Indian J. pure appl. Math., **29**(11) : 1137-1141, November 1998

SOME HYPERGEOMETRIC TRANSFORMATIONS

P. A. PADMANABHAM

Department of Mathematics, Pondicherry Engineering College, Pondicherry, 605 014

(*Received* 8 *September* 1997; *Accepted* 22 *May* 1998)

In this paper we prove five hypergeometric transformations involving quadruple hypergeometric series K_9, K_{10}, K_{11}, K_{12} and K_{13}.

Key Words : Transformations, Hypergeometric Series

1. INTRODUCTION

No specific study has been made of hypergeometric function of four variables except probably Lauricella functions F_A, F_B, F_C and F_D, until Exton[1] in the year 1972 defined and examined a few of their properties. It is surprising that these hypergeometric functions of four variables have been overlooked over the years. In fact, Srivastava and Karlsson [3, p. 274] remarked in their excellent book titled "Multiple Gaussian Hypergeometric Series" that "Two obvious subjects of future investigations are the quadruple Gaussian hypergeometric series and the triple Clausenian hypergeometric series."

We begin by recalling the definitions and their integral representations of four variable hypergeometric functions of order two given by Exton [2, page 78, eqs. (3.3.9), to (3.3.13) and page 82, eqs. (3.3.2.2) to (3.3.2.6)]:

$$K_9(\alpha, \alpha, \alpha, \alpha; \beta, \beta, \gamma_1, \gamma_2; \varepsilon_1, \varepsilon_2, \delta, \delta; x, y, z, t)$$

$$= \sum_{p,q,r,s=0}^{\infty} \frac{(\alpha)_{p+q+r+s}(\beta)_{p+q}(\gamma_1)_r(\gamma_2)_s}{(\varepsilon_1)_p(\varepsilon_2)_q(\delta)_{r+s}} \frac{x^p y^q z^r t^s}{p!\,q!\,r!\,s!} \qquad \text{... (1.1)}$$

$$K_{10}(\alpha, \alpha, \alpha, \alpha; \beta, \beta, \gamma_1, \gamma_2; \delta_1, \delta_2, \delta_3, \delta_4; x, y, z, t)$$

$$= \sum_{p,q,r,s=0}^{\infty} \frac{(\alpha)_{p+q+r+s}(\beta)_{p+q}(\gamma_1)_r(\gamma_2)_s}{(\delta_1)_p(\delta_2)_q(\delta_3)_r(\delta_4)_s} \frac{x^p y^q z^r t^s}{p!\,q!\,r!\,s!} \qquad \text{... (1.2)}$$

$$K_{11}(\alpha, \alpha, \alpha, \alpha; \beta_1, \beta_2, \beta_3, \beta_4; \gamma, \gamma, \gamma\, \delta; x, y, z, t)$$

$$= \sum_{p,q,r,s=0}^{\infty} \frac{(\alpha)_{p+q+r+s}(\beta_1)_p(\beta_2)_q(\beta_3)_r(\beta_4)_s}{(\gamma)_{p+q+r}(\delta_s)} \frac{x^p y^q z^r t^s}{p!\, q!\, r!\, s!} \quad \dots (1.3)$$

$$K_{12}(\alpha, \alpha, \alpha, \alpha; \beta_1, \beta_2, \beta_3, \beta_4; \gamma_1, \gamma_1; \gamma_2, \gamma_2; x, y, z, t)$$

$$= \sum_{p,q,r,s=0}^{\infty} \frac{(\alpha)_{p+q+r+s}(\beta_1)_p(\beta_2)_q(\beta_3)_r(\beta_4)_s}{(\gamma_1)_{p+q}(\gamma_2)_{r+s}} \frac{x^p y^q z^r t^s}{p!\, q!\, r!\, s!} \quad \dots (1.4)$$

$$K_{13}(\alpha, \alpha, \alpha, \alpha; \beta_1, \beta_2, \beta_3, \beta_4; \gamma, \gamma, \delta_1, \delta_2; x, y, z, t)$$

$$= \sum_{p,q,r,s=0}^{\infty} \frac{(\alpha)_{p+q+r+s}(\beta_1)_p(\beta_2)_q(\beta_3)_r(\beta_4)_s}{(\gamma)_{p+q}(\delta_1)_r(\delta_2)_s} \frac{x^p y^q z^r t^s}{p!\, q!\, r!\, s!}. \quad \dots (1.5)$$

For convergence and other related results see Exton [2, p. 78] and

$$\Gamma(\alpha) K_9(\alpha, \alpha, \alpha, \alpha, \beta, \beta, \gamma_1, \gamma_2; \varepsilon_1, \varepsilon_2, \delta, \delta; x, y, z, t)$$

$$= \int_0^{\infty} e^{-s} s^{\alpha-1} \psi_2(\beta; \varepsilon_1, \varepsilon_2; xs, ys)\, \phi_2(\gamma_1, \gamma_2; \delta; zs, ts) ds \quad \dots (1.6)$$

$$\Gamma(\alpha) K_{10}(\alpha, \alpha, \alpha, \alpha, \beta, \beta, \gamma_1, \gamma_2; \delta_1, \delta_2, \delta_3, \delta_4; x, y, z, t)$$

$$= \int_0^{\infty} e^{-s} s^{\alpha-1} \psi_2(\beta; \delta_1, \delta_2; xs, ys)\, {}_1F_1(\gamma_1; \delta_3; zs)\, {}_1F_1(\gamma_2; \delta_4; ts)\, ds \quad \dots (1.7)$$

$$\Gamma(\alpha) K_{11}(\alpha, \alpha, \alpha, \alpha, \beta_1, \beta_2, \beta_3, \beta_4, \gamma, \gamma, \gamma, \delta; x, y, z, t)$$

$$= \int_0^{\infty} e^{-s} s^{\alpha-1} \phi_2^{(3)}(\beta_1, \beta_2, \beta_3; \gamma, xs, ys, zs)\, {}_1F_1(\beta; \delta; ts)\, ds \quad \dots (1.8)$$

and

$$\Gamma(\alpha) K_{12}(\alpha, \alpha, \alpha, \alpha, \beta_1, \beta_2, \beta_3, \beta_4, \gamma_1, \gamma_1, \gamma_2, \gamma_2; x, y, z, t)$$

$$= \int_0^{\infty} e^{-s} s^{\alpha-1} \phi_2(\beta_1, \beta_2; \gamma_1; xs, ys)\, \phi_2(\beta_3, \beta_4; \gamma_2; zs, ts)\, ds \quad \dots (1.9)$$

$$\Gamma(\alpha) K_{13}(\alpha, \alpha, \alpha, \alpha; \beta_1, \beta_2, \beta_3, \beta_4; \gamma, \gamma, \delta_1, \delta_2; x, y, z, t)$$

$$= \int_0^{\infty} e^{-s} s^{\alpha-1} \phi_2(\beta_1, \beta_2; \gamma, xs, ys)\, {}_1F_1(\beta_3; \delta_1; zs)\, {}_1F_1(\beta_4; \delta_2; ts) ds \quad \dots (1.10)$$

Where $Re(\alpha)$ is positive and the functions in the integrand and Kummer functions and Humbert functions.

2. RESULTS

$$K_9(\alpha, \alpha, \alpha, \alpha; \beta, \beta, \gamma, \delta-\gamma, \beta, \beta, \delta, \delta; x, y, z, t)$$

$$= (1-x-y-t)^{-\alpha} H_4\left(\alpha, \gamma, \beta, \delta; \frac{xy}{(1-x-y-t)^2}, \frac{z-t}{(1-x-y-t)}\right), \qquad \text{... (2.1)}$$

$$K_{10}(\alpha, \alpha, \alpha, \alpha; \beta, \beta, \delta_1-\gamma_1; \delta_2-\gamma_2; \beta, \beta, \delta_1, \delta_2; x, y, -y, -x)$$

$$= x_8(\alpha, \gamma_1, \gamma_2; \beta, \delta_1, \delta_2; xy, x, y), \qquad \text{... (2.2)}$$

$$K_{11}(\alpha, \alpha, \alpha, \alpha; \beta_1, \beta_2, \beta_3, \beta_4; \gamma, \gamma, \gamma, \delta; x, x, x, t)$$

$$= F_2(\alpha; \beta_1+\beta_2+\beta_3, \beta_4; \gamma, \delta; x, t), \qquad \text{.. (2.3)}$$

$$K_{12}(\alpha, \alpha, \alpha, \alpha; \beta_1, \gamma_1-\beta_1, \beta_2, \gamma_2-\beta_2; \gamma_1, \gamma_1, \gamma_2, \gamma_2; x, y, z, -y)$$

$$= F_2(\alpha; \beta_1, \beta_2; \gamma_1, \gamma_2; (x-y), (y+z)) \qquad \text{... (2.4)}$$

$$K_{13}(\alpha, \alpha, \alpha, \alpha; \beta_1, \gamma_1-\beta_1, \delta_1-\beta_2, \beta_3; \gamma_1, \gamma_1, \delta_1, \delta_2; x, y, -y, t)$$

and

$$= F_A^{(3)}(\alpha, \alpha, \alpha; \beta_1, \beta_2, \beta_3; \gamma_1, \delta_1, \delta_2; x-y, y, t), \qquad \text{... (2.5)}$$

where the functions on the right of the above are Horn H_4, Exton X_8, Appell F_2 and Lauricella F_A. For details, see for example, Srivastava and Karlsson [3, p. 4, p. 84, p. 23 and p. 78].

Using the relation [3, p. 305, eq. (108)] viz.,

$$F_2(\alpha, \beta, \beta^1; \alpha; \alpha; x, y)$$

$$= (1-x)^{-\beta}(1-y)^{-\beta^1}\, {}_2F_1\left(\beta, \beta^1; \alpha; \frac{xy}{(1-x)(1-y)}\right) \qquad \text{... (2.6)}$$

in (2.3) and (2.4), we observe that

$$K_{11}(\alpha, \alpha, \alpha, \alpha; \beta_1, \beta_2, \beta_3, \beta_4; \alpha, \alpha, \alpha, \alpha; x, x, x, t)$$

$$= (1-x)^{-\beta_1-\beta_2-\beta_3}(1-t)^{-\beta_4}\, {}_2F_1\left(\beta_1+\beta_2+\beta_3, \beta_4; \alpha; \frac{xt}{(1-x)(1-t)}\right) \qquad \text{... (2.7)}$$

and

$$K_{12}(\alpha, \alpha, \alpha, \alpha; \beta_1, \alpha-\beta_1, \beta_2, \alpha-\beta_2; \alpha, \alpha, \alpha, \alpha; x, y, z, -y)$$

$$= (1-x+y)^{-\beta_1}(1-y-z)^{-\beta_2}\, {}_2F_1\left(\beta_1, \beta_2; \alpha; \frac{(x-y)(y+z)}{(1-x+y)(1-y-z)}\right). \qquad \text{... (2.8)}$$

Letting $t \to 0$ in (2.5), we get

$$F_G(\alpha, \alpha, \alpha; \beta, \gamma-\beta, \delta-\varepsilon; \gamma, \gamma, \delta; x, y, -y)$$
$$= F_2(\alpha; \beta, \varepsilon; \gamma, \delta; x-y, y) \quad \text{... (2.9)}$$

which in view of (2.6) can be put in the form

$$F_G(\alpha, \alpha, \alpha; \beta, \alpha-\beta, \alpha-\varepsilon; \alpha, \alpha, \alpha; x, y, -y)$$
$$= (1-x+y)^{-\beta}(1-y)^{-\varepsilon}\, {}_2F_1\left(\beta, \varepsilon; \alpha; \frac{y(x-y)}{(1-x+y)(1-y)}\right). \quad \text{... (2.10)}$$

Proofs of (2.1) to (2.5): If we apply the results [3, p. 322. eqs. 181 & 182], viz.

$$\psi_2(\alpha, \alpha, \alpha; x, y) = e^{x+y}\, {}_0F_1(-;\alpha; xy) \quad \text{... (2.11)}$$

and
$$\phi_2(\alpha, \gamma-\alpha, \gamma; x, y) = e^{y}\, {}_1F_1(\alpha; \gamma; x-y) \quad \text{... (2.12)}$$

to the ψ_2 and ϕ_2 functions in the integrand of (2.1), and after suitable specialisation of parameters, we get

$$\Gamma(\alpha)\, K_9(\alpha, \alpha, \alpha, \alpha; \beta, \beta, \gamma, \delta-\gamma, \beta, \beta, \delta, \delta; x, y, z, t)$$

$$= \int_0^\infty e^{-s(1-x-y-t)} s^{\alpha-1}\, {}_0F_1(-;\beta; xys^2)\, {}_1F_1(\gamma; \delta; s(z-t))ds.$$

Now, expanding the integrand, changing the order of integration summations, which is justified, and interpreting the resultant as Horn function H_4 we get the right side of (2.1).

Eq. (2.2) can be proved in a similar manner if we apply (2.11) and Kummer transformation viz.,

$${}_1F_1(\alpha; \gamma; z) = e^{z}\, {}_1F_1(\gamma-\alpha; \gamma; -z). \quad \text{... (2.13)}$$

Using (2.12) for both the ϕ_2 functions, involved in the integrand of (1.9) we get (2.4).

Applying (2.12) and (2.13) for $F_1(\delta_1-\beta_2; \delta_1; -ys)$ in the integral representation of K_{13} given by (1.10) we get (2.5).

PROOF OF (2.3) : We have by (1.3),

$$K_{11}(\alpha, \alpha, \alpha, \alpha; \beta_1, \beta_2, \beta_3, \beta_4; \gamma, \gamma, \gamma, \delta; x, x, x, t)$$

$$= \sum_{p=0}^{\infty} \frac{(\alpha)_p (\beta_4)_p}{(\delta)_p} \frac{t^p}{p!} F_D(\alpha+p; \beta_1, \beta_2, \beta_3; \gamma, x, x, x),$$

where F_D is the Lauricella function of fourth kind, see [3, p. 33].

Since [3, p. 34],

$$F_D(\alpha, \beta_1, \beta_2, \beta_3; \gamma, x, x, x) = {}_2F_1(\alpha; \beta_1+\beta_2+\beta_3; \gamma, x)$$

we have

$$K_{11}(\alpha, \alpha, \alpha, \alpha; \beta_1, \beta_2, \beta_3, \beta_4; \gamma, \gamma, \gamma, \delta; x, x, x, t)$$

$$= \sum_{p,q=0}^{\infty} \frac{(\alpha)_{p+q} (\beta_1 + \beta_2 + \beta_3)_p (\beta_4)_q}{(\gamma)_p (\delta)_q} \frac{x^p t^q}{p!\, q!}$$

$$= F_2 (\alpha; \beta_1 + \beta_2 + \beta_3, \beta_4; \gamma, \delta; x, t).$$

Hence the result.

REFERENCES

1. H. Exton, *Bull. Soc. Math. grece. N. S.* **13** (1972) pp. 104-114.
2. H. Exton, *Multiple Hypergeometric Function and Applications*, Halsted Press, John-Wiley & Sons, New York, 1976.
3. H. M. Srivastava and P. W. Karlsson, *Multiple Gaussian Hypergeometric Series*, Halsted Press, John-Wiley & Sons, New York, 1985.