Indian J. pure appl. Math., 29(10) : 1025-1028, October 1998
© Printed in India.

ON THE SUBSPACE $\Gamma(\lambda)$ OF ENTIRE FUNCTIONS

D. SOMASUNDARAM

Professor & Head, Department of Mathematics, Madras University
PG Centre Salem 636 011, *Tamil Nadu*

(*Received* 10 *September* 1997; *Accepted* 1 *May* 1998)

Different subspace of the spaces of entire functions in Nanda *et al.*[2] were of much interest in recent times. The object of this paper is to study the FK-space properties of the subspace $\Gamma(\lambda)$ introduced by Titus[3].

Key Words : Sequence Spaces; Entire Function; Subspace $G(\Gamma)$

1. INTRODUCTION

In the study of certain subspace of entire functions of Ganapathy Iyer[1], Titus introduced $\Gamma(\lambda)$ for a fixed sequence $\lambda = (l_0, l_1, l_2, \ldots l_n \ldots)$ where $n_0\ l_n = 0$ for $n = 0, 1, 2, 3, \ldots$. We shall study FK-space aspect of the space $\Gamma(\lambda)$ which he had not considered in his study. Developing necessary preliminaries in §2, we shall prove the main theorems of this note in §3 and in §4, we shall explain how the representation of continuous linear functionals of $\Gamma(\lambda)$ can be obtained treating $\Gamma(\lambda)$ as an EK-AK space.

2. ENTIRE FUNCTIONS

Let Γ denote the space of entire functions of Ganapathy Iyer[1] and let $\Gamma(\lambda) = \{\alpha \in \Gamma : \lambda \cdot \alpha \in \Gamma\}$, where $\lambda = (l_0, l_1, l_2, \ldots l_n \ldots)$ is a fixed sequence such that no element of λ is zero and

$$\lambda \cdot \alpha = \sum_{n=0}^{\infty} l_n a_n z^n \quad \text{where} \quad \alpha = \sum_{n=0}^{\infty} a_n z^n.$$

As noted by Titus[3], $\Gamma(\lambda) \neq \phi$ and $\Gamma(\lambda)$ is a linear subspace of Γ. As in Γ, we can introduce a topology on $\Gamma(\lambda)$ by the metric d as follows :

$$d_\lambda(\alpha, \beta) = \text{Sup } \{|l_0|\,|a_0 - b_0|,\ |l_n|^{1/n}\,|a_n - b_n|^{1/n},\quad n \geq 1\}.$$

Denoting this topology by T_λ, Titus[3] establishes the following theorem.

Theorem A — *$(\Gamma(\lambda), T_\lambda)$ is a complete metric space if and only if lim inf* $\{|l_0|,\ |l_n|^{1/n}\, n \geq 1\} = L = 0$.

The topology T of Γ is given by the metric d where

$$d(\alpha, \beta) = \text{Sup } \{|a_0 - b_0|, \ |a_n - b_n|^{1/n} \ n \geq 1\}, \quad f, g \in \Gamma$$

where we take $f = \sum_{n=0}^{\infty} a_n z^n$ and $g = \sum_{n=0}^{\infty} b_n z^n$.

3. SEQUENCE SPACES

3.1. Since $\Gamma(\lambda)$ is a subspace of Γ, it has two topologies T_λ on its own and another topology inherited from T of Γ. First we note the following property of Γ.

Theorem 1 — *The space (Γ, T) is an FK-AK space.*

PROOF : As proved by Ganapathy Iyer[1], (Γ, T) is a complete linear metric space. So it is an F-space. Hence, we shall show that it is a K-space. For this it is enough if we prove that the coordinate wise projections P_n's are continuous.

Let $x = (x_n) \in \Gamma$ and $a = (a_n)$ be a fixed element of Γ. Then $d(x, a) = \text{Sup } \{|x_0 - a_0|, |x_n - a_n|^{1/n}, \ n > 1\}$. Given $\varepsilon > 0$, choose a $\delta = \varepsilon^{1/n}$ for a fixed n. Hence, $d(x, a) < \delta$ implies $|x_n - a_n|^{1/n} < \delta = \varepsilon^{1/n}$. This proves that $|x_n - a_n| < \varepsilon$ which proves that $|P_n(x) - P_n(a)| < \varepsilon$ for each fixed n. Thus P_n's are continuous at $a \in X$. So P_n's are continuous for all $n = 0, 1, 2, 3, \ldots$. Thus is a K-space.

To prove that it is AK-space, first note that $\phi \in \Gamma$, where ϕ is the set of all finite sequences.

If $a = (a_n)$, the nth section of a is defined as $a^{(n)} = (a_0, a_1, a_2, \ldots, a_n, 0, 0, 0, \ldots)$. To prove that Γ is a AK-space, we have to prove $a^{(n)} \to a$ as $n \to \infty$. Now we have

$$d(a^{(n)}, a) = \sup_{k \geq n+1} |a_k|^{1/k} \to 0 \quad \text{as} \quad n \to \infty.$$

This proves that Γ is a AK-space. Thus Γ is an FK-AK space.

3.2. Taking $\lambda = (l_n)$, $l_n \neq 0$ for any n, we shall define the space $\bar{\Gamma}(\lambda)$ as follows :

$$\bar{\Gamma}(\lambda) = \{(x_n) \in \omega : x_n l_n \in \Gamma\}$$

where ω is the set of sequences of complex numbers.

From the definition of $\Gamma(\lambda)$ and $\bar{\Gamma}(\lambda)$, we note that $\Gamma(\lambda) = \Gamma \cap \bar{\Gamma}(\lambda)$. Further for the space $\bar{\Gamma}(\lambda)$, we have the following theorem.

Theorem 2 — *The space $\bar{\Gamma}(\lambda)$ is complete if and only if*

$$\liminf \{|l_0|, |l_n|^{1/n}, \ n \geq 1\} = M > 0.$$

PROOF : Similar to the proof of Theorem 2.1 of Titus[3].

In the background of the FK-space theory, we have the following theorem for the space $\bar{\Gamma}(\lambda)$:

Theorem 3 — *The space* $(\overline{\Gamma}(\lambda), T_\lambda)$ *where* $\overline{T}_\lambda$ *denotes the topology of* $\overline{\Gamma}(\lambda)$ *is an FK-AK space.*

PROOF : Though we can prove the theorem as in Theorem 1, the following proof is more elegant :

First note that by Theorem 1, (Γ, T) is an *FK-AK* space where T is given by the metric

$$d(f, g) = \text{Sup } \{|a_0 - b_0|, |a_n - b_n|^{1/n}, \quad n \geq 1\},$$

where $f \sim (a_n)$ and $g \sim (b_n)$.

Using the notation as in Wilansky[4], let $\Gamma_M = \{x : Mx \in \Gamma\}$ where M is the diagonal matrix given by (l_n) with no l_n is zero. From the definitions of $\overline{\Gamma}(\lambda)$, $\Gamma(\lambda)$ is the domain of the diagonal matrix M so that $\overline{\Gamma}(x) = \Gamma_M = M^{-1}(\Gamma)$. Further $l_n \neq 0$ for any n so that M is a normal matrix. Since M is normal, we get $(\overline{\Gamma}(\lambda), \overline{T}_\lambda)$ is an *FK-AK* space.

Corollary — The space $(\Gamma(\lambda), T_\lambda)$ is an *FK-AK* space.

PROOF : From the definition of Γ, $\overline{\Gamma}(\lambda)$, $\Gamma(\lambda)$, we get $\Gamma(\lambda) = \Gamma \cap \overline{\Gamma}(\lambda)$ where Γ and $\overline{\Gamma}(\lambda)$ are *FK-AK* spaces. Since the intersection of two *FK-AK* spaces is an *FK-AK* space $(\Gamma(\lambda), T_\lambda)$ is an *FK-AK* space where the topology T is the coarset locally convex topology on $\Gamma(\lambda)$ which is finer than $T/\Gamma(\lambda)$ and $\overline{T}_\lambda/\Gamma(\lambda)$. From Wilansky [4, Theorem 15, 59] such a topology T is given by the metric d where

$$d_\lambda(f, g) = d(f, g) + \overline{d}_\lambda(f, g).$$

Using the known properties of *FK*-spaces as found in Wilansky we prove the following theorems :

Theorem 4 — *The following statements are equivalent :*

(*i*) $(\Gamma(\lambda), \overline{T}_\lambda/\Gamma(\lambda))$ *is an FK-space.*

(*ii*) $\Gamma(\lambda) = \overline{\Gamma}(\lambda)$.

(*iii*) $\overline{\Gamma}(\lambda) \subset \Gamma$.

(*iv*) $\overline{T}_\lambda/\Gamma(\lambda)$ *is finer than* $T/\Gamma(\lambda)$ *on* $\Gamma(\lambda)$.

PROOF : (*i*) $\Rightarrow$ (*ii*) Since *FK*-topologies are unique we get $T_\lambda = \overline{T}_\lambda/\Gamma(\lambda)$. Hence $\Gamma(\lambda)$ is closed in $(\overline{\Gamma}(\lambda), \overline{T}_\lambda)$ and since $(\overline{\Gamma}(\lambda), \overline{T}_\lambda)$ is an *FK-AK* space, we get $\Gamma(\lambda) = \overline{\Gamma}(\lambda)$

(*ii*) $\Rightarrow$ (*iii*) is obvious.

(*iii*) $\Rightarrow$ (*iv*) is on account of the monotony of *FK*-topologies.

(*iv*) $\Rightarrow$ (*i*) (*iv*) implies that $T_\lambda = \overline{T}_\lambda/\Gamma(\lambda)$.

Theorem 5 — *The following statements are equivalent* :

(*i*) $\Gamma = \Gamma(\lambda)$,

(*ii*) $\Gamma \subset \overline{\Gamma}(\lambda)$,

(*iii*) $T/\Gamma(\lambda) = T_\lambda$ *on* $\Gamma(\lambda)$.

PROOF : (*i*) $\Rightarrow$ (*ii*) is obvious.

$(i) \Rightarrow (iii)$ Since FK-topologies are unique we have $T = T_\lambda$.

$(iii) \Rightarrow (i)$ $T/\Gamma(\lambda) = T_\lambda$ implies that $\Gamma(\lambda)$ is a closed subspace of the FK-AK space (Γ, T) which implies $\Gamma = \Gamma(\lambda)$.

4. We shall explain how the representation of continuous linear functional on the complete linear metric space $(\Gamma(\lambda), T_\lambda)$ can be obtained by using its FK-AK property. From the Corollary of Theorem 3, $(\Gamma(\lambda), T_\lambda)$ is an FK-AK space and therefore the topological dual of $\Gamma(\lambda)$ is isomorphic to its β-dual $(\Gamma(\lambda))^\beta$ from which we get the set of all continuous linear functional on $\Gamma(\lambda)$.

From the definition, the β-dual of $\Gamma(\lambda)$ is

$$(\Gamma(\lambda))^\beta = \left\{ (c_n) : \sum_{n=0}^{\infty} a_n c_n \text{ is convergent for every } (a_n) \in \Gamma(\lambda) \right\}.$$

So it is enough if we determine the β-dual of $\Gamma(\lambda)$.

Theorem 6 — *Let* $(a_n) \in \Gamma(\lambda)$. *Then we have*

$$(\Gamma(\lambda))^\beta = \left\{ (c_n) : \left| \frac{c_n}{n} \right|^{1/n} \text{ as bounded} \right\}.$$

PROOF : It follows from the Lemma 4.2 of Titus[4].

REFERENCES

1. V. Ganapathy Iyer, *J. Indian math. Soc.* **12** (1948) 13-30.
2. S. Nanda, P. D. Srivastava and K. C. Nayak, *Indian J. pure appl. Math.* **12** (1981) 971-76.
3. J. P. J. Titus, *Math. Student*, **47** (1979) 360-69.
4. A. Wilansky, *Summability Through Functional Analysis*, Elsevier Science Publisher, North Holland (1984).