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ON A CERTAIN BANACH SPACE IN CONNECTION WITH MINIMAX SERIES

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In this paper, a new functional space is introduced. This space is connected with error series and approximation theory and it can be considered as an abstract generalization of several normed spaces whose norms are defined in terms of infinite error series. Convergence and completeness properties are studied.

Key Words : Banach Space; Error Series

1. INTRODUCTION

In the literature of approximation theory one can find numerous examples of norms, semi-norms and quasi-norms defined in terms of infinite series. In order to motivate the introduction of our function space, we briefly describe some of the above mentioned norms, for more details see e.g.[1]

Suppose $f \in L_p(0, 1), 0 < p, q < \infty$ and $\lambda > 0$. Taking $k = [\lambda] + 1$ is defined

$$\|f\|_{B^{\lambda}_{p,q}} = \left(\int_0^1 (t^{-\lambda} w_k(f; t)_p)^q \frac{dt}{t} \right)^{1/q}$$

where $w_k(f; t)_p$ is the usual modulus of smoothness of f in $L_p(0, 1)$. The Besov space $B^{\lambda}_{p,q}$ is defined as the set of those functions $f \in L_p(0, 1)$ such that the quasi-norm $\|f\|_{B^{\lambda}_{p,q}}$ is finite. More frequently, Besov spaces $B^{\lambda}_{p,q}$ are considered with the quasi-norm

$$\|f\| = \|f\|_p + \|f\|_{B^{\lambda}_{p,q}} . \qquad \ldots (1.1)$$

In spline approximation, the following variant of $B^{\lambda}_{p,q}$ is used. Denote by $B^{\lambda}_{p,q;k}$ the set of all functions $f \in L_p(0, 1)$ such that the following quasi-norm is finite.

$$\|f\|_{B^{\lambda}_{p,q;k}} = \left(\int_0^1 (t^{-\lambda} w_k(f;t)_p)^q \frac{dt}{t} \right)^{1/q}, \qquad \text{... (1.2)}$$

where $0 < p, q < \infty$, $\lambda > 0$, $k \geq 1$. Obviously $B^{\lambda}_{p,q;k}$ coincides with $B^{\lambda}_{p,q}$ when $k = [\lambda] + 1$ but k in the definition of $B^{\lambda}_{p,q;k}$ is not directly connected with λ. An equivalent quasi-norm in $B^{\lambda}_{p,q;k}$ is

$$\|f\|_{B^{\lambda}_{p,q;k}} = \left(\sum_{\nu=0}^{\infty} \left(2^{\lambda\nu} w_k\left(f; \frac{1}{2^{\nu}} \right)_p \right)^q \right)^{1/q}, \qquad \text{... (1.3)}$$

$0 < p, q < \infty$, $\lambda > 0$, $k \geq 1$. See e.g. [1 pp. 201-203].

On the other hand, let X be a normed (or quasi-normed) linear space and let $\{G_n\}_{n=1}^{\infty}$ be a family of subset of X, being each G_n an existence set in X, that is, every $f \in X$ has an element of best approximation in G_n denoted by $E_n(f)_X$:

$$E_n(f)_X = \inf \{\|f - g\|_X : g \in G_n\}.$$

The approximation space

$$A^{\gamma}_q(X) = \{f \in X : \|f\|_{A^{\gamma}_q(X)} < \infty\}$$

where

$$\|f\|_{A^{\gamma}_q(X)} = \|f\|_X + \left(\sum_{\nu=0}^{\infty} (2^{\nu\gamma} E_{2^{\nu}}(f)_X)^q \right)^{1/q} \qquad \text{... (1.4)}$$

and $0 < q < \infty$, $\gamma > 0$, appear in the theory of interpolation spaces and is closely related with the k-functional introduced by Peetre. See e.g. [2]. Particular cases of (1.4) are treated in [1].

After the above considerations, we can now introduce the following abstract generalization. Let X and $(X, \| \|)$ be a function space and a normed linear space respectively. We consider that the functions of X are defined on a domain T of the complex plane $\mathbb{C}$. Furthermore, we consider that the constant function is in X and we suppose associated to each $f \in X$ a sequence of approximants $\{P_k(f)\}_{k=0}^{\infty} \subset X$, where $P_0(f)$ is usually the constant function and satisfying

$$P_k(\mu f) = \mu P_k(f), \quad \forall f \in X, \mu \in \mathbb{C}, k = 0, 1, ..., \qquad \text{... (1.5)}$$

$$\|f + g - P_k(f+g)\| \leq \|f - P_k(f)\| + \|g - P_k(g)\|, \quad \forall f, g \in X, k = 0, 1, ... \qquad \text{... (16)}$$

Furthermore, let $\alpha \equiv \{\alpha_k\}_{k=0}^{\infty} \subset \mathbb{R}$ be such that α_0 is usually strictly positive, $\alpha_k \geq 0$, $k = 1, 2, ...$, and define

$$S_{\alpha, q}(f) = \left(\sum_{k=0}^{\infty} (\alpha_k \| f - P_k(f) \|)^q \right)^{1/q} \quad \ldots (1.7)$$

where $0 < q < \infty$.

In the above case, $P_k(f) = E_k(f)_X \in G_k, k = 0, 1, \ldots$. Some remarks can be made. As a consequence of Theorem 3.3 in [1] one has if

$$E_n(f)_p = E_n(f) L_{p(0,1)} = \inf \{ \| f - t \|_{L_p(0,1)} : t \in T_n \}$$

where T_n denotes the subspace of all trigonometric polynomials of degree n at most that

$$E_n(f)_p \leq s(k) w_k \left(f; \frac{1}{n} \right)_p$$

being $k, n \geq 1$ natural numbers, $p \geq 1$ ans $s(k)$ a constant. Thus, taking $\lambda, q > 0, p > 1, k \geq 1$ a natural number, $\alpha_{2^\nu} = 2^{\lambda\nu}$ and $\alpha_k = 0$ otherwise, and $P_k(f)$ the best approximation of f in T_k with respect to the L_p-norm, one obtain

$$[S_{\alpha, q}(f)]^q = \sum_{u=0}^{\infty} (2^{\lambda\nu} E_{2^\nu}(f)_p)^q \leq s^q(k) \sum_{n=0}^{\infty} \left(2^{\lambda\nu} w_k \left(f; \frac{1}{2^\nu} \right)_p \right)^q.$$

Hence,

$$S_{\alpha, q}(f) \leq s(k) \| f \|_{B^{\lambda}_{p, q; k}}.$$

On the other hand, for $q = 1$, $S_{\alpha, 1}(f)$ can be seen as a *weighted sum of errors*. For example, let us consider

$$(X, \| \|) = (C[a, b], \| \|) \quad \ldots (1.8)$$

where $C[a, b]$ denotes as usual the space of all continuous function on $[a, b]$ and $\| \|_\infty$ the uniform norm, defined for all $f \in C[a, b]$ by

$$\| f \|_\infty = \max_{x \in [a, b]} | f(x) | \quad \ldots (1.9)$$

If $P_k(f)$ is the best approximation of f in Π_k, (Π_k denotes the space of all polynomials of degree k at most) and

$$E_k(f) = \| f - P_k(f) \|_\infty = \inf_{P \in \Pi_k} \| f - P \|_\infty \quad \ldots (1.10)$$

then

$$S_{\alpha, 1}(f) = \sum_{k=0}^{\infty} \alpha_k \| f - P_k(f) \|_\infty = \sum_{k=0}^{\infty} \alpha_k E_k(f).$$

In [5], $S_{\alpha,1}$ is called *weighted minimax serie.*

Let $C_{\alpha,q} = \{f \in X : S_{\alpha,q} < \infty\}$ and x_0 be a fixed point in T. Define

$$C_{\alpha,q}(x_0) = \{f \in M(x_0) : S_{\alpha,q}(f) < \infty\} \qquad \ldots (1.11)$$

where

$$M(x_0) = \{f \in X : f(x_0) = 0\}. \qquad \ldots (1.12)$$

$(C_{\alpha,q}, S_{\alpha,q})$ is a semi-normed space for $1 \le q < \infty$. Certainly, for $1 \le q < \infty$ and similarly than in (1.1) and (1.4) the following function

$$T_{\alpha,q}(f) = \|f\| + \left(\sum_{k=0}^{\infty} (\alpha_k \|f - P_k(f)\|)^q \right)^{1/q}$$

is a norm (although $P_0(f)$ is not a constant function and $\alpha_0 = 0$) but in this paper we shall be concerned with the normed space $(C_{\alpha,q}(x_0), S_{\alpha,q})$ with $\alpha_0 > 0$. Our interest here is to prove that under certain conditions $(C_{\alpha,q}(x_0), S_{\alpha,q})$ is a Banach space.

First, in Section 2, we give necessary and sufficient conditions to characterize the convergence and the Cauchy sequences in $(C_{\alpha,q}(x_0), S_{\alpha,q})$. After that, we search conditions to assure that every Cauchy sequence converge to a function of this space. Finally, in Section 3, we present several examples, chosing $(X, \|\ \|)$ and $\{P_k\}_{k=0}^{\infty}$

2. THEORETICAL RESULTS

In the sequel, we shall consider $q \ge 1$. Furthermore, we shall make use of the inequality $(a+b)^q \le 2^{q-1}(a^q + b^q)$, $a, b \ge 0$, $q \ge 1$.

Proposition 2.1 — If $f, f_n \in C_{\alpha,q}(x_0)$, $n = 1, 2, \ldots$, and $f_n \overset{S_{\alpha,q}}{\longrightarrow} f$ then there exists a sequence $\{c_n\}_{n=1}^{\infty} \subset \mathbb{C}$ such that $f_n + c_n \overset{\|\ \|}{\longrightarrow} f$.

PROOF : Let $\varepsilon > 0$ be given. For sufficiently large $n_0 \in \mathbb{N}$, it holds

$$S_{\alpha,q}(f_n - f) < \varepsilon, \quad \forall\ n \ge n_0.$$

Thus

$$\|f_n - f - P_0(f_n - f)\| \le \frac{S_{\alpha,q}(f_n - f)}{\alpha_0} \le \frac{\varepsilon}{\alpha_0}, \qquad \forall\ n \ge n_0.$$

Taking $c_n = -P_0(f_n - f)$, the proof is achieved.

In our treatment, the following condition is required

$$f_n \overset{\|\ \|}{\to} f \Rightarrow f_n(x) \to f(x), \quad \forall\, x \in T; f, f_n \in X. \qquad \ldots (2.1)$$

On the other hand, we shall say that $\left\{ S_{\alpha, q}(f_n) \right\}_{n=1}^{\infty}$ is uniformly convergent in $\mathbb{N}$, if for each $\varepsilon > 0$ there exists $k_0 \in \mathbb{N}$ such that

$$\sum_{k \geq k_0} (\alpha_k \| f_n - P_k(f_n) \|)^q < \varepsilon, \ n = 1, 2, \ldots . \qquad \ldots (2.2)$$

Proposition 2.2 — Suppose that $f, f_n \in C_{\alpha, q}(x_0), n = 1, 2, \ldots$, and (2.1) is satisfied. If $f_n \overset{S_{\alpha, q}}{\longrightarrow} f$ then

1. $f_n \overset{\|\ \|}{\longrightarrow} f$,
2. $\left\{ S_{\alpha, q}(f_n) \right\}_{n=1}^{\infty}$ is uniformly convergent in $\mathbb{N}$.

PROOF :

1. By virtue of Proposition 2.1, there is $\left\{ c_n \right\}_{n=1}^{\infty} \subset \mathbb{C}$ such that $f_n + c_n \overset{\|\ \|}{\longrightarrow} f$. Now, from (2.1) with $x = x_0$ is deduced that $c_n \to 0$ and hence, the proof of item 1 is complete.
2. Let $\varepsilon > 0$ be given. By hypothesis, for all n greater or equal than a certain integer $n_0 \in \mathbb{N}$ one has $[S_{\alpha, q}(f_n - f)]^q < \varepsilon$. ·

On the other hand, for $i = 1, 2, \ldots, n_0 - 1$, $S_{\alpha, q}(f_i - f) \leq S_{\alpha, q}(f_i) + S_{\alpha, q}(f) < \infty$.

Thus, for appropriate $k_i \in \mathbb{N}$

$$\sum_{k \geq k_i} (\alpha_k \| f_i - f - P_k(f_i - f) \|)^q < \varepsilon, \quad i = 1, 2, \ldots, n_0 - 1.$$

Taking $\nu = \max_{1 \leq i \leq n_0 - 1} k_i$ one obtain

$$\sum_{k \geq k_\nu} (\alpha_k \| f_n - f - P_k(f_n - f) \|)^q < \varepsilon, \ n = 1, 2, \ldots,$$

that is, $\left\{ S_{\alpha, q}(f_n - f) \right\}_{n=1}^{\infty}$ is uniformly convergent. Now, take into account that for all $k \geq 0$ and $n \geq 1$

$$(\alpha_k \| f_n - P_k(f_n) \|)^q = (\alpha_k \| f_n - f + f - P_k(f_n - f + f) \|)^q.$$

By virtue of eq. (1.6) and the inequality at the beginning of this Section, one can write

$$(\alpha_k \| f_n - P_k(f_n) \|)^q \le 2^{q-1} [(\alpha_k \| f_n - f - P_k(f_n - f) \|)^q + (\alpha_k \| f - P_k(f) \|)^q].$$

The proof follows since $\left\{ S_{\alpha, q}(f_n - f) \right\}_{n=1}^{\infty}$ is uniformly convergent and $S_{\alpha, q}(f) < \infty$. □

In the following, we restrict our attention to sequences of approximants $\left\{ P_k \right\}_{k=0}^{\infty}$ fulfilling that for all fixed function $f \in C_{\alpha, q}(x_0)$ and $k \in \mathbb{N}$, there is a constant $\sigma_k(f)$ such that

$$\| P_k(f) - P_k(g) \| \le \sigma_k(f) \| f - g \|, \qquad \forall\, g \in C_{\alpha, q}(x_0). \qquad \text{... (2.3)}$$

Theorem 2.1 — *Suppose that* $f, f_n \in C_{\alpha, q}(x_0)$, $n = 1, 2, \ldots$, *and eqns. (2.1), (2.3) are satisfied. Then* $f_n \overset{S_{\alpha, q}}{\longrightarrow} f$ *if and only if*

1. $f_n \overset{\| \|}{\longrightarrow} f$

2. $\left\{ S_{\alpha, q}(f_n) \right\}_{n=1}^{\infty}$ *is uniformly convergent in* $\mathbb{N}$.

PROOF : In view of Proposition 2.2, it suffices to show that $f_n \overset{S_{\alpha, q}}{\longrightarrow} f$ when item 1 and 2 are satisfied. By virtue of (2.3) one has

$$\| f_n - f - P_k(f_n - f) \| \le \| f_n - f \| + \| P_k(f_n - f) \| \le (1 + \sigma_k(0)) \| f_n - f \|. \qquad \text{... (2.4)}$$

Take into account that from (1.5) is deduced $P_k(0) = 0$. Let $\varepsilon > 0$ be given. By a similar reasoning to the one given at the end of the proof of item 2 in Proposition 2.2, now with $\left\{ S_{\alpha, q}(f_n) \right\}_{n=1}^{\infty}$ uniformly convergent and $S_{\alpha, q}(f) < \infty$, one can deduce the existence of an integer k_0 satisfying

$$\sum_{k \ge k_0} (\alpha_k \| f_n - f - P_k(f_n - f) \|)^q < \varepsilon, \qquad \forall\, n \in \mathbb{N}. \qquad \text{... (2.5)}$$

That is $\left\{ S_{\alpha, q}(f_n - f) \right\}_{n=1}^{\infty}$ is uniformly convergent. Thus, taking into account (2.4) and (2.5)

$$[S_{\alpha, q}(f_n - f)]^q \le \| f_n - f \|^q \left(\sum_{k < k_0} ((1 + \sigma_k(0))\alpha_k)^q \right) + \varepsilon, \quad \forall\, n \in \mathbb{N}.$$

The proof follows as a consequence of item 1. □

Hereafter, we shall assume the existence of a constant $d > 0$ such that for all $g \in C_{\alpha, q}(x_0)$ it holds

$$| g(x) | \leq d \| g \|, \qquad \forall\ x \in T. \qquad \ldots (2.6)$$

Theorem 2.2 — *Assume* $\{f_n\}_{n=1}^{\infty} \subset C_{\alpha,q}(x_0)$ *and conditions (2.3), (2.6) are satisfied. Then* $\{f_n\}_{n=1}^{\infty}$ *is a Cauchy sequence in* $(C_{\alpha,q}(x_0), S_{\alpha,q})$ *if and only if*

1. $\{f_n\}_{n=1}^{\infty}$ *is a Cauchy sequence in* $(X, \| \|)$,

2. $\{S_{\alpha,q}(f_m - f_n)\}_{m,n=1}^{\infty}$ *is uniformly convergent in* $\mathbb{N} \times \mathbb{N}$.

PROOF : We first prove item 1. Let $\varepsilon > 0$ be given. Since $\{f_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $(C_{\alpha,q}(x_0), S_{\alpha,q})$, for a certain integer $n_0 \in \mathbb{N}$ one has $S_{\alpha,q}(f_m - f_n) < \varepsilon$, $\forall\ m, n \geq n_0$. Hence

$$\| f_m - f_n - P_0(f_m - f_n) \| \leq \frac{S_{\alpha,q}(f_m - f_n)}{\alpha_0} < \frac{\varepsilon}{\alpha_0}, \qquad \forall\ m, n \geq n_0. \qquad \ldots (2.7)$$

On the other hand, from (2.6) replacing x and g by x_0 and $f_m - f_n - P_0(f_m - f_n)$ respectively, we get

$$| P_0(f_m - f_n)(x_0) | \leq d \| f_m - f_n - P_0(f_m - f_n) \|. \qquad \ldots (2.8)$$

Now, in view of (2.7) and (2.8) one obtain since P_0 is a constant function

$$\| f_m - f_n \| \leq \| f_m - f_n - P_0(f_m - f_n) \| + \| P_0(f_m - f_n) \| < \frac{\varepsilon}{\alpha_0} + \frac{d\varepsilon \| 1 \|}{\alpha_0}, \qquad \forall\ m, n \geq n_0.$$

Thus the proof of item 1 is achieved.

Let n_0 be such that $S_{\alpha,q}(f_m - f_n) < \varepsilon^{1/q}/2$, $\forall\ m, n \geq n_0$. Define $J = \{(m, n) : 1 \leq m, n \leq n_0\}$. For every $(m, n) \in J$, since $S_{\alpha,q}(f_m - f_n) < \infty$, there is a constant $k_{m,n}$ satisfying

$$\sum_{k \geq k_{m,n}} (\alpha_k \| f_m - f_n - P_k(f_m - f_n) \|)^q < \frac{\varepsilon}{2^q}.$$

For $r = \max_{(m,n) \in J} \{k_{m,n}\}$ one has

$$\sum_{k \geq r} (\alpha_k \| f_m - f_n - P_k(f_m - f_n) \|)^q < \frac{\varepsilon}{2^q} < \varepsilon, \qquad \forall\ (m, n) \in J.$$

If $m, n \geq n_0$ then

$$\sum_{k \geq r} (\alpha_k \| f_m - f_n - P_k (f_m - f_n) \|)^q \leq [S_{\alpha, q} (f_m - f_n)]^1 < \frac{\varepsilon}{2^q} < \varepsilon.$$

In the case $m > n_0$ and $n < n_0$,

$$\sum_{k \geq r} (\alpha_k \| f_m - f_n - P_k (f_m - f_n) \|)^q \leq 2^{q-1} \sum_{k \geq r} (\alpha_k \| f_m - f_{n_0} - P_k (f_m - f_{n_0}) \|)^q$$

$$+ 2^{q-1} \sum_{k \geq r} (\alpha_k \| f_n - f_{n_0} - P_k (f_n - f_{n_0}) \|)^q < 2^{q-1} \left(\frac{\varepsilon}{2^q} + \frac{\varepsilon}{2^q} \right) = \varepsilon.$$

Case $n > n_0$ and $m < n_0$ is similar to the last one. Thus

$$\sum_{k \geq r} (\alpha_k \| f_m - f_n - P_k (f_m - f_n) \|)^q < \varepsilon, \ \forall \ m, \ n,$$

and this means that $\left\{ S_{\alpha, q} (f_m - f_n) \right\}_{m, n = 1}^{\infty}$ is uniformly convergent in $\mathbb{N} \times \mathbb{N}$.

Let us now prove that $\left\{ f_n \right\}_{n=1}^{\infty}$ is a Cauchy sequence in $(C_{\alpha, q} (x_0), S_{\alpha, q})$. Let $\varepsilon > 0$ be given. By item 2, there exists $k_0 \in \mathbb{N}$ such that

$$\sum_{k \geq k_0} (\alpha_k \| f_m - f_n - P_k (f_m - f_n) \|)^q < \varepsilon, \quad \forall m, n, \qquad \ldots (2.9)$$

According to item 1, one can assure the existence of an integer $n_0 \in \mathbb{N}$ such that

$$\| f_m - f_n \| < \left(\frac{\varepsilon}{2 \sum_{0 \leq k < k_0} ((1 + \sigma_k (0)) \alpha_k)^q} \right)^{1/q}, \quad \forall \ m, n \geq n_0. \qquad \ldots (2.10)$$

Taking into account (2.3), (2.9) and (2.10) one can obtain

$$[S_{\alpha, q} (f_m - f_n)]^q = \sum_{0 \leq k < k_0} (\alpha_k \| f_m - f_n - P_k (f_m - f_n) \|)^q + \sum_{k \geq k_0} (\alpha_k \| f_m - f_n - P_k (f_m - f_n) \|)^q$$

$$< \left(\sum_{0 \leq k < k_0} ((1 + \sigma_k (0)) \alpha_k)^q \right) \| f_m - f_n \|^q + \frac{\varepsilon}{2} < \varepsilon, \quad \forall \ m, n > n_0.$$

Proposition 2.3 — Suppose $\left\{ f_n \right\}_{n=1}^{\infty} \subset C_{\alpha, q} (x_0)$ and properties (2.1), (2.3) are satisfied. Furthermore, assume $f_n \xrightarrow{\| \ \|} f \in X$ and $S_{\alpha, q} (f_n) \leq M < \infty$, $n = 1, 2, \ldots$, then $S_{\alpha, q} (f) \leq 2^{(q-1)/q} M$ and $f \in C_{\alpha, q} (x_0)$.

PROOF : One can write

$$\| f - P_k(f) \| \leq \| f - f_n \| + \| f_n - P_k(f_n) \| + \| P_k(f_n) - P_k(f) \|. \qquad \ldots (2.11)$$

Making use of (2.3) into (2.11) we obtain

$$\| f - P_k(f) \| \leq (1 + \sigma_k(f)) \| f - f_n \| f - f_n \| + \| f_n - P_k(f_n) \|.$$

By hypothesis, for a given $\varepsilon > 0$ the following inequality

$$\| f - f_n \| < \frac{\varepsilon}{2^{k+1}(1 + \sigma_k(f))\alpha_k}$$

holds for all $n > n_k$ and $\alpha_k \neq 0$, where is taken $n_k \geq n_{k-1}$, $k = 0, 1, \ldots$. Thus

$$\| f - P_k(f) \| \leq \frac{\varepsilon}{2^{k+1}\alpha_k} + \| f_n - P_k(f_n) \|, \qquad \forall\ n > n_k, \alpha_k \neq 0, k = 0, 1, \ldots .$$

Hence,

$$\sum_{k=0}^{m} (\alpha_k \| f - P_k(f) \|)^q \overset{\prime}{=} \sum_{k=0,\, \alpha_k \neq 0}^{m} (\alpha_k \| f - P_k(f) \|)^q$$

$$\leq 2^{q-1} \sum_{k=0}^{m} \left(\frac{\varepsilon}{2^{k+1}} \right)^q + 2^{q-1} \sum_{k=0}^{m} (\alpha_k \| f_n - P_k(f_n) \|)^q, \qquad \forall\ n > n_m.$$

So we have

$$\sum_{k=0}^{m} (\alpha_k \| f - P_k(f) \|)^q \leq 2^{q-1} \sum_{k=0}^{\infty} \left(\frac{\varepsilon}{2^{k+1}} \right)^q + 2^{q-1} [S_{\alpha, q}(f_n)]^q \leq 2^{q-1} (\varepsilon^q + M^q).$$

Consequently,

$$S_{\alpha, q}(f) \leq (2^{q-1} (\varepsilon^q + M^q))^{1/q}, \ \forall\ \varepsilon > 0 \Rightarrow S_{\alpha, q}(f) \leq 2^{\frac{q-1}{q}} M.$$

$$\| \ \|$$

On the other hand, since $f_n(x_0) = 0$ and $f_n \longrightarrow f$, it results from (2.1) that $f(x_0) = 0$ and hence $f \in C_{\alpha, q}(x_0)$. □

Theorem 2.3 — *If* $(X, \| \ \|)$ *is a Banach space and (2.1), (2.3), (2.6) are satisfied, then* $(C_{\alpha, q}(x_0), S_{\alpha, q})$ *is a Banach space.*

PROOF : Let $\{ f_n \}_{n=1}^{\infty}$ be a Cauchy sequence in $(C_{\alpha, q}(x_0), S_{\alpha, q})$. Our interest is to prove

that there is $f \in C_{\alpha,q}(x_0)$ such that $f_n \overset{S_{\alpha,q}}{\longrightarrow} f$. Since $\{f_n\}_{n=1}^{\infty}$ is a Cauchy sequence, $\{f_n\}_{n=1}^{\infty}$ is bounded, so $S_{\alpha,q}(f_n) \leq M$, $n = 1, 2, \ldots$. On the other hand, by virtue of Theorem, 2.2, $\{f_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $(X, \| \|)$. Since $(X, \| \|)$ is a Banach space, there is $f \in X$ such that $f_n \overset{\| \|}{\longrightarrow} f$. We conclude from Proposition 2.3 that $S_{\alpha,q}(f) \leq 2^{(q-1)/q} M$ and $f \in C_{\alpha,q}(x_0)$. In order to show that $f_n \overset{S_{\alpha,q}}{\longrightarrow} f$, we only need to prove that $\{S_{\alpha,q}(f_n)\}_{n=1}^{\infty}$ is uniformly convergent in $\mathbb{N}$, by virtue of Theorem 2.1. Indeed, let $\varepsilon > 0$ be given. Since $\{f_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $(C_{\alpha,q}(x_0), S_{\alpha,q})$, equation

$$S_{\alpha,q}(f_n - f_m) < \frac{\varepsilon^{1/q}}{2}$$

holds for all m, n greater or equal than a certain integer n_0. In particular

$$S_{\alpha,q}(f_n - f_{n_0}) < \frac{\varepsilon^{1/q}}{2}, \qquad \forall\ n \geq n_0. \qquad \ldots (2.12)$$

Now, since $S_{\alpha,q}(f_{n_0}) < \infty$, there is $k_0 \in \mathbb{N}$ such that

$$\sum_{k \geq k_0} (\alpha_k \| f_{n_0} - P_k(f_{n_0}) \|)^q < \frac{\varepsilon}{2^q}. \qquad \ldots (2.13)$$

On the other hand, from (1.6) with f and g being replaced by $f_n - f_{n_0}$ and f_{n_0} respectively, one gets

$$\| f_n - P_k(f_n) \| \leq \| f_n - f_{n_0} - P_k(f_n - f_{n_0}) \| + \| f_{n_0} - P_k(f_{n_0}) \|. \qquad \ldots (2.14)$$

Hence, from (2.12), (2.13) and (2.14) for $n \geq n_0$

$$\sum_{k \geq k_0} (\alpha_k \| f_n - P_k(f_n) \|)^q \leq 2^{q-1} [S_{\alpha,q}(f_n - f_{n_0})]^q + 2^{q-1} \sum_{k \geq k_0} (\alpha_k \| f_{n_0} - P_k(f_{n_0}) \|)^q$$

$$< 2^{q-1} \left(\frac{\varepsilon}{2^q} + \frac{\varepsilon}{2^q} \right) = \varepsilon.$$

For $n < n_0$ we use $f_n \in C_{\alpha,q}(x_0)$ to deduce the existence of integers $k_j, j = 1, 2, \ldots, n_0 - 1$ satisfying

$$\sum_{k \geq k_j} (\alpha_k \| f_j - P_k(f_j) \|)^q < \varepsilon, \qquad j = 1, 2, .., n_0 - 1.$$

Finally, for $r = \max \{k_0, k_1, ..., k_{n_{0-1}}\}$ we get the desired result

$$\sum_{k \geq r} (\alpha_k \| f_n - P_k(f_n) \|)^q < \varepsilon, \qquad n = 1, 2, \ldots .$$

Corollary 2.1 — If $(X, \| \|)$ is a Banach space and (2.1), (2.3), (2.6) are satisfied, then $(C_{\alpha, q}(x_0), T_{\alpha, q})$ is a Banach space.

PROOF : Take into account the definition of $T_{\alpha, q}$ and Theorem 2.1.

3. EXAMPLES

In this Section we present several examples of approximants and norms satisfying (2.1), (2.3) and (2.6) in order to illustrate the theoretical results in Section 2. In the examples 1-4 we shall take the space in eq. (1.8). Observe that the uniform norm (1.9) satisfies properties (2.1) and (2.6). Furthermore $(C[a, b], \| \|_\infty)$ is a Banach space. Thus, if the approximants $\{P_k\}_{k=0}^{\infty}$ hold (2.3) we may conclude from Theorem, 2.3 that $(C_{\alpha, q}(x_0), S_{\alpha, q})$ is a Banach space. Note that in examples 2-6 the constant σ_k in (2.3) does not depend on the functions f and g.

Example 1 — Let $P_k(f)$ be the best approximation of f in Π_k and consider $E_k(f)$, $k = 0$, 1, ..., given by equation (1.10). $E_k(f)$ is the so-called minimax of f in Π_k, see e.g. [3]. Now, if $q = 1$ and $\alpha_k = 1$, $k = 0, 1, ..., S_{\alpha, 1}(f) = \sum_{k=0}^{\infty} E_k(f)$, is a minimax serie. For additionals results about this topic see e.g. [3, 4]. For property (2.3) see [3, p. 82].

Example 2 — For every $n \in \mathbb{N}$, consider the system of nodes given by

$$a \leq x_0^{(n)} < x_1^{(n)} < \ldots < x_n^{(n)} \leq b,$$

and let now $P_n(f)$ be the Lagrange polynomial interpolating f at $\{x_i^{(n)}\}_{i=0}^{n}$. It is well known that $P_n(f)$ can be written in the form

$$P_n(f) = P_n(f; x) = \sum_{i=0}^{n} f(x_i^{(n)}) L_i^{(n)}(x),$$

where $L_i^{(n)}(x)$ are the so-called fundamental Lagrange polynomials, see e.g. [3]. Property (2.3) is also satisfied by Lagrange polynomials. Indeed,

$$| P_n(f;x) - P_n(g;x) | \leq \sum_{i=0}^{m} | L_i^{(n)}(x) | \, | f(x_i^{(n)}) - g(x_i^{(n)}) |, \quad \forall\, x \in [a, b].$$

Thus

$$\| P_n(f) - P_n(g) \|_\infty \leq \sigma_n \| f - g \|_\infty$$

where

$$\sigma_n = \left\| \sum_{i=0}^{n} | L_i^{(n)}(x) | \right\|_\infty$$

is the Lebesgue constant (see e.g. [3]).

Example 3 — Take $[a, b] = [0, 1]$ and let now $P_n(f)$ be the Bernstein polynomial of degree at most n corresponding to f :

$$P_n(f) = P_n(f;x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} x^k (1-x)^{n-k}.$$

Clearly, equation (2.3) is satisfied. Indeed,

$$\| P_n(f;x) - P_n(g;x) \|_\infty = \| P_n(f-g;x) \|_\infty \leq \| f - g \|_\infty.$$

Example 4 — Let $[a, b] = [-1, 1]$ and $P_n(f) = H_n(f;x)$ be the Hermite-Fejér interpolation polynomial corresponding to f :

$$H_n(f;x) = \sum_{k=1}^{n} \left(\frac{T_n(x)}{n(x - x_k^{(n)})} \right)^2 (1 - x x_k^{(n)}) f(x_k^{(n)}).$$

They satisfy

$$H_n(f;x_k^{(n)}) = f(x_k^{(n)}), \; H_n'(f;x_k^{(n)}) = 0, \; k = 1, 2, \ldots, n,$$

where $x_k^{(n)}$, $k = 1, 2, \ldots, n$, are the roots of the nth Tschebysheff polynomials of first kind $T_n(x)$ (see e.g. [3]). Define

$$A_k^{(n)}(x) = \left(\frac{T_n(x)}{n(x - x_k^{(n)})} \right)^2 (1 - x x_k^{(n)}), \qquad k = 1, 2, \ldots, n.$$

Then

$$| H_n(f;x) - H_n(g;x) | = \left| \sum_{k=1}^{n} A_k^{(n)}(x) \, (f(x_k^{(n)}) - g(x_k^{(n)})) \right|, \qquad x \in [-1, 1].$$

Thus

$$\| H_n(f;x) - H_n(g;x) \|_\infty \le \| f-g \|_\infty \left\| \sum_{k=1}^{n} | A_k^{(n)}(x) | \right\|_\infty \le f-g \|_\infty. \quad \ldots. (3.1)$$

For the last inequality in (3.1) take into account that $A_k^{(n)}(x) \ge 0$, $\forall\ x \in [-1, 1]$ and $\sum_{k=1}^{n} A_k^{(n)}(x) = 1$.

Example 5 — Let X be the space of all 2π-periodic continuous functions. For $f \in X$, consider $\| f \|_{C_0} = \sup_{\mathbb{R}} | f(x) |$. Clearly, eqs. (2.1) and (2.6) are satisfied and $(X, \| \|_{C^0})$ is a Banach space. If $f \in X$, let $P_n(f) = S_n(f)$ be the nth partial sum of its Fourier series, see e.g. [4, pp. 99-107]. $S_n(f)$ can be represented by

$$S_n(f;x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x+t) \frac{\sin((2n+1)t/2)}{\sin(t/2)}\, dt, \qquad x \in \mathbb{R}.$$

Approximants S_n satisfy property (2.3). For if,

$$| S_n(f;x) - S_n(g;x) | \le \frac{1}{2\pi} \int_{-\pi}^{\pi} | f(x+t) - g(x+t) | \left| \frac{\sin((2n+1)t/2)}{\sin(t/2)} \right| dt, \ \forall\ x \in \mathbb{R}.$$

Thus

$$| S_n(f;x) - S_n(g;x) | \le \sigma_n \| f-g \|_{C^0}, \qquad \forall\ x \in \mathbb{R},$$

where

$$\sigma_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \frac{\sin((2n+1)\, t/2)}{\sin(t/2)} \right| dt.$$

Hence,

$$\| S_n(f) - S_n(g) \|_{C^0} \le \frac{\sigma_n}{2\pi} \| f-g \|_{C^0}.$$

Example 6 — Let I be the interval $I = I_1 \cup I_2$ where $I = [a, b]$ and $I_2 = (b, c]$. Let X be the space of all functions defined in I such that, they are constants in I_1 and constants in I_2. Thus for $f \in X$, if the value of f in I_1 is equal to the value in I_2, then f is a constant function in I and hence continuous. In the other case, f is discontinuous in I.

For $f \in X$, consider $\| f \| = \max_{x \in I} | f(x) |$. Clearly, eqs. (2.1) and (2.6) are satisfied. Furthermore, $(X, \| \|)$ is a Banach space. Indeed, if $\{ f_n \}_{n=1}^{\infty} \subset X$ is a Cauchy sequence in $(X, \| \|)$ then, following

its definition one deduce that for all $x \in I$, $\left\{ f_n(x) \right\}_{n=1}^{\infty}$ is a Cauchy sequence in $\mathbb{R}$ with the usual norm. Since this one is a Banach space, we can assure that, $f_n(x) \to f(x)$, for all $x \in I$. The function f is constant in I_1 because for all $x \in I_1$ and a fixed point $x_1 \in I_1$, one has

$$f_n(x) \to f(x) \text{ and } f_n(x) = f_n(x_1) \to f(x_1) \Rightarrow f(x) = f(x_1).$$

Analogously in I_2. So, $f \in X$. Furthermore, $f_n \xrightarrow{\| \|} f$ by virtue of the point to point convergence and the fact that they are constant functions in both subintervals I_1 and I_2.

On the other hand, for $f \in X$, let $f_{(i)}$ be the value of the function f in I_i, $i = 1, 2$. Consider the approximants $P_k(f) = (f_{(1)} + f_{(2)})/2$, $k = 0, 1, 2 \ldots$. They hold (2.3). Indeed,

$$\| P_k(f) - P_k(g) \| \leq \left| \frac{f_{(1)} - g_{(1)}}{2} \right| + \left| \frac{f_{(2)} - g_{(2)}}{2} \right| \leq \| f - g \| .$$

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