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SOME SANDWICH TYPE NEAR-RINGS

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Some sandwich type near-rings are constructed using modern topology.

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1. INTRODUCTION

A right near-ring is a triple $(N, +, \cdot)$, where $+$ and $\cdot$ are binary operations on the set N, where $(N, +)$ is a group (not necessarily Abelian) and $(N, \cdot)$ is a semi-group and satisfies the right distributive law :

$$\forall \quad n_1, n_2, n_3 \in N : (n_1 + n_2) \cdot n_3 = n_1 \cdot n_3 + n_2 \cdot n_3.$$

A subgroup $(M, +)$ of a near-ring $(N, +, \cdot)$ is called a subnear-ring of N if $M \cdot M \subseteq M$ and we write $M \leqq N$ following Pilz[1]. A mapping h from near-ring $(N, \oplus, \odot)$ into another near-ring $(N', +, \cdot)$ is called a near-ring homomorphism if

$$\forall \quad m, n \in N : h(m \oplus n) = h(m) + h(n) \quad \text{and} \quad h(m \odot n) = h(m) \cdot h(n).$$

A bijective near-ring homomorphism from a near-ring N onto a near-ring N' is called a near-ring isomorphism and we say that N and N' are isomorphic and symbolically we write $N \cong N'$.

A topological group is a triple $(X, *, \tau)$ such that $(X, *)$ is a group, (X, τ) is a topological space such that the functions $x \to x^{-1}$ and $(x, y) \to x * y$ are continuous. Suppose (X, τ) is a topological space and $(G, *, \sigma)$ is a topological group.

Let $C(X, G) = \{f | f : X \to G$ is continuous$\}$. If $(X, \tau) = (G, \sigma)$, then $(C(X, G), \oplus, o)$, where $(f \oplus g)(x) = f(x) * g(x)$ for any $f, g \in C(X, G)$ and for any $x \in X$; 'o' is the composition of maps is a right near-ring. But K. D. Magill Jr., considered the case $(X, \tau) \neq (G, \sigma)$. Fixing a continuous map $\phi : G \to X$, he defined a binary operation on $C(X, G)$ as $f *_\phi g = fo\phi og$. With such a ϕ, $(C(X, G), \oplus, *_\phi)$ is a near-ring. Such near-rings are called sandwich near-rings. He[2] played with them.

2. PREREQUISTITES

If A and B are any two subsets of a group $(X, *)$, then we take $A^{-1} = \{x^{-1}/x \in A\}$ and $A * B = \{x * y/x \in A,\ y \in B\}$.

Theorem 2.1 — *Let $(X, *, \tau)$ be a topological group and $A \subseteq X$. Then A is open if and only if A^{-1} is open.*

PROOF : Since $(A^{-1})^{-1} = A$, we only need to show that if A is open, then so is A^{-1}. Let $x \in A^{-1}$. $\Rightarrow x^{-1} \in A$. Since $x \to x^{-1}$ is a continuous map, there exists an open set B_x such that $x \in B_x$ and $B_x^{-1} \subseteq A$. This implies that B_x is contained in A^{-1}. Hence, we can express A^{-1} as $A^{-1} = \bigcup_{x \in A^{-1}} B_x$. Thus A^{-1} is a union of open sets. So A^{-1} is open.

Corollary 2.1 — If $(X, *, \tau)$ is a topological group, then the mapping $x \to x^{-1}$ is a homeomorphism.

PROOF : Already we have that $x \to x^{-1}$ is continuous. From the above theorem $x \to x^{-1}$ is an open mapping. Now if $x_1^{-1} = x_2^{-1} = y$ (say) for some $x_1, x_2 \in X$, then $x_1 * x_2^{-1} = x_1 * x_1^{-1} = e = x_1^{-1} * x_1 = x_2^{-1} * x_1$, where e is the identity element of the group $(X, *)$. This means $x_1 = x_2$. Hence, the mapping $x \to x^{-1}$ is one-to-one. We have no doubt about the fact that $x \to x^{-1}$ is onto. Therefore, the mapping $x \to x^{-1}$ is a homeomorphism.

Theorem 2.2 — *Let $(X, *, \tau)$ be any topological group and $A \subseteq X$. Then for any $b \in X$, $b * A$ and $A * b$ are open if and only if A is open.*

PROOF : Since if $b * A = C$, then $A = b^{-1} * C$, it will suffice to show that if A is open, then so are $b * A$ and $A * b$. First, we note that if $b * x_1 = b * x_2$, for some $x_1, x_2 \in X$, then $x_1 = x_2$. Now consider the mapping $f: X \to X$ defined by $f(x) = b^{-1} * x$. Clearly f is a continuous mapping. Since $f^{-1}(A) = b * A$, we have that $b * A$ is open. Similarly, it can be shown that $A * b$ is open if and only if A is open.

Corollary 2.2 — For each point x in a topological group $(X, *, \tau)$, the left and right translations L_a and R_a defined as $L_a(x) = a * x$ and $R_a(x) = x * a$ for any fixed $a \in X$ are homeomorphisms.

PROOF : It is an immediate consequence from the above theorem.

3. SECTION

If (X, τ) is a topological space and $A \subseteq X$, then $cl(A)$, the intersection of all closed sets containing A, is called the closure of A and $Int(A)$, the union of all open sets contained in A, is called the interior of A. Levine[3] introduced the concept of semi-open sets and semi-continuous functions for topological spaces.

Definition 3.1 — Let (X, τ) be a topological space and $A \subseteq X$. A is called a semi-open subset of X iof $A \subseteq cl\,(Int(A))$.

The collection of all semi-open subsets of X is denoted by $SO(X)$.

Definition 3.2 — A function f from a topological space (X, τ) into another topological space (Y, σ) is called semi-continuous if the inverse image of each open set in Y is semi-open in X.

Definition 3.3 — Let (X, τ) and (Y, σ) be any two topological spaces. A mapping $f: X \to Y$ is called an irresolute[4] if the inverse image of each semi-open set in Y is semi-open in X.

Note that every continuous map is semi-continuous and every irresolute map is also semi-continuous.

Theorem 3.1 — *Let (X, τ) and (Y, σ) be any two topological spaces and $f: X \to Y$ be an open, semi-continuous mapping. Then f is an irresolute.*[5]

Corollary 3.1 — Let (X, τ), (Y, σ) and (Z, η) be any three topological spaces. If $f: X \to Y$ is open, semi-continuous and $g: Y \to Z$ is semi-continuous, then $g \circ f: X \to Z$ is semi-continuous.[6]

Theorem 3.2 — *Let $(X, *, \tau)$ be a topological group and $A \subseteq X$. If $b \in X$, then $b * A$ and $A * b$ are semi-open if and only if A is semi-open.*

PROOF : Suppose A is a semi-open in X. Define $f: X \to X$ by $f(x) = b^{-1} * x$. It is not only a continuous mapping but by the Corollary 2.4, it is also a homeomorphism. This implies that f is open and semi-continuous.

So f is an irresolute. Hence, $f^{-1}(A)$ is a semi-open in X. But $f^{-1}(A) = b * A$. Therefore, $b * A$ is semi-open. Similarly, it can be shown that $A * b$ is also semi-open.

Conversely suppose that $b * A$ is semi-open. If we let $b * A = C$, then $A = b^{-1} * C$. By the above part of the proof $b^{-1} * C$ is semi-open in X. In the same way one can show that if $A * b$ is semi-open, then A is semi-open.

Let $(G, +, \sigma)$ be a topological group and (X, τ) be a topological space. Let $S\ C(G, X) = \{f / f : X \to G$ is semi-continuous$\}$. But we cannot define a binary operation $\oplus$ on $S\ C(G, X)$ as $f \oplus g = f + g$ since $SO(X)$ need not be closed under finite intersections. The composition of maps also is not closed. Suppose $(G, \sigma) \neq (X, \tau)$ and fix some semi-continuous map $\phi: G \to X$. Then if we define $*_\phi (f, g) = f \circ \phi \circ g$ for any $f, g \in S\ C(G, X)$, we can observe that $*_\phi$ need not be closed.

The main purpose of this paper is to overcome these obstructions and to construct sandwich type near-rings using some concepts from modern topology. For this we impose two more conditions, one on X and the other by considering a super class of semi-continuous functions.

Definition 3.4 — A topological space (X, τ) is said to an extremally disconnected[7] if the closure of every open set in it is open.

If (X, τ) is a topological space, then we know that $SO(X)$ need not be a topology on X in general. Njåstad[8] in the corollary to the Proposition 7, proved that $SO(X)$ forms a topology on X if and only if (X, τ) is an extremally disconnected space. With this we can overcome the first obstruction.

Let (X, τ) be an extremally disconnected space and $(G, +, \sigma)$ be a topological group with $(X, \tau) \neq (G, \sigma)$.

Consider $OG_S^X = \{f / f : X \to G$ is open and semi-continuous$\}$.

Define $\oplus\ OG_S^X \times OG_S^X \to OG_S^X$ by $f \oplus g = f + g : \forall\ f, g \in OG_S^X$. Let $f, g \in OG_S^X$. Clearly, $f \oplus g$ is open. Now we show that $f \oplus g$ is also semi-continuous. Let S be any open subset of G. First it is easy to see that $(f \oplus g)^{-1}(S) = f^{-1}(S - g(x)) \cap g^{-1}(S - f(x))$. By the Theorem 2.3,

$S-g(x)$ and $S-f(x)$ are open. Since f and g are semi-continuous mappings, $f^{-1}(S-g(x))$ and $g^{-1}(S-f(x))$ are semi-open. Now $f^{-1}(S-g(x)) \cap g^{-1}(S-f(x))$ is also a semi-open set in X as X is an extremally disconnected space. Hence, $(f \oplus g)^{-1}(S)$ is semi-open.

Moreover, it is easy to see that $(OG_S^X, \oplus)$ is a group. Fix some $\phi: G \to X$, an open, semi-continuous map. Let f and g be any two members of OG_S^X. Consider $f \circ \phi \circ g$. Clearly, $f \circ \phi \circ g$ is open and semi-continuous by the Corollary 3.5. Hence, we can define a binary operation on OG_S^X by $f *_\phi g = f \circ \phi \circ g$ for any $f, g \in OG_S^X$. It is evident that $*_\phi$ is associative. So $(OG_S^X, *_\phi)$ is a semi group. Let $f, g, h \in OG_S^X$. $(f \oplus g) *_\phi h = (f \oplus g) \circ \phi \circ h = ((f+g) \circ h = (f \circ \phi + g \circ \phi) \circ h = f \circ \phi \circ h + g \circ \phi \circ h = f *_\phi h + g *_\phi h$. Therefore, $(OG_S^X, \oplus, *_\phi)$ is a right near-ring.

We cal also overcome the second obstruction in another way. For that let (X, τ) be an extremally disconnected space and $(G, +, \sigma)$ be a topological group.

Consider $IG^X = \{\{f/f : X \to G$ is an irresolute$\}$

For any $f, g \in IG^X$, define $f \oplus g = f + g$. It is not difficult to verify that $(IG^X, \oplus)$ is a group. Fix an irresolute $\phi: G \to X$. For any $f, g \in IG^X$, $f *_\phi g = f \circ \phi \circ g$. Since the composition of irresolute mappings is again an irresolute, we have that $*_\phi$ is well-defined. Finally, we can observe that $(IG^X, \oplus, *_\phi)$ is a right near-ring.

Definition 3.5 — Let (X, τ) be a topological space and $A \subseteq X$. A is called an α-open set[8] if $A \subseteq int(cl(int(A)))$.

Definition 3.6 — Let (X, τ) and (Y, σ) be any two topological spaces. A function $f: X \to Y$ is said to be α-continuous[5] if the inverse image of each open set in Y is α-open in X.

Definition 3.7 — Let (X, τ) and (Y, σ) be any two topological spaces. A function $f: X \to Y$ is called an α-irresolute[9] if the inverse image of each α-open set in Y is α-open in X.

If (X, τ) is a topological space, then the class of all α-open subsets of X is denoted by τ^α. Njåstad[8] in Proposition 2 showed that τ^α is a topology on X.

Theorem 3.3 — *Let* $(X, *, \tau)$ be a topological group $A \subseteq X$. If $b \in X$, then $b * A$ and $A * b$ are α-open if and only if A is α-open.

PROOF : Similar to that of Theorem 3.2.

Definition 3.8 — Let (X, τ) be a topological spaces and $A \subseteq X$. A is said to be pre-open[10] in X if $A \subseteq int(cl(A))$.

Definition 3.9 — Let (X, τ) and (Y, σ) be any two topological spaces. A function $f: X \to Y$ is said to be pre-open[10] if the image of each open set is pre-open.

Note that every open map is pre-open.

Theorem 3.4 — *Let* (X, τ), (Y, σ), (Z, η) *be any three topological spaces. Let* $f: X \to Y$ *and* $g: Y \to Z$ *be any two mappings. If* f *is pre-open,* α*-continuous, then* $g \circ f$ *: is* α*-continuous.*[11]

Suppose (X, τ) is a topological space and $(G, +, \sigma)$ is a topological group. Consider $OG_S^X = \{f/f : X \to G$ is open, α-continuous$\}$. Fix some open, α-continuous $\phi: G \to X$. For any $f, g \in OG_\alpha^X$, define $f \oplus g = f + g$ and $f *_\phi g = f \circ \phi \circ g$. Clearly, $\oplus$ and $*_\phi$ are well-defined and we

can verify that $(OG_{\alpha}^{X}, \oplus, *_{\phi})$ is a right near-ring. Observe that here we did not put any condition that X must be an extremally disconnected space. If we consider $IG_{\alpha}^{X} = \{f/f : X \to G$ is an α-irresolute$\}$ and ffx an irresolute $\phi : G \to X$, then if we define $f \oplus g = f + g$ and $f *_{\phi} g = f \circ \phi \circ g$ for any $f, g \in IG^{X}$, then one can see that $(IG_{\alpha}^{X}, \oplus, *_{\phi})$ is a right near-ring without any difficulty.

Remark 1 : Suppose (X, τ) is a topological space and $(G, +, \sigma)$ be a topological group. Then $IG_{\alpha}^{X} \leqq IG^{X}$ and $OG_{\alpha}^{X} \leqq IG_{\alpha}^{X}$.

Remark 2 : Suppose (X, τ) is an extremally disconnected space and $(G, +, \sigma)$ is a topological group. Then $OG_{\alpha}^{X} \leqq OG_{S}^{X}$ and $OG_{S}^{X} \leqq IG^{X}$.

Definition 3.10 — Let (X, τ) and (Y, σ) be any two topological spaces. A function $f : X \to Y$ is called pre-semi-open[12] if the image of every semi-open set is semi-open.

Definition 3.11 — Let (X, τ) and (Y, σ) be any two topological spaces. A function $f : X \to Y$ is said to be a semi-homomorphism (C.H.)[12] if (1)f is irresolute, (2)f is pre-semi-open and (3)f is bijective.

Theorem 3.5 — *Let $(IG_{S}^{X}, \oplus, *_{\phi})$ and $(IH_{S}^{Y}, \oplus, *_{\psi})$ be any two sandwich near-rings. Suppose $h : X \to Y$ is a semi-homeomorphism (C.H.) and $t : G \to H$ is a group isomorphism as well as a semi-homeomorphism (C.H.) such that $h \circ \phi = \psi \circ t$. Then $IG_{S}^{X} \cong IH_{S}^{Y}$ as near-rings.*

PROOF : Define $\alpha : IG_{S}^{X} \to IH_{S}^{Y}$ by

$$\alpha(f) = t \circ f \circ h^{-1} : \forall f \in IG_{S}^{X}.$$

Let $f \in IG_{S}^{X}$. Since $t : G \to H, f : X \to G$ and $h^{-1} : Y \to X$ are irresolute maps, we have that $t \circ f \circ h^{-1} : Y \to H$ is also an irresolute. Hence, α is well defined.

Let $f \in IG_{S}^{X}$.

$$\alpha(f \oplus g) = t \circ (f \oplus g) \circ h^{-1} = t \circ ((f \oplus g) \circ h^{-1})$$

$$= t \circ (f \circ h^{-1} \oplus g \circ h^{-1}) = t \circ f \circ h^{-1} \oplus t \circ g \circ h^{-1} = \alpha(f) \oplus \alpha(g).$$

$$\alpha(f *_{\phi} g) = t \circ (f *_{\phi} g) \circ h^{-1} = t \circ (f \circ \phi \circ g) \circ h^{-1} = t \circ f \circ \phi \circ g \circ h^{-1}$$

$$= t \circ f \circ (h^{-1} \circ h) \circ \phi \circ g \circ h^{-1} = (t \circ f \circ h^{-1}) \circ (h \circ \phi) \circ (g \circ h^{-1})$$

$$= (t \circ f \circ h^{-1}) \circ (\psi \circ t) \circ (g \circ h^{-1})$$

$$= (t \circ f \circ h^{-1}) \circ \psi \circ (t \circ g \circ h^{-1}) = \alpha(f) \circ \psi \circ \alpha(g) = \alpha(f) *_{\psi} \alpha(g).$$

Definition 3.12 — Let (X, τ) and (Y, σ) be any two topological spaces. A function $f : X \to Y$ is called semi-open[3] if the image of every open set is semi-open.

Definition 3.13 — Let (X, τ) and (Y, σ) be any two topological spaces. A function $f: X \to Y$ is said to be a semi-homeomorphism (B)[13] if (1)f is continuous, (2)f is semi-open and (3)f is bijective.

Theorem 3.5 — *Let* $(OG_S^X, \oplus, *_\phi)$ *and* $(OH_S^Y, \oplus, *_\psi)$ *be any two sandwich near-rings. Suppose* $h: X \to Y$ *be an open, semi-homeomorphism (B) and* $t: G \to H$ *is a group isomorphism as well as a semi-homeomorphism (B) such that* $h \circ \phi = \psi \circ t$. *Then* $IG_S^X \cong IH_S^Y$.

PROOF : Omitted.

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