Indian J. pure appl. Math., **29**(11) : 1143-1149, November 1998
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STATIC CHARGED ISOTHERMAL GASEOUS SPHERES IN GENERAL RELATIVITY

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(*Received* 11 *March* 1998; *accepted* 27 *May* 1998)

Einstein's field equations for spherically symmetric static charged perfect fluid distributions have been rewritten using new variables $u = 4\pi\delta r^2$, $v = 4\pi p r^2$, $w = (4\pi/3)(\rho + \varepsilon)r^2$ where δ $(= (-1/2) F_{14} F^{14})$, ρ and ε denote respectively the energy densities of the electrostatic matter and free gravitational fields whereas p represents the fluid pressure. Assuming (i) $u = v (\rightarrow T_1^1 = 0)$ and (ii) $v = kw (\rightarrow p = (k/3)(\rho + \varepsilon))$, where k is a constant we have integrated the field equations. It is shown that for this class of solutions $\varepsilon > 2\rho$ and hence they represent strong gravitational fields. Two special cases corresponding to (*i*) charged black-body radiaiton (k = 1) and (*ii*) charged stiff gas (k = 3) are discussed in detail. It is shown that, whereas for neutral gas p and ρ fall off as inverse square of r, for the charged gas spheres these physical quantities fall off much faster. In fact, for the charged black-body radiation and stiff gas with charge δ, p, ρ and ε all fall off respectively as r^{-7} and r^{-9} indicating a stronger in general, and particularly singularity in the latter case.

Key Words : Einstein-Maxwell's Equations; Isothermal Gas Spheres; Energy Density of the Free Gravitational Field; Singularity.

1. INTRODUCTION

In general theory of relativity spherically symmetric static models filled with a charged perfect fluid (or dust) under various assumptions have been studied by several authors including Bonnor[1], Nduka[2], Cooperstock and De la Cruz[3], Mehra[4], Tikekar[5-6], Xingxiang[7], Patino and Rago[8], Pant and Sah[9] and Shi-Chang[10]. Bonnor[1] and independently, De and Raychaudhuri[11] have shown that in the case of a static distribution of pressureless charged fluid (dust) the matter density ρ is equal to the absolute value of the charge density $|\sigma|$. In a recent paper, Krishna Rao and Trivedi[12] gave a reformulation of the Einstein field equations for spherically symmetric static distribution of charged perfect fluids by introducing three functions

$$u = 4\pi\delta r^2, \qquad \ldots (1)$$

$$v = 4\pi p r^2 \qquad \ldots (2)$$

and

$$w = (4\pi/3)(\rho + \varepsilon)r^2. \qquad \ldots (3)$$

Here $\delta = (-1/2) F_{14} F^{14}$, p and ρ being the pressure and density of the charged fluid respectively, whereas ε is the eigen value of the conformal Weyl tensor representing the energy density of the free gravitational field (Krishna Rao[13-14]). This reformulation helps to choose some simple relationship between u, v, w and solve the field equations. In the present paper we obtained a general solution for charged isothermal gas spheres obeying the relations.

$$u = v = kw. \qquad \ldots (4)$$

In (4) the first equality implies that the radial component of the energy momentum tensor T_1^1 vanishes identically (see (12) of Section 2) and hence the charged particles experience only tangential stresses. The second equality provides an equation of state (Krishna Rao and Annapurna[15-17]).

In Section 2, we introduce the field equations for charged perfect fluid filled spheres and give their equivalent form in terms of u, v, w. In Section 3, we derive the new solutions with (4) as the equation of state and in Section 4 we discuss two particular cases with $k = 1$ and 3. The paper ends with concluding remarks in Section 5.

2. FIELD EQUATIONS

For the spherically symmetric static space-time described by the metric

$$ds^2 = -\exp \lambda\, dr^2 - r^2 (d\theta^2 + \sin^2 \theta\, d\phi^2) + \exp \nu\, dt^2, \qquad \ldots (5)$$

assuming that the matter content is a charged perfect fluid of null conductivity, with pressure p and fluid density ρ, the energy momentum tensor is given by,

$$T_a^b = (\rho + p) u_a u^b - p\, g_a^b - F_{ac} F^{bc} + (1/4)\, g_a^b F_{cd} F^{cd}, \qquad \ldots (6)$$

where $u^a = (0, 0, 0, \exp(-\nu/2))$ and the skew-symmetric Maxwell tensor F_{ab} satisfies the equations

$$F_{ab,c} + F_{bc,a} + F_{ca,b} = 0, \qquad \ldots (7)$$

$$\{(-g)^{1/2} F^{ab}\}_{,b} = (-g)^{1/2} J^a, \qquad \ldots (8)$$

$$J^a = \sigma u^a. \qquad \ldots (9)$$

In (9) σ denotes the charge density. In view of spherical symmetry the only non-vanishing component of F_{ab} is F_{14}. We express F_{14} and J_4 as below :

$$F_{14} = -\exp\{(\lambda + \nu)/2\}\, Q(r)/r^2 \qquad \ldots (10)$$

and

$$J_4 = -\exp\{(\nu - \lambda)/2\}\, Q'(r)/r^2, \qquad \ldots (11)$$

where $Q(r) = \int_0^r 4\pi r^2 \sigma \exp(\lambda/2)\, dr$, and a prime for Q denotes a differentiation with respect to r.

With the help of Einstein's field equations we obtain from (5) and (6) the following relations :

$$8\pi(p-\delta) = -r^{-2}\{1-\exp(-\lambda)\} + r^{-1}\exp(-\lambda)\nu', \quad \ldots (12)$$

$$8\pi(p+\delta) = -8\pi\varepsilon + r^{-2}\{1-\exp(-\lambda)\} + r^{-1}\exp(-\lambda)(\nu'-\lambda') \quad \ldots (13)$$

and

$$8\pi(\rho+\delta) = r^{-2}\{1-\exp(-\lambda)\} + r^{-1}\exp(-\lambda)\lambda', \quad \ldots (14)$$

where

$$8\pi\varepsilon = r^{-2}\{1-\exp(-\lambda)\} - (1/4)\exp(-\lambda)\{2\nu'' + (\nu'-\lambda')(\nu'-2r^{-1})\} \quad \ldots (15)$$

and a prime for λ and ν denotes a differentiation with respect to r. The expression given in (15) denotes the eigen value of the conformal Weyl tensor for the space-time described by (5) and is interpreted as the energy density of the free gravitational field (Krishna Rao[13-14]). When $\varepsilon = 0$ the space-time described by (5) is conformally flat and conversely.

The combination of $\{(13)+(14)-(12)\}$ gives

$$\exp(-\lambda) = 1 - (8\pi/3)(\rho+\varepsilon+3\delta)r^2. \quad \ldots (16)$$

In (16), we note that ε is coupled to ρ and hence the energy density of the free gravitational field as represented by ε is always added to the material energy density ρ. In strong gravitational fields of ε dominates over ρ.

The field equations (12), (13) and (14) can be rewritten as below using the variables u, v, w defined by (1), (2), and (3) respectively (Krishna Rao and Trivedi[12]):

$$\exp(-\lambda) = 1 - 2(w+u), \quad \ldots (17)$$

$$\frac{(w'+u')}{r} = \frac{4\pi}{3}(2\rho-\varepsilon) \quad \ldots (18)$$

and

$$\frac{dr}{r} = \frac{(v+w)(du+dw) + (1-2(u+w))(dv-du)}{(2(u+v) - 4(u+w)(u+v) - (v+w)^2)}. \quad \ldots (19)$$

We note that (18) provides the criteria for strong gravitational fields. In our earlier paper[12] we have noted that for $\varepsilon \geq 2\rho$ the solutions are singular whereas for $\varepsilon < 2\rho$ the solutions are regular. From (17) and (18) we get

$$\varepsilon \begin{matrix} > \\ = \\ < \end{matrix} 2\rho \Rightarrow \lambda' \begin{matrix} < \\ = \\ > \end{matrix} 0, \quad \ldots (20)$$

providing a criteria for strong gravitational fields in terms of the first derivative of the metric component $g_{11} = -\exp(\lambda)$. The equation (19) is nothing but the radial equation of motion for hydrostatic equilibrium. In the next section we solve (17)-(19) assuming that the models are filled with charged isothermal gas.

3. CHARGED ISOTHERMAL GASEOUS SPHERES

The study of isothermal neutral gas spheres is interesting because they have no exteriors and such a study was undertaken by Bisnovatyi-Kogan and Zeldovich[18], and Bisnovatyi-Kogan and Thorne[19]. These models exhibit strong gravitational fields with $\varepsilon = 2\rho$ with both p and ρ falling as inverse

square of radial distance. However, when the gas carries a small amount of charge the above situation alters dramatically and the singularity at the centre is much more stronger. We note that the mathematical problem we have to solve has five unknown functions, namely λ, v, u, v, w but with only three eqs. (17), (18) and (19). Hence, we impose on u, v, w two more conditions given by (4).

The first condition $u = v$ gives $T_1^1 = 0$ whereas ... (21)

$$v = kw \Rightarrow (k/3)\ (\rho + \varepsilon). \qquad \text{... (22)}$$

In (22) we have modified the pressure-density relation for isothermal gas by adding the energy density of the free gravitational field ε to material energy density ρ since ε is always coupled to ρ (Krishna Rao and Annapurna[15-17]).

Now, when (4) is substituted in (17), (18) and (19) we get

$$\exp\ (-\lambda) = 1 - 2(1+k)w, \qquad \text{... (23)}$$

$$\frac{w'}{r} = \frac{4\pi}{3(1+k)}(2\rho - \varepsilon), \qquad \text{... (24)}$$

$$-\frac{dr}{r} = \frac{(1+k)}{(1+9k)}\frac{dw}{[\,w - (4k/(1+k)\,(1+9k))]} \qquad \text{... (25)}$$

Integrating (25), we get

$$w = \frac{\alpha}{r^{(1+9k)/(1+k)}} + \frac{4k}{(1+k)\,(1+9k)}, \qquad \text{... (26)}$$

where $\alpha > 0$ is a constant of integration. Since $k > 0$ always from (20) we note that w given by (26) denotes strong gravitational fields with $\varepsilon > 2\rho$.

Since from (23) and (26) the expression for $\exp(-\lambda)$ is known we can integrate (12) (since $u = v \Rightarrow p = \delta$) and the expression for the v is given by

$$\exp\ (v) = \frac{8\beta}{(1+)r}\left[\, r^{(1+9k)/(1+k)} - 2\alpha(1+9k)\,\right], \qquad \text{... (27)}$$

where $\beta > 0$ is a constant of integration. Since λ and v are known we now write the explicit expressions for $\delta, p, \rho, \varepsilon$ as below :

$$4\pi\delta = 4\pi p = \frac{k\alpha}{r^{(3+11k)/(1+k)}} + \frac{4k^2}{(1+k)\,(1+9k)r^2}, \qquad \text{... (28)}$$

$$4\pi\rho = \frac{4k}{(1+k)\,(1+9k)r^2} - \frac{9k\alpha}{r^{(3+11k)/(1+k)}} \qquad \text{... (29)}$$

and

$$4\pi\varepsilon = \frac{8k}{(1+k)\,(1+9k)r^2} + \frac{3\alpha\,(1+3k)}{r^{(3+11k)/(1+k)}}. \qquad \text{... (30)}$$

Also, the expressions for the physical components $F_{(14)}$, $J_{(4)}$ and σ are given by

$$F_{(14)} = -Q(r)/r^2, \qquad \text{... (31)}$$

$$J_{(4)} = \sigma = -\left\{ \frac{(1+k)}{(1+9k)} - \frac{2\alpha(1+k)}{r^{(1+9k)/(1+k)}} \right\}^{1/2} \frac{Q'(r)}{r^2}, \quad \ldots (32)$$

where

$$Q(r) = \left(\frac{k}{2\pi} \right)^{1/2} r \left\{ \frac{\alpha}{r^{(1+9k)/(1+k)}} + \frac{4k}{(1+k)(1+9k)} \right\}^{1/2}. \quad \ldots (33)$$

PARTICULAR CASES

The general solution for charged isothermal gas spheres given in section 3 will be now specialized for the following particular cases :

(*i*) $k = 1 \Rightarrow \delta = p = (1/3)(\rho + \varepsilon)$ ... (34)

and (*ii*) $k = 3 \Rightarrow \delta = p = (\rho + \varepsilon)$. ... (35)

Eqs. (34) and (35) represent the modified equations of state for charged black-body radiation and charged stiff gas respectively (Krishna Rao and Annapurna[15-17]). We have added ε to ρ because the former is always coupled to the latter. We give below the expressions for exp λ, exp ν, δ, p, ρ, ε, $F_{(14)}$, $J_{(4)}$, σ and $Q(r)$ for both these cases.

Geometrical/Physical variables	$3p = (\rho+\varepsilon)$ $(k = 1)$	$p = (\rho+\varepsilon)$ $(k = 3)$
$\exp(-\lambda)$	$\frac{1}{5} - \frac{4\alpha}{r^5}$	$\frac{1}{7} - \frac{8\alpha}{r^7}$
$\exp(\nu)$	$4\beta\left(r^4 - \frac{20\alpha}{r} \right)$	$2\beta\left(r^6 - \frac{56\alpha}{r} \right)$
$4\pi\delta = 4\pi p$	$\frac{1}{r^2}\left(\frac{1}{5} + \frac{\alpha}{r^5} \right)$	$\frac{3}{r^2}\left(\frac{3}{28} + \frac{\alpha}{r^7} \right)$
$4\pi\rho$	$\frac{1}{r^2}\left(\frac{1}{5} + \frac{9\alpha}{r^5} \right)$	$\frac{3}{r^2}\left(\frac{1}{28} + \frac{9\alpha}{r^7} \right)$
$4\pi\varepsilon$	$\frac{2}{r^2}\left(\frac{1}{5} + \frac{6\alpha}{r^5} \right)$	$\frac{3}{r^2}\left(\frac{1}{14} + \frac{10\alpha}{r^7} \right)$
$F_{(14)}$	$-\left(\frac{1}{2\pi} \right)^{1/2} \frac{1}{r}\left(\frac{1}{5} + \frac{\alpha}{r^5} \right)^{1/2}$	$-\left(\frac{3}{2\pi} \right)^{1/2} \frac{1}{r}\left(\frac{3}{28} + \frac{\alpha}{r^5} \right)^{1/2}$
$J_{(4)} = \sigma$	$-\left(\frac{1}{5} - \frac{4\alpha}{r^5} \right)^{1/2} \frac{f'(r)}{r^2}$	$-\left(\frac{1}{7} - \frac{8\alpha}{r^5} \right)^{1/2} \frac{Q'(r)}{r^2}$
$Q(r)$	$\frac{r}{(2\pi)^{1/2}}\left(\frac{1}{5} + \frac{\alpha}{r^5} \right)^{1/2}$	$\left(\frac{3}{2\pi} \right)^{1/2} r\left(\frac{3}{28} + \frac{\alpha}{r^7} \right)^{1/2}$

From the above formulae we note that all geometrical and physical variables are singular at $r = 0$. We may point out here that Bisnovatyi-Kogan and Zeldovich[18] and Bisnovatyi-Kogan and Thorne[19] considered neutral isothermal gas spheres for which p and ρ $(= \varepsilon/2)$ fall off as inverse squares of r. However, as shown above for charged isothermal gas spheres $p(=\delta)$, ρ and ε all fall off much faster than inverse square of the radial distance from the centre. In fact, in the expressions for δ, p, ρ, ε besides the usual r^{-2} terms for neutral gas, we have respectively r^{-7} and r^{-9} terms for charged black-body radiation and stiff fluid with charge indicating stronger singularity in the case when the gas carries a small amount of charge.

5. CONCLUDING REMARKS

Florides[20] and Kofinti[21] considered static spherically symmetric distributions of neutral matter with the radial pressure of the fluid vanishing $\left(T_1^1 = -p = 0\right)$. A charged generalization of Florides' solution was given by Gron[22] in which the fluid pressure p balances the electrostatic energy density δ making $T_1^1 = 0$. It is well known that the presence of an electric field along the radial direction gives raise to anisotropy. Hence, the assumption $T_1^1 = 0$ is justified and the charged particles are acted upon by tangential stresses only. In the present paper we assumed two relations between u, v and w the first assumption $u = v\left(\Rightarrow T_1^1 = 0\right)$ gives vanishing radial stress whereas the other assumption $v = kw$ leads to charged isothermal gas spheres which have no exteriors. The above simplifications were possible due to reformulation of Einsten's field equations in terms of u, v and w which also provides a criteria for strong gravitational fields $(\varepsilon \geq 2\rho)$ leading to singularities. From the two analytical situations presented here we understand that when matter is packed closely as in the case of stiff gas with charge the singularity is more stronger.

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