0, q and Γ_0.

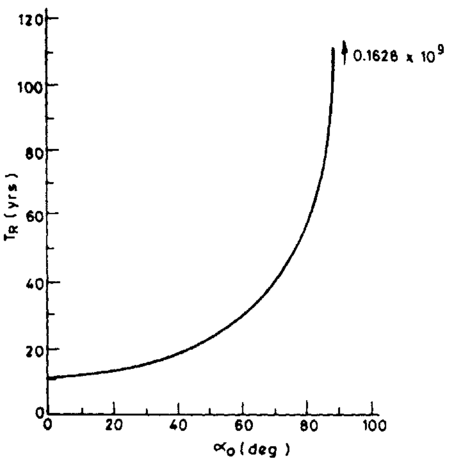

FIG. 1. Regression period for fixed orbital radius ($J_2^{(2)} = 0$, $q = 1$, $r_0/R_0 = 4$).

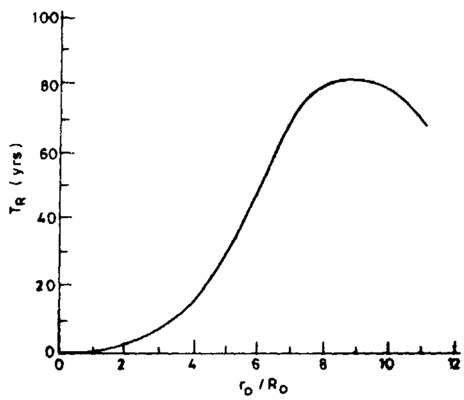

FIG. 2. Regression period for fixed orbital inclination ($J_2^{(2)} = 0$, $q = 1$, $\alpha_0 = 30°$).

Case 2 : $J_2^{(2)} \neq 0$, $q = 1$ (Figs. 3 and 4) — We observe that due to the earth's equational ellipticity there is no change in T_R when $|\Gamma_0| = 45°$ as then the effect of ellipticity is nil but for $0° \leq |\Gamma_0| \leq 180°$ the change in $|T_R|$ i.e. $|\Delta T_R|$ oscillates between the values corresponding to $|\Gamma_0| = 0°$ and $|\Gamma_0| = 90°$ with the mean position at $|\Gamma_0| = 45°$. For example, for fixed $r_0 = 4R_0$, $0° \leq \alpha_0 \leq 80°$ and $0° \leq |\Gamma_0| \leq 180°$, $|\Delta T_R|$ oscillates between 0.1190 years and 0.6860 years and for fixed $\alpha_0 = 0°$ and different r_0, $0° \leq |\Gamma_0| \leq 180°$, $|\Delta T_R|$ oscillates between 0 year and 0.3660 years.

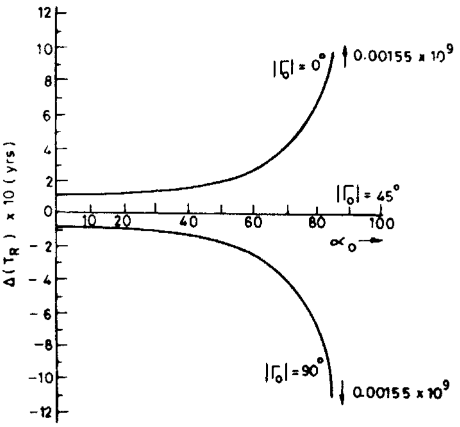

FIG. 3. Change in regression period due to ellipticity from values when $|\Gamma_0| = 45^0$ for fixed orbital radius ($J_2^{(2)} = 0$, $q = 1$, $r_0/R_0 = 4$).

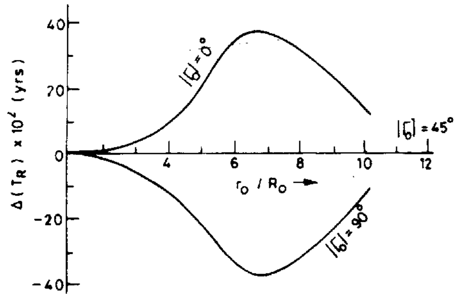

FIG. 4. Change in regression period due to ellipticity for fixed orbital inclination ($J_2^{(2)} \neq 0$, $q = 1$, $\alpha_0 = 0^0$).

Case 3 : $J_2^{(2)} = 0$, $q \neq 1$ (Figs. 5 and 6) — We observe that due to the solar radiation pressure, T_R decreases as the orbital inclination α_0 increases. It further decreases with the decrease in the solar radiation parameter q. For surface orbits $(1 \leq \frac{r_0}{R_0} \leq 3)$ the change is insignificant but for $(4 \leq \frac{r_0}{R_0} \leq 10)$, it is significant attaining a maximum in the vicinity of $r_0 = 7R_0$. For $1 \leq q \leq .9$, $0^0 \leq \alpha_0 \leq 80^0$, $1 \leq \frac{r_0}{R_0} \leq 10$, $|\Delta T_R|$ varies between 0 year and .42 years.

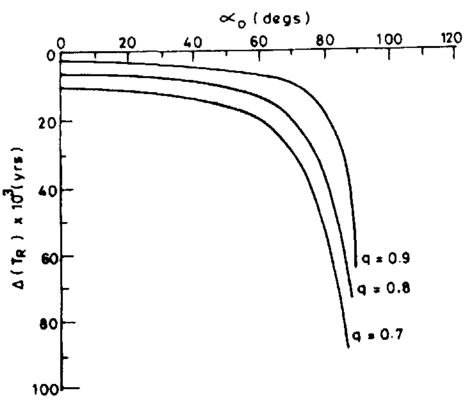

FIG. 5. Change in regression period due to solar radiation pressure for fixed orbital radius ($J_2^{(2)} = 0$, $q \neq 1$, $r_0/R_0 = 4$).

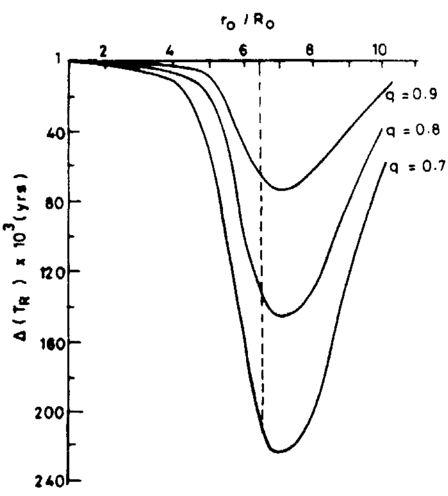

FIG. 6. Change in regression period due to solar radiation pressure for fixed orbital inclination ($J_2^{(2)} = 0$, $q \neq 1$, $\alpha_0 = 0^0$).

Case 4 : $J_2^{(2)} \neq 0$, $q \neq 0$ — We observe that as $|\Gamma_0|$ increases from 45° to 90° and q decreases from 1 to 0.9, $|\Delta T_R|$ decreases by 0.1160 years at $\alpha_0 = 0°$ to 0.6670 years at $\alpha_0 = 80°$. As $|\Gamma_0|$ changes from 45° to 0° and q decreases from 1 to 0.9, $|\Delta T_R|$ increases. For $90° \leq |\Gamma_0| \leq 180°$ and $q = 0.9$, $|\Delta T_R|$ oscillates between the values for $|\Gamma_0| = 0°$, $q = .9$ and $|\Gamma_0| = 90°$, $q = 0.9$ with the mean position almost at $|\Gamma_0| = 45°$, $q = 1$. Also, we observe that $|\Delta T_R|$ increases with r_0 attaining a maximum in the vicinity of $r_0 = 7R_0$ where as the change $|\Delta T_R|$ is 0.4350 years and then this change starts decreasing. We have further observed from our computations that T_R varies between 0.0978 years and 411.16 years for different orbital inclinations and altitudes.

Conclusion

We conclude that due to the earth's equatorial ellipticity T_R oscillates between the position corresponding to $|\Gamma_0| = 0°$ and $|\Gamma_0| = 90°$ with mean position at $|\Gamma_0| = 45°$. Due to the solar radiation pressure, it always decreases. Due to the combined effect of the earth's equatorial ellipticity and the solar radiation pressure, T_R for $1 \leq \frac{r_0}{R_0} \leq 10$, $0° \leq \alpha_0 \leq 80°$, $0° \leq |\Gamma_0| \leq 180°$ varies between 0.0978 years (min. at $r_0 = R_0$, $\alpha_0 = 0°$, $|\Gamma_0| = 90°$, $q = .7$) and 411.16 years (max. at $r_0 = 9R_0$, $\alpha_0 = 80°$, $|\Gamma_0| = 0°$, $q = 1$). However, it is maximum at $r_0 = 9R_0$ but the effect of earth's equatorial ellipticity and the solar radiation pressure is maximum at $r_0 = 7R_0$.

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