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A NOTE ON DIFFERENTIAL SUBORDINATION ASSOCIATED WITH *H*-FUNCTION

JAE HO CHOI

Department of Applied Mathematics, Fukuoka University, Fukuoka 814-0180, *Japan*

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In the present paper, we consider a best dominant of the first-order differential subordination and investigate some distortion inequalities for the *H*-function of general hypergeometric type introduced by Fox[1].

Key Words : *H*- and *G*-functions; Subordination

1. INTRODUCTION AND DEFINITIONS

Let $\mathcal{A}'$ denote the class of all analytic functions $f(z)$ with $f(0) = 1$ in the open unit disk $\mathcal{U} = \{z : |z| < 1\}$. Further, let $f(z)$ be an analytic function and $g(z)$ be a univalent function satisfying $f(0) = g(0)$ and $f(\mathcal{U}) \subset g(\mathcal{U})$, then $f(z)$ is said to be subordinate to $g(z)$, and is written as $f(z) \prec g(z)$.

Let $m, n, p, q \in \mathcal{N}_0 = \mathcal{N} \cup \{0\}$ such that $0 \le m \le q$, $0 \le n \le p$, and let $a_i, b_j \in C$ and $\alpha_i, \beta_j \in \mathcal{R}_+ = (0, \infty)$ $(1 \le i \le p;\ 1 \le j \le q)$. The *H*-function occurring in this paper is defined via a Mellin-Barnes type integral in the following way :

$$H^{m,n}_{p,q}(z) \equiv H^{m,n}_{p,q}\left[z \,\middle|\, \begin{matrix} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{matrix} \right] \equiv H^{m,n}_{p,q}\left[z \,\middle|\, \begin{matrix} (a_1, \alpha_1), \ldots, (a_p, \alpha_p) \\ (b_1, \beta_1), \ldots, (b_q, \beta_q) \end{matrix} \right]$$

$$= \frac{1}{2\pi i} \int_{\mathcal{L}} \Theta(s)\, z^s\, ds, \qquad \ldots (1.1)$$

where

$$\Theta(s) = \frac{\prod_{j=1}^{m} \Gamma(b_j - \beta_j s) \prod_{i=1}^{n} \Gamma(1 - a_i + \alpha_i s)}{\prod_{j=m+1}^{q} \Gamma(1 - b_j + \beta_j s) \prod_{i=n+1}^{p} \Gamma(a_i - \alpha_i s)}, \qquad \ldots (1.2)$$

the contour $\mathcal{L}$ is so chosen that the integral (1.1) is absolutely convergent in the whole complex plane, in the unit disk $\mathcal{U}$ or outside of it: that is, $\mathcal{L}$ starts at $-i\infty$ and runs to $+i\infty$ in the s-plane, curving if necessary to put the poles of $\Gamma(b_j - \beta_j s)$ ($j = 1, 2, ..., m$) to the left of path and to put the poles of $\Gamma(1 - a_i + \alpha_i s)$ ($i = 1, 2, n$) to the right of the path. An empty product in (1.2), if it occurs, is taken to be one. For details about the theory of this functions, one may find in [3], [4], [8, §8.3] and [9, Chapter 2]. When $\alpha_i = \beta_j = 1$ ($1 \le i \le p$; $1 \le j \le q$), it reduces to the Meijer G-function[4].

We note that the Bessel function J_η can be represented by the H-function as follows :

$$J_\eta(z) = \frac{(z/2)^\eta}{\Gamma(\eta+1)}\, {}_0F_1\left(-;\eta+1;-\frac{z^2}{4}\right) = \left(\frac{2}{z}\right)^a H^{1,0}_{0,2}\left[\frac{z^2}{4}\,\middle|\, \overline{\left(\frac{a}{2}+\frac{\eta}{2},1\right)\left(\frac{a}{2}-\frac{\eta}{2},1\right)}\right]. \quad ... (1.3)$$

2. PRELIMINARY RESULTS

Before assessing and proving our main results, we state several lemmas to be used in the sequel.

Lemma 1 — (Ponnusamy[7]) Let $p \in \mathcal{A}$, $\alpha \in C$ with $Re\{\alpha\} \ge 0$ ($\alpha \ne 0$) and $\beta < 1$ be such that $Re\,\{p(z) + \alpha z p'(z)\} > \beta$, then

$$Re\{p(z)\} > \beta + (1-\beta)\,[2\delta - 1], \quad ... (2.1)$$

where

$$\delta = \delta(Re\{\alpha\}) = \int_0^1 \frac{dt}{1+t^{Re\{\alpha\}}}, \quad ... (2.2)$$

which is an increasing function of $Re\{\alpha\}$ with $(1 + Re\{\alpha\})/(2 + Re\{\alpha\}) \le \delta < 1$. The estimate (2.1) cannot be improved in general.

Incidentally, the value of δ in (2.2) for $Re\alpha\beta > 0$ can be expressed as the Gauss hypergeometric function

$${}_2F_1\left(1, \frac{1}{Re\{\alpha\}}; 1 + \frac{1}{Re\{\alpha\}}; -1\right), \quad ... (2.3)$$

which may be also rewritten in terms of the difference of two Digamma (or ψ–) functions :

$$\delta = \frac{1}{2\,Re\{\alpha\}}\left[\psi\left(\frac{1+Re\{\alpha\}}{2\,Re\{\alpha\}}\right) - \psi\left(\frac{1}{2\,Re\{\alpha\}}\right)\right] \qquad \left(\psi(z) := \frac{\Gamma'(z)}{\Gamma(z)}\right).$$

Lemma 2 — (Miller-Mocanu[5]; see also Owa-Srivastava[6]) Let $p(z)$ be analytic in $\mathcal{U}$ and $\beta, \gamma \in C$. Let $h(z)$ be convex and univalent in $\mathcal{U}$ with $Re\{\beta h(z) + \gamma\} > 0$ and $p(0) = h(0)$. If

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec h(z), \quad ... (2.4)$$

then $p(z) \prec h(z)$. Furthermore, if the Briot-Bouquet differential equation

$$q(z)+\frac{zq'(z)}{\beta q(z)+\gamma}=h(z) \qquad (q(0)=h(0))$$

has a univalent solution $q(z)$, then $p(z) \prec q(z) \prec h(z)$ and $q(z)$ is the best dominant of (2.4).

Lemma 3 — (Miller-Mocanu[5]) Let $0 \le \beta < 1$. Then the differential equation

$$q_\beta(z)+\frac{zq'_\beta(z)}{q_\beta(z)}=\frac{1+(1-2\beta)z}{1-z} \qquad \text{... (2.5)}$$

has a univalent solution $q_\beta(z)$ given by

$$q_\beta(z)=\begin{cases}\dfrac{(1-2\beta)z}{(1-z)\,(1-(1-z)^{1-2\beta})} & \left(\beta\ne\dfrac{1}{2}\right)\\[2ex] \dfrac{z}{(z-1)\log(1-z)} & \left(\beta=\dfrac{1}{2}\right)\end{cases} \qquad \text{... (2.6)}$$

Lemma 4 — (Jack[2]) Let $w(z)$ be analytic in $\mathcal{U}$ with $w(0)=0$. Then if $|w(z)|$ attains its maximum value on the circle $|z|=r$ $(r<1)$ at a point z_0, we can write

$$z_0\,w'(z_0)=s\,w(z_0), \qquad \text{... (2.7)}$$

where s is real and $s \ge 1$.

Lemma 5 — (Prudnikov *et al.*[8]) Let conditions

$$\alpha_i(b_j+l)\ne\beta_j(a_i-k-1) \qquad (1\le i\le m;\quad 1\le j\le n;\quad k,l\in\mathcal{N}_0) \qquad \text{... (2.8)}$$

and

$$\beta_j(b_i+k)\ne\beta_i(b_j+l) \qquad (i\ne j;\quad 1\le i,\ j\le m;\ k,l\in\mathcal{N}_0) \qquad \text{... (2.9)}$$

be satisfied and either $\Delta>0$, $Z\ne 0$ or $\Delta=0, 0<|z|<\lambda$, where

$$\Delta=\sum_{j=1}^{q}\beta_j-\sum_{j=1}^{p}\alpha_i \qquad \text{and} \qquad \lambda=\prod_{i=1}^{p}\alpha_i^{-\alpha_i}\prod_{j=1}^{q}\beta_j^{-\beta_j}.$$

Then the H-function (1.1) has the power series expansion

$$H_{p,q}^{m,n}(z)$$
$$=\sum_{j=1}^{m}\sum_{l=0}^{\infty}\frac{(-1)^l\prod\limits_{i=1,\,i\ne j}^{m}\Gamma(b_i-[b_j+l]\beta_i/\beta_j)\prod\limits_{i=1}^{n}\Gamma(1-a_i+[b_j+l]\alpha_i/\beta_j)\,z^{(b_j+l)/\beta_j}}{l!\,\beta_j\prod\limits_{i=n+1}^{p}\Gamma(a_i-[b_j+l]\alpha_i/\beta_j)\prod\limits_{i=m+1}^{q}\Gamma(1-b_i+[b_j+l]\alpha_i/\beta_j)}.$$

The next differentiation formula follows from the definition of the H-function given in (1.1) and (1.2).

Lemma 6 — (Prudnikov *et al.*[8]; see also Kilbas-Saigo[3]) There holds the differentiation formula for $\omega, c \in \mathcal{C}, \sigma > 0$:

$$\left(\frac{d}{dz}\right)^k \left\{ z^{\omega} H^{m,n}_{p,q}\left[cz^{\sigma} \left| \begin{matrix} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{matrix} \right.\right]\right\}$$

$$= z^{\omega - k} H^{m,n+1}_{p+1,q+1}\left[cz^{\sigma} \left| \begin{matrix} (-\omega, \sigma), (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q}, (k-\omega, \sigma) \end{matrix} \right.\right]. \qquad \text{... (2.10)}$$

Now we denote the differential operator $\mathcal{D}^*$ by

$$\mathcal{D}^* = z^{k-\omega} \frac{d}{dz} z^{-k+\omega+1} = z\frac{d}{dz} + (-k+\omega+1). \qquad \text{... (2.11)}$$

Then (2.10) implies

$$\mathcal{D}^* H^{1,n+1}_{p+1,q+1}\left[cz^{\sigma} \left| \begin{matrix} (-\omega, \sigma), (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q}, (k-\omega-1, \sigma) \end{matrix} \right.\right]$$

$$= H^{1,n+1}_{p+1,q+1}\left[cz^{\sigma} \left| \begin{matrix} (-\omega, \sigma), (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q}, (k-\omega, \sigma) \end{matrix} \right.\right]. \qquad \text{... (2.12)}$$

3. Main Results

Throughout this paper, we denote

$$H(cz^{\sigma}) := H^{1,n+1}_{p+1,q+1}\left[cz^{\sigma} \left| \begin{matrix} (-\omega, \sigma), (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q}, (k-\omega-1, \sigma) \end{matrix} \right.\right]$$

and the Meijer G-function

$$G(cz) := G^{1,n+1}_{p+1,q+1}\left[cz \left| \begin{matrix} -\omega, (a_i)_{1,p} \\ (b_j)_{1,q}, k-\omega-1 \end{matrix} \right.\right].$$

Applying the above lemmas, we prove.

Theorem 1 — *Let condition (2.8) be satisfied and either $\Delta > 0$, $z \neq 0$ or $\Delta = 0$, $\lambda \geq 1$, and let $\omega, c, b_1, \sigma \in \mathcal{R}_+$, $k \in \mathcal{N}$ and $z \in \mathcal{U}$. Suppose that $n, p, q \in \mathcal{N}_0$ with $q \geq 1$, $0 \leq n \leq p$ and that $a_i, b_j \in \mathcal{R}$ and $\alpha_i, \beta_j \in \mathcal{R}_+$ with $b_1\alpha_i/\beta_1 < a_i \leq b_1\alpha_i/\beta_1 + 1$ and $b_j < b_1\beta_j/\beta_1 + 1$ $(1 \leq i \leq p;\ 1 \leq j \leq q)$. Then the inequality*

$$Re\left\{\frac{\mathcal{D}^* H(cz^{\sigma})}{z^{b_1\sigma/\beta_1}}\right\} > \gamma \qquad \left(1 \leq k < 1 + \omega + \frac{b_1\sigma}{\beta_1};\ 0 \leq \gamma < \Phi(k)\right) \qquad \text{... (3.1)}$$

implies that

$$Re\left\{\frac{H(cz^{\sigma})}{z^{b_1\sigma/\beta_1}}\right\} > \frac{2\beta_1\gamma(1-\delta_1)}{b_1\sigma+\beta_1(\omega-k+1)} + (2\delta_1-1)\,\Phi(k-1), \quad \text{... (3.2)}$$

where, for convenience,

$$\Phi(k) = \frac{\Gamma(1+\omega+b_1\sigma/\beta_1)\prod_{i=1}^{n}\Gamma(1-a_i+b_1\alpha_i/\beta_1)\,c^{b_1/\beta_1}}{\beta_1\prod_{i=n+1}^{p}\Gamma(a_i-b_1\,\alpha_i/\beta_1)\prod_{i=2}^{q}\Gamma(1-b_i+b_1\,\beta_i/\beta_1)\,\Gamma(1-k+\omega+b_1\sigma/\beta_1)} \quad \text{... (3.3)}$$

and

$$\delta_1 = \int_0^1 \frac{dt}{1+t^{\beta_1/\{b_1\sigma+(\omega-k+1)\beta_1\}}}. \quad \text{... (3.4)}$$

The estimate (3.2) is sharp in general.

PROOF : We set

$$p(z) = \frac{H(cz^{\sigma})}{z^{b_1\,\sigma/\beta_1}\,\Phi(k-1)},$$

where $\Phi(k)$ is given by (3.3).

By using Lemma 5 and Lemma 6, we can see that $p(z) \in \mathcal{A}'$ and

$$\frac{\mathcal{D}^{*}\,H(cz^{\sigma})}{z^{b_1\,\sigma/\beta_1}\,\Phi(k)} = p(z) + \frac{\beta_1}{b_1+(\omega-k+1)\beta_1}\,zp'(z).$$

From the inequality (3.1) and Lemma 1, we have

$$Re\{p(z)\} > \frac{\gamma}{\Phi(k)} + \left(1-\frac{\gamma}{\Phi(k)}\right)2(\delta_1-1), \quad \text{... (3.5)}$$

where δ_1 is given by (3.4).

Hence, from (3.5) we obtain (3.2), which completes the proof of Theorem 1.

By setting $\alpha_i = \beta_j = \sigma = 1$ $(1 \le i \le p;\ 1 \le j \le q)$ in Theorem 1, we have

Corollary 1 — Under the hypothesis of Theorem 1, let $\alpha_i = \beta_j = \sigma = 1$ $(1 \le i \le p;\ 1 \le j \le q)$. If

$$Re\left\{\frac{\mathcal{D}^{*}\,G(cz)}{z^{b_1}}\right\} > \gamma \qquad (1 \le k < b_1+\omega+1;\ 0 \le \gamma < \Psi(k)), \quad \text{... (3.6)}$$

then

$$Re\left\{\frac{G(cz)}{z^{b_1}}\right\} > \frac{2\gamma(1-\delta_2)}{b_1+\omega-k+1} + (2\delta_2-1)\,\Psi(k-1), \qquad \ldots (3.7)$$

where

$$\Psi(k) = \frac{\Gamma(1+\omega+b_1)\prod_{i=1}^{n}\Gamma(1-a_i+b_1)\,c^{b_1}}{\prod_{i=n+1}^{p}\Gamma(a_i-b_1)\prod_{i=2}^{q}\Gamma(1-b_i+b_1)\,\Gamma(1-k+\omega+b_1)} \qquad \ldots (3.8)$$

and $\delta_2 = {}_2F_1\,(b_1+\omega-k+1, 1; b_1+\omega-k+2; -1)$. The estimate (3.7) is sharp in general.

Corollary 2 — Let $a+\eta > 0$, $\eta > -1$ and $z \in \mathcal{U}$ $(z \neq 0)$ and let $0 \le \gamma < (a+\eta)/\,(2^{a+\eta}\,\Gamma(1+\eta))$. If

$$Re\left\{\frac{H^{1,1}_{1,3}\left[\frac{z^2}{4}\,\middle|\,\begin{matrix}(0,2)\\ \left(\frac{a}{2}+\frac{\eta}{2},1\right),\left(\frac{a}{2}-\frac{\eta}{2},1\right),(1,2)\end{matrix}\right]}{2^{a+\eta}}\right\} > \gamma, \qquad \ldots (3.9)$$

then

$$Re\left\{\frac{J_\eta(z)}{z^\eta}\right\} > \frac{2^a\gamma}{a+\eta} + \left(\frac{1}{2^\eta\,\Gamma(1+\eta)} - \frac{2^a\alpha}{a+\eta}\right)\left(\int_0^1 \frac{2\,dt}{1+t^{1/(a+\eta)}} - 1\right), \qquad \ldots (3.10)$$

where the function $J_\eta(z)$ is Bessel function defined by (1.3). The estimate (3.10) is sharp in general.

Theorem 2 — *Under the hypothesis of Theorem 1, let, for* $0 \le \rho < 1$,

$$Re\left\{\frac{\mathcal{D}^* H(cz^\sigma)}{H(cz^\sigma)}\right\} > \begin{cases} \frac{b_1}{\beta_1}\sigma+\omega-k+\frac{3}{2}-\frac{1}{2(1-\rho)} & \left(0 \le \rho \le \frac{1}{2}\right) \\ \frac{b_1}{\beta_1}\sigma+\omega-k+\frac{3}{2}-\frac{1}{2\rho} & \left(\frac{1}{2} \le \rho < 1\right). \end{cases} \qquad \ldots (3.11)$$

Then

$$\frac{1}{\Phi(k-1)}\,Re\left\{\frac{H(cz^\sigma)}{z^{b_1\sigma/\beta_1}}\right\} > \rho, \qquad \ldots (3.12)$$

where $\Phi(k)$ *is given by (3.3).*

PROOF : If we define

$$\frac{H(cz^\sigma)}{z^{b_1\sigma/\beta_1}\,\Phi(k-1)} = \frac{1+(1-2\rho)\,w(z)}{1-w(z)},$$

then

$$\frac{\mathcal{D}^* H(cz^{\sigma})}{H(cz^{\sigma})} - \left(\frac{b_1}{\beta_1}\sigma + \omega - k + 1\right) = \frac{zw'(z)}{w(z)}\left\{\frac{(1-2\rho)\,w(z)}{1+(1-2\rho)\,w(z)} + \frac{w(z)}{1-w(z)}\right\}.$$

Suppose that there exists $z_0 \in \mathcal{U}$ such that $\max_{|z| \le |z_0|} |w(z)| = |w(z_0)| = 1$ $(w(z_0) \neq 1)$. By using Lemma 4, we have

$$Re\left\{\frac{\mathcal{D}^* H(cz^{\sigma})}{H(cz^{\sigma})}\right\} = \frac{b_1}{\beta_1}\sigma + \omega - k + 1 + s\,Re\left\{\frac{(1-2\rho)\,e^{i\theta}}{1+(1-2\rho)\,e^{i\theta}} + \frac{e^{i\theta}}{1-e^{i\theta}}\right\}$$

$$= \frac{b_1}{\beta_1}\sigma + \omega - k + 1 + s\left\{\frac{(1-2\rho)\{(1-2\rho)+\cos\theta\}}{1+(1-2\rho)^2 + 2(1-2\rho)\cos\theta} - \frac{1}{2}\right\}$$

for $s \ge 1$ and $\omega(z_0) = e^{i\theta}$ $(0 \le \theta \le 2\pi)$. Hence, let

$$f(t) = \frac{1-2\rho+t}{1+(1-2\rho)^2+2(1-2\rho)t} \qquad (-1 \le t \le 1),$$

then $f(t)$ is increasing if $0 \le \rho < 1$.

If $0 \le \rho \le 1/2$ and $s \ge 1$, then

$$Re\left\{\frac{\mathcal{D}^* H(cz^{\sigma})}{H(cz^{\sigma})}\right\} \le \frac{b_1}{\beta_1}\sigma + \omega - k + 1 - \frac{\rho}{2(1-\rho)}.$$

If, on the other hand, $1/2 \le \rho < 1$, then

$$Re\left\{\frac{\mathcal{D}^* H(cz^{\sigma})}{H(cz^{\sigma})}\right\} \le \frac{b_1}{\beta_1}\sigma + \omega - k + 1 + \frac{\rho - 1}{2\rho}.$$

These contradict (3.11), which completes the proof of Theorem 2.

Corollary 3 — Under the hypothesis of Theorem 1, let $\alpha_i = \beta_j = \sigma = 1$ $(1 \le i \le p;\ 1 \le j \le q)$ and $0 \le \rho < 1$. If

$$Re\left\{\frac{\mathcal{D}^* G(cz)}{G(cz)}\right\} > \begin{cases} b_1 + \omega - k + (2-3\rho)/2(1-\rho) & (0 \le \rho \le 1/2) \\ b_1 + \omega - k + (3\rho - 1)/2\rho & (1/2 \le \rho < 1), \end{cases} \qquad \dots (3.13)$$

then

$$\frac{1}{\Psi(k-1)}\,Re\left\{\frac{G(cz)}{z^{b_1}}\right\} > \rho, \qquad \dots (3.14)$$

where $\Psi(k)$ is given by (3.8).

Theorem 3 — *Under the hypothesis of Theorem 1, let $h(z)$ be a convex function in $\mathcal{U}$ with $h(0) = 1$ and $Re\{h(z)\} > 0$. If*

$$\frac{\mathcal{D}^{*} H(cz^{\sigma})}{z^{b_1\sigma/\beta_1}\,\Phi(k)} \prec h(z), \qquad \dots (3.15)$$

then $p(z) \prec h(z)$, where $\Phi(k)$ and $p(z)$ are defined by (3.3) and (3.5). In addition, if the differential equation

$$q(z) + \frac{\beta_1 z q'(z)}{b_1\sigma + (\omega - k + 1)\beta_1} = h(z) \qquad \dots (3.16)$$

has a univalent solution $q(z)$ in $\mathcal{U}$ that satisfies $q(0) = h(0)$, then $p(z) \prec q(z) \prec h(z)$ and $q(z)$ is the best dominant of (3.15).

PROOF : It follows from (3.5) and (3.15), that

$$\frac{\mathcal{D}^{*} H(cz^{\sigma})}{z^{b_1\sigma/\beta_1}\,\Phi(k)} = p(z) + \frac{\beta_1 z q'(z)}{b_1\sigma + (\omega - k + 1)\beta_1} \prec h(z). \qquad \dots (3.17)$$

Applying Lemma 2, we obtain $p(z) \prec q(z)$. Further, if the differential eq. (3.16) has a univalent solution $q(z)$ in $\mathcal{U}$ that satisfies $q(0) = h(0)$, then the assertion (3.17) implies $p(z) \prec q(z) \prec h(z)$. Thus the proof of Theorem 3 is completed.

***Theorem* 4** — *Under the hypothesis of Theorem 1, let, for $0 \le \gamma < 1$,*

$$Re\left\{\frac{\mathcal{D}^{*} H(cz^{\sigma})}{H(cz^{\sigma})} + p(z)\right\} > \gamma + \frac{b_1}{\beta_1}\sigma + \omega - k + 1 \quad (z \in \mathcal{U}), \qquad \dots (3.18)$$

where $p(z)$ is given by (3.16). Then $Re\{H(cz^{\sigma})/z^{b_1\sigma/\beta_1}\} > \delta(\gamma)$, where

$$\delta(\gamma) = \begin{cases} (2\gamma - 1)\,\Phi(k-1)/2(1 - 2^{1-2\gamma}) & (\gamma \ne 1/2) \\ \Phi(k-1)/2\log 2 & (\gamma = 1/2) \end{cases}$$

and $\Phi(k)$ is given by (3.3). The result is sharp.

PROOF : Making use of (2.11) and (3.16), we have

$$\frac{zp'(z)}{p(z)} = \frac{\mathcal{D}^{*} H(cz^{\sigma})}{H(cz^{\sigma})} - \left(\frac{b_1}{\beta_1}\sigma + \omega - k + 1\right)$$

From (3.18) we obtain $Re\{p(z) + zp'(z)/p(z)\} > \gamma$. Hence, if we set

$$h_{\gamma}(z) = \frac{1 + (1 - 2\gamma)z}{1 - z} \qquad (0 \le \gamma < 1),$$

then Lemma 2 gives $p(z) \prec h_{\gamma}(z)$. Applying Lemma 3, we conclude that

$$q_{\gamma}(z) + \frac{zq_{\gamma}'(z)}{q_{\gamma}(z)} = h_{\gamma}(z) \qquad (q_{\gamma}(0) = h_{\gamma}(0) = 1)$$

has a univalent solution $q_{\gamma}(z)$ given by (2.6). Using Lemma 2 again, we get $p(z) \prec q_{\gamma}(z) \prec h_{\gamma}(z)$. Since

$$\min_{|z|\leq 1} Re\{q_\gamma(z)\} = q_\gamma(-1) = \begin{cases} (2\gamma-1)/2(1-2^{1-2\gamma}) & (\gamma\neq 1/2) \\ 1/2\log 2 & (\gamma = 1/2). \end{cases}$$

Therefore, we find

$$Re\{p(z)\} > \begin{cases} (2\gamma-1)/2(1-2^{1-2\gamma}) & (\gamma\neq 1/2) \\ 1/2\log 2 & (\gamma = 1/2), \end{cases}$$

which completes the proof of Theorem 4.

Corollary 4 — Under the hypothesis of Theorem 1, let $\alpha_i = \beta_j = \sigma = 1$ $(1 \leq i \leq p;\ 1 \leq j \leq q)$ and $0 \leq \gamma < 1$. If

$$Re\left\{\frac{\mathcal{D}^* G(cz)}{G(cz)} + \frac{G(cz)}{z^{b_1}}\right\} > \gamma + b_1 + \omega - k + 1 \qquad (z \in \mathcal{U}),$$

then $Re\{G(cz)/z^{b_1}\} > \delta(\gamma)$, where

$$\delta(\gamma) = \begin{cases} (2\gamma-1)\,\Psi(k-1)/2(1-2^{1-2\gamma}) & (\gamma\neq 1/2) \\ \Psi(k-1)/2\log 2 & (\gamma = 1/2) \end{cases}$$

and $\Phi(k)$ is given by (3.8). The result is sharp.

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