

## ON A MATRIX TRANSFORMATION OVER NON-ARCHIMEDIAN FIELDS

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An inclusion theorem of the matrix transformation of the sequence space  $l_\alpha$  to  $l_\alpha$  over a non-archimedian field is obtained by using functional analysis techniques.

**Key Words :** Matrix Transformation; Non-archimedian Field; Functional Analysis

§1. In a recent paper, Natarajan<sup>1</sup> characterized all the matrices transforming the sequence space  $l_\alpha$  to  $l_\alpha$  over non-archimedian fields. This result is of interest because we do not have classical analogue of such a characterization of matrices for the sequence space  $l_\alpha$  defined over the field of complex numbers or real numbers. His method of proof is a manipulation of the original Schur technique<sup>2</sup>. The object of this paper is to obtain a more general condition characterising such matrices by using functional analysis techniques. We shall define the required sequence space over non-archimedian fields in section 2 and describe required functional analysis preliminaries in section 3. The main theorem of this short note is established in section 4.

§2. Let  $K$  be a non-trivial non-archimedian field and  $K$  be complete under the metric of valuation. Let  $x = (x_n), x_n \in K$  be a sequence defined over  $K$ . Unless and otherwise stated the variation for  $n$  is 1, 2, 3, ... . In what follows, the notion of convergence and boundedness will be in relation to the metric of valuation.

$$\text{Let } l_\alpha = \left\{ x = (x_n), x_n \in K : \sum_{n=1}^{\infty} |x_n|^\alpha < \infty \right\}.$$

It is easy to see that  $l_\alpha$  is a non-archimedian complete linear metric space with the metric induced by the norm

$$\|x\| = \left[ \sum_{n=1}^{\infty} |x_n|^\alpha \right]^{1/\alpha}.$$

§3. Functional analysis Preliminaries : It is known that a complete linear metric space is of second category Monna<sup>3</sup> (p. 1137). Since  $l_\alpha$  is a complete metric space over  $K$ ,  $l_\alpha$  is of second category. For such a space the following analogue of a Theorem of Banach<sup>4</sup> (p. 19) is valid.

**Theorem 1** — Let  $\{F_n(x)\}$  be a sequence of non-negative real valued continuous functions defined on a complete linear metric space  $E$  such that for each  $x \in E$  there exists a real number  $m(x)$  with the property that  $F_n(x) \leq m(x)$  for all  $n$ . Then there exists a real number  $M$  and a closed sphere  $S$  in  $E$  such that

$$F_n(x) \leq M \text{ if } x \in S \text{ for all } n \geq 1.$$

Let  $\{f_n\}$  be a sequence of continuous linear functionals on a non-archimedean Banach space  $E$  with values in  $K$ . The following convergence theorem can be proved as in the classical case.

**Theorem 2** — If  $f(x) = \lim_{n \rightarrow \infty} f_n(x)$  exists for each  $x \in E$ , then  $f$  is a continuous linear functional on  $E$ .

§4. Matrix transformation of  $l_\alpha$  into  $l_\alpha$  over  $K$ .

Let us consider the matrix transformation

$$y_n = \sum_{p=1}^{\infty} a_{np} x_p, \quad n = 1, 2, 3, \dots, a_{np} \in K. \quad \dots (4.1)$$

By using functional analysis techniques, we shall find a necessary and sufficient condition on the infinite matrix  $A = (a_{np})$  in order that the matrix should transform  $l_\alpha$  into  $l_\alpha$ . The class of all such matrices is denoted by  $(l_\alpha, l_\alpha)$ .

**Theorem 3** — A necessary and sufficient condition that  $(y_n) \in l_\alpha$  whenever  $(x_n) \in l_\alpha$  is

$$\sup_{n,p} |a_{np}|^\alpha \leq M < \infty, \quad 1 \leq p < \infty, \quad 1 \leq n < \infty.$$

PROOF : The condition is sufficient.

From (4.1), we have

$$|y_n|^\alpha \leq \sup_{1 \leq p < \infty} |a_{np}|^\alpha |x_p|^\alpha.$$

Hence,  $\sum |y_n|^\alpha \leq M \sum |x_p|^\alpha$  from the hypothesis.

Hence, if  $(x_p) \in l_\alpha$ , from the above step we get  $\sum_n |y_n|^\alpha \leq M'$ .

Hence, if  $\sup_{n,p} |a_{np}|^\alpha \leq M$ , then  $A \in (l_\alpha, l_\alpha)$ .

To prove the necessity of the condition, consider the functional  $f_{mn}$  on  $l_\alpha$  defined by

$$f_{mn}(x) = \sum_{p=1}^m a_{np} x_p, \quad x = (x_p) \in l_\alpha.$$

These are continuous linear functionals on  $l_\alpha$  to  $K$ . The existence of  $A$  transform of  $l_\alpha$  implies for each  $x \in l_\alpha$

$$\lim_{m \rightarrow \infty} f_{mn}(x) = f_n(x) \text{ exists.}$$

Hence, by Theorem 2,  $f_n(x)$  is a continuous linear functional on  $l_\alpha$  into  $K$ . Since  $A \in (l_\alpha, l_\alpha)$  we have for each  $x$ ,

$$\sum_n |f_n(x)|^\alpha < \infty. \tag{4.2}$$

(4.2) implies that  $|f_n(x)|^\alpha$  is bounded as  $n \rightarrow \infty$ .

Hence,  $g_n(x) = |f_n(x)|^\alpha$  is bounded for all  $x$  and all  $n$ .

Now let us consider the functional

$$g_n(x) = |f_n(x)|^\alpha.$$

Then  $g_n(x)$  are non-negative real valued continuous functions on  $l_\alpha$ . For each  $x \in l_\alpha$ ,  $g_n(x)$  is bounded for all  $n$ . Hence by Theorem 1, there is a constant  $M$  and a closed sphere  $S$  in  $l_\alpha$  such that

$$g_n(x) = |f_n(x)|^\alpha \leq M \text{ for all } x \in S, n = 1, 2, 3, \dots$$

Now let us take the sphere as

$$S = \left\{ x = (x_p) / \left( \sum_{p=1}^\infty |x_p|^\alpha \right)^{1/\alpha} < d \text{ for all } p. \right\}$$

Let  $z$  be an element of  $K$  such that  $|z| = d_1 < d$ . Choose the sequence  $x = (x_n)$  as follows.

$$x_p = z \text{ when } n = p$$

$$x_p = 0 \text{ when } n \neq p.$$

Then 
$$\|x\| = \left( \sum_p |x_p|^\alpha \right)^{1/\alpha} = d_1 < d.$$

Therefore,  $x = (x_p) \in S \subset l_\alpha$

Hence, 
$$|f_n(x)|^\alpha = |a_{np} z|^\alpha \leq M.$$

From the above we get 
$$|a_{np}|^\alpha \leq \frac{M}{|z|^\alpha} = \frac{M}{d_1} \leq C.$$

Hence  $|a_{np}|^\alpha \leq C$  which is independent of  $n$  and  $p$ .

So we have  $\sup_{n,p} |a_{np}|^\alpha \leq C$ ,  $1 \leq p < \infty$ ,  $1 \leq n < \infty$ .

This completes the proof of the theorem.

*Remark* : The condition proved above is not only more general than that of Natarajan<sup>1</sup> but it also reflects the basic nature of the valuation of the field  $K$ .

#### REFERENCES

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