Indian J pure appl Math, **29**(12) 1235-1244, December 1998

ON THE COMMUTATIVE NEUTRIX PRODUCT OF $x_+^{r-1/2}$ AND $x_+^{-r-1/2}$

JOEL D. NICHOLAS AND BRIAN FISHER

Department of Mathematics and Computer Science, Leicester University, Leicester, LE1 7RH, England

(*Received* 1 *December* 1997; *after revision* 12 *June* 1998; *Accepted* 14 *July* 1998)

The commutative neutrix product of the distributions $x_+^{r-1/2}$ and $x_+^{-r-1/2}$ is evaluated for $r = 0, 1, 2, \ldots$ and the product of $x_+^{r-1/2}$ and $x_+^{-r-3/2}$ is deduced for $r = 0, 1, 2, \ldots$

Key Words : Distribution; Delta-Function; Neutrix; Neutrix Limit; Neutrix Product

In the following, we let $\rho(x)$ be an infinitely differentiable function having the following properties :

(i) $\rho(x) = 0$ for $|x| \geq 1$,

(ii) $\rho(x) \geq 0$,

(iii) $\rho(x) = \rho(-x)$

and (iv) $\int_{-1}^{1} \rho(x)\, dx = 1.$

Putting $\delta_n(x) = n\rho(nx)$ for $n = 1, 2, \ldots$, it follows that $\{\delta_n(x)\}$ is a regular sequence of infinitely differentiable functions converging to the Dirac delta-function $\delta(x)$.

Now let $\mathcal{D}$ be the space of infinitely differentiable functions with compact support and let $\mathcal{D}'$ be the space of distributions defined on $\mathcal{D}$. Then if f is an arbitrary distribution in $\mathcal{D}'$, we define

$$f_n(x) = (f * \delta_n)(x) = \langle f(t), \delta_n(x-t) \rangle$$

for $n = 1, 2, \ldots$. It follows that $\{f_n(x)\}$ is a regular sequence of infinitely differentiable functions converging to the distribution $f(x)$.

A first extension of the product of a distribution and an infinitely differentiable function is the following, see for example[2].

Definition 1 — Let f and g be distributions in $\mathcal{D}'$ for which on the interval (a, b), f is the kth derivative of a locally summable function F in $L^p(a, b)$ and $g^{(k)}$ is a locally summable function in $L^q(a, b)$ with $1/p + 1/q = 1$. Then the product $fg = gf$ of f and g is defined on the interval (a, b) by

$$fg = \sum_{i=0}^{k} \binom{k}{i} (-1)^i [Fg^{(i)}]^{(k-i)}.$$

In the following, we let N be the neutrix, see van der Corput[1], having domain $N' = \{1, 2, \ldots, n, \ldots\}$ and range the real numbers, with negligible functions finite linear sums of the functions

$$n^{\lambda} \ln^{r-1} n, \quad \ln^r n : \quad \lambda > 0, \; r = 1, 2, \ldots$$

and all functions which converge to zero in the normal sense as n tends to infinity. A function $f(n)$ is then said to have the neutrix limit L as n tends to infinity if $f(n) - L$ is a negligible function and we then write $\underset{n \to \infty}{N-\lim} f(n) = L$. It is clear that the nuetrix limit is equal to the usual limit if it exists.

Neutrix limits can be used to define functions and distributions. For example, it was proved in [4] that the Gamma function $\Gamma(x)$ can be defined for $x < 0$ by

$$\Gamma(x) = \underset{n \to \infty}{N-\lim} \int_{1/n}^{\infty} t^{x-1} e^{-t}\, dt, \quad x \neq 1, -2, \ldots .$$

This is because if $-r < x < -r + 1$, then

$$\int_{1/n}^{\infty} t^{x-1} e^{-t}\, dt = \int_{1/n}^{\infty} t^{x-1} \left[e^{-t} - \sum_{i=0}^{r-1} \frac{(-t)^i}{i!} \right] dt - \sum_{i=0}^{r-1} \frac{(-t)^i}{i!(x+i)} n^{-x-i}.$$

Since the second sum in this equation is a sum of negligible functions, it follows that

$$\underset{n \to \infty}{N-\lim} \int_{1/n}^{\infty} t^{x-1} e^{-t}\, dt = \int_{0}^{\infty} t^{x-1} \left[e^{-t} - \sum_{i=0}^{r-1} \frac{(-t)^i}{i!} \right] dt = \Gamma(x).$$

Similarly, if x_+^{λ} is the locally summable function defined by

$$x_+^{\lambda} = \begin{cases} x^{\lambda}, & x \geq 0, \\ 0, & x < 0 \end{cases}$$

for $\lambda > -1$ and x_+^{λ} is the distribution defined inductively for $\lambda < -1$ and $\lambda \neq -2, -3, \ldots$ by the equation

$$x_+^{\lambda - 1} = \lambda^{-1} (x_+^{\lambda})',$$

then

$$\langle x_+^{\lambda}, \varphi \rangle = N-\lim_{n\to\infty} \int_{1/n}^{\infty} x^{\lambda} \varphi(x)\, dx, \quad \lambda \neq -1, -2, \ldots$$

for arbitrary function φ in $\mathcal{D}$.

The above results gave the motivation for the following definition of the commutative neutrix product of two distributions which was given in [2] and generalizes Definition 1.

Definition 2 — Let f and g be distributions in $\mathcal{D}'$ and let $f_n(x) = (f * \delta_n(x))$, $g_n(x) = (g * \delta_n)(x)$. We say that the nuetrix product $f \square g$ of f and g exists and is equal to the distribution h on the interval (a, b) if

$$N-\lim_{n\to\infty} \langle f_n(x)\, g_n(x), \varphi(x) \rangle = \langle h(x), \varphi(x) \rangle$$

for all functions φ in $\mathcal{D}$ with support contained in the interval (a, b).

We now prove the following theorem.

Theorem 1 — *The nuetrix product* $x_+^{r-1/2} \square x_+^{-r-1/2}$ *exists and*

$$x_+^{r-1/2} \square x_+^{-r-1/2} = x_+^{-1} + a_r\, \delta(x) \qquad \ldots (1)$$

for $r = 0, 1, 2, \ldots$, *where*

$$a_0 = 2 \ln 2 - 2c(\rho),$$

$$a_r = 2 \ln 2 - 2c(\rho) - 2 \sum_{i=1}^{r} \frac{1}{2i-1}$$

for $r = 1, 2, \ldots$ *and*

$$c(\rho) = \frac{1}{2} \int_{-1}^{1} \int_{-1}^{1} \rho(v)\, \rho(u)\, \ln|u - v|\, du\, dv.$$

PROOF : We first of all consider the case $r = 0$ and put

$$(x_+^{-1/2})_n = x_+^{-1/2} * \delta_n(x)$$

$$
= \begin{cases} \displaystyle\int_{-1/n}^{1/n} (x-t)^{-1/2}\,\delta_n(t)\,dt, & x > 1/n, \\[2ex] \displaystyle\int_{-1/n}^{x} (x-t)^{-1/2}\,\delta_n(t)\,dt, & -1/n \le x \le 1/n, \\[2ex] 0, & x < -1/n. \end{cases}
$$

Then

$$
\int_{-1/n}^{1} [(x_+^{-1/2})_n]^2\,dx = \int_{-1/n}^{1/n} \int_{-1/n}^{x} \int_{-1/n}^{x} (x-t)^{-1/2}\,\delta_n(t)\,(x-s)^{-1/2}\,\delta_n(s)\,ds\,dt\,dx
$$

$$
+ \int_{1/n}^{1} \int_{-1/n}^{1/n} \int_{-1/n}^{1/n} (x-t)^{-1/2}\,\delta_n(t)\,(x-s)^{-1/2}\,\delta_n(s)\,ds\,dt\,dx
$$

$$
= \int_{-1/n}^{1/n} \delta_n(t) \int_{t}^{1/n} \delta_n(s) \int_{s}^{1/n} (x-t)^{-1/2}(x-s)^{-1/2}\,dx\,ds\,dt
$$

$$
+ \int_{-1/n}^{1/n} \delta_n(t) \int_{-1/n}^{t} \delta_n(s) \int_{t}^{1/n} (x-t)^{-1/2}(x-s)^{-1/2}\,dx\,ds\,dt
$$

$$
+ \int_{-1/n}^{1/n} \delta_n(t) \int_{-1/n}^{1/n} \delta_n(s) \int_{1/n}^{1} (x-t)^{-1/2}(x-s)^{-1/2}\,dx\,ds\,dt
$$

$$
= \int_{-1}^{1} \rho(v) \int_{v}^{1} \rho(u) \int_{u}^{1} (w-v)^{-1/2}(w-u)^{-1/2}\,dw\,du\,dv
$$

$$
+ \int_{-1}^{1} \rho(v) \int_{-1}^{v} \rho(u) \int_{v}^{1} (w-v)^{-1/2}(w-u)^{-1/2}\,dw\,du\,dv
$$

$$
+ \int_{-1}^{1} \rho(v) \int_{-1}^{1} \rho(u) \int_{1}^{n} (w-v)^{-1/2}(w-u)^{-1/2}\,dw\,du\,dv,
$$

where the substitutions $ns = u$, $nt = v$ and $nx = w$ have been made.

Now

$$\int_u^1 (w-v)^{-1/2}(w-u)^{-1/2}\,dw = 2\Big[\ln[(w-v)^{1/2}+(w-u)^{1/2}]\Big]_u^1$$

$$= 2\ln[(1-v)^{1/2}+(1-u)^{1/2}] - \ln|u-v|,$$

$$\int_v^1 (w-v)^{-1/2}(w-u)^{-1/2}\,dw = 2\ln[(1-v)^{1/2}+(1-u)^{1/2}] - \ln|u-v|,$$

$$\int_1^n (w-v)^{-1/2}(w-u)^{-1/2}\,dw = 2\ln[(n-v)^{1/2}+(n-u)^{1/2}]$$

$$-\ 2\ln[(1-v)^{1/2}+(1-u)^{1/2}]$$

and it follows that

$$\int_{-1/n}^1 [(x_+^{-1/2})_n]^2\,dx = -\int_{-1}^1\int_{-1}^1 \rho(v)\,\rho(u)\ln|u-v|\,du\,dv$$

$$+\int_{-1}^1\int_{-1}^1 \rho(v)\,\rho(u)\,\{\ln n + 2\ln[(1-v/n)^{1/2}+(1-u/n)^{1/2}]\}\,du\,dv.$$

Thus

$$\underset{n\to\infty}{N-\lim}\int_{-1/n}^1 [(x_+^{-1/2})_n]^2\,dx = 2\ln 2 - \int_{-1}^1\int_{-1}^1 \rho(v)\,\rho(u)\ln|u-v|\,du\,dv$$

$$= 2\ln 2 - 2c(\rho) = a_0. \qquad \ldots (2)$$

Further

$$\int_{-1/n}^{1/n} |[(x_+^{-1/2})_n]^2 x|\,dx$$

$$= \int_{-1/n}^{1/n}\int_{-1/n}^{x}\int_{-1/n}^{x} [x(x-t)^{-1/2}\delta_n(t)\,(x-s)^{-1/2}\delta_n(s)|\,ds\,dt\,dx$$

$$= n^{-1}\int_{-1}^1 \rho(v)\int_v^1 \rho(u)\int_u^1 |w(w-v)^{-1/2}(w-u)^{-1/2}|\,dw\,du\,dv$$

using the above substitutions again. It follows that if ψ is any continuous function, then

$$\lim_{n\to\infty} \int_{-1/n}^{1/n} [(x_+^{-1/2})_n]^2 x\, \psi(x)\, dx = 0. \qquad \text{... (3)}$$

If now $0 < \eta < 1$ and $n > \eta^{-1}$, then

$$\left| \int_{1/n}^{\eta} [(x_+^{-1/2})_n]^2 x\, dx \right|$$

$$= \int_{1/n}^{\eta} \int_{-1/n}^{1/n} \int_{-1/n}^{1/n} x(x-t)^{-1/2}\, \delta_n(t)\, (x-s)^{-1/2}\, \delta_n(s)\, ds\, dt\, dx$$

$$= n^{-1} \int_{-1}^{1} \rho(v) \int_{-}^{1} \rho(u) \int_{1}^{n\eta} w(w-v)^{-1/2} (w-u)^{-1/2}\, dw\, du\, dv,$$

where

$$\int_{1}^{n\eta} w(w-v)^{-1/2} (w-u)^{-1/2}\, dw = \sum_{i=0}^{\infty} \binom{-1/2}{i} \int_{1}^{n\eta} \left[\frac{u+v}{w} - \frac{uv}{w^2} \right]^i dw$$

$$= n\eta - 1 + \frac{1}{2}(u+v)\, ln(n\eta) + \ldots$$

and it follows that

$$\lim_{n\to\infty} \int_{1/n}^{\eta} \cdot [(x_+^{-1/2})_n]^2 x\, dx = \eta.$$

Thus, if ψ is a continuous function

$$\lim_{n\to\infty} \int_{1/n}^{\eta} [(x_+^{-1/2})_n]^2 x\psi(x)\, dx = O(\eta). \qquad \text{... (4)}$$

Now let φ be an arbitrary function in $\mathcal{D}$ with support contained in the interval $[-1, 1]$. By the mean value theorem

$$\varphi(x) = \varphi(0) + x\varphi'(\xi x), \qquad \text{... (5)}$$

where $0 < \xi < 1$ and so

$$\langle [(x_+^{-1/2})_n]^2, \varphi(x) \rangle = \int_{-1/n}^{1} [(x_+^{-1/2})_n]^2 \varphi(x)\,dx$$

$$= \varphi(0) \int_{-1/n}^{1} [(x_+^{-1/2})_n]^2\,dx + \int_{-1/n}^{1/n} [(x_+^{-1/2})_n]^2 x\varphi'(\xi x)\,dx$$

$$+ \int_{1/n}^{\eta} [(x_+^{-1/2})_n]^2 x\varphi'(\xi x)\,dx + \int_{\eta}^{1} [(x_+^{-1/2})_n]^2 x\varphi'(\xi x)\,dx.$$

Using eqs. (2), (3), (4) and (5) and noting that the sequence $\{[(x_+^{-1/2})_n]^2\}$ converges uniformly to x^{-1} on the interval $[\eta, 1]$, it follows that

$$\underset{n\to\infty}{N-\lim} \langle [(x_+^{-1/2})_n]^2, \varphi(x) \rangle = a_0\varphi(0) + O(\eta) + \int_{\eta}^{1} x^{-1}[\varphi(x) - \varphi(0)]\,dx$$

$$= a_0\varphi(0) + O(\eta) + \int_{0}^{1} x^{-1}[\varphi(x) - \varphi(0)]\,dx.$$

Since η can be made arbitrarily small, it follows that

$$\underset{n\to\infty}{N-\lim} \langle [(x_+^{-1/2})_n]^2, \varphi(x) \rangle = a_0\varphi(0) + \langle x^{-1}, \varphi(x) \rangle,$$

giving eq. (1) on the interval $[-1, 1]$ when $r = 0$. However, eq. (1) clearly holds on any closed interval not containing the origin.

Now suppose that eq. (1) holds for a non-negative integer r. The product of $x_+^{r+1/2}$ and $x_+^{-r-1/2}$ exists by Definition 1 and is equal to H. Differentiating the equation

$$x_+^{r+1/2}\, x_+^{-r-1/2} = H(x)$$

we get

$$(r + 1/2)\, x_+^{r-1/2} \,\Box\, x_+^{-r-1/2} - (r + 1/2)\, x_+^{r+1/2} \,\Box\, x_+^{-r-3/2} = \delta(x)$$

and so

$$x_+^{r+1/2} \,\Box\, x_+^{-r-3/2} = x_+^{-1} + a_r\,\delta(x) - \frac{2}{2r+1}\,\delta(x)$$

$$= x_+^{-1} + a_{r+1}\,\delta(x).$$

Eq. (1) now follows by induction for $r = 0, 1, 2, \ldots$.

Corollary 1.1 — The neutrix product $x_-^{r-1/2} \square x_-^{-r-1/2}$ exists and

$$x_-^{r-1/2} \square x_-^{-r-1/2} = x_-^{-1} + a_r \delta(x) \qquad \ldots (6)$$

for $r = 0, \pm 1, \pm 2, \ldots$

PROOF : Eq. (6) immediately follows from Theorem 1 on replacing x by $-x$.

To prove our next corollary, we need the following result proved in [2] :

$$x_+^{\lambda} \cdot x_-^{-1-\lambda} = -\frac{1}{2}\pi \operatorname{cosec}(\pi\lambda)\, \delta(x)$$

for $\lambda \neq 0, \pm 1, \pm 2, \ldots$ so that in particular

$$x_+^{r-1/2} \cdot x_-^{-r-1/2} = \frac{1}{2}(-1)^r \pi\delta(x) \qquad \ldots (7)$$

for $r = 0, \pm 1, \pm 2, \ldots$

Corollary 1.2 — The neutrix product $(x + i0)^{r-1/2} \square (x + i0)^{-r-1/2}$ exists and

$$(x + i0)^{r-1/2} \square (x + i0)^{-r-1/2} = (x + i0)^{-1}$$

for $r = 0, \pm 1, \pm 2, \ldots$.

PROOF : The distribution $(x + i0)^{\lambda}$ is defined by

$$(x + i0)^{\lambda} = x_+^{\lambda} + e^{i\lambda\pi} x_-^{\lambda}$$

for $\lambda \neq -1, -2, \ldots$ and

$$(x + i0)^{-r} = x^{-r} + \frac{i\pi(-1)^r}{(r-1)!} \delta^{(r-1)}(x)$$

for $r = 1, 2, \ldots$, see Gel'fand and Shilov[5]. Thus

$$\begin{aligned}
&(x + i0)^{r-1/2} \square (x + i0)^{-r-1/2} \\
&= [x_+^{r-1/2} - (-1)^r i x_-^{r-1/2}] \square [x_+^{-r-1/2} - (-1)^r i x_-^{-r-1/2}] \\
&= x_+^{r-1/2} \square x_+^{-r-1/2} - (-1)^r i x_+^{r-1/2} \square x_-^{-r-1/2} \\
&\qquad - (-1)^r i x_-^{r-1/2} \square x_+^{-r-1/2} - x_-^{r-1/2} \square x_-^{-r-1/2} \\
&= (x + i0)^{-1}
\end{aligned}$$

on using eqs. (1), (6) and (7).

The following results can be proved similarly :

$$|x|^{r-1/2} \square [\operatorname{sgn} x \cdot |x|^{-r-1/2}] = x^{-1},$$

$$|x|^{r-1/2} \square |x|^{-r-1/2} = |x|^{-1} + [2a_r + (-1)^r \pi]\, \delta(x),$$

$$[\operatorname{sgn} x \cdot |x|^{r-1/2}] \square [\operatorname{sgn} x \cdot |x|^{-r-1/2}] = |x|^{-1} + [2a_r - (-1)^r \pi]\, \delta(x),$$

for $r = 0, \pm 1, \pm 2, \ldots$, since by definition

$$x^{-1} = x_+^{-1} - x_-^{-1},\ |x|^{\lambda} = x_+^{\lambda} + x_-^{\lambda},\ \operatorname{sgn} x \cdot |x|^{\lambda} = x_+^{\lambda} - x_-^{\lambda}.$$

Theorem 2 — *The neutrix product* $x_+^{r-1/2} \square x_+^{-r-3/2}$ *exists and*

$$x_+^{r-1/2} \square x_+^{-r-3/2} = x_+^{-2} - b_r\, \delta'(x) \qquad \ldots (8)$$

for $r = 0, \pm 1, \pm 2, \ldots$, *where*

$$b_0 = a_0,$$

$$b_r = \frac{a_0}{2r+1} + 2 \sum_{i=1}^{r} \frac{a_i}{2r+1}$$

$$= 2 \ln 2 - 2c(\rho) - \frac{4}{2r+1} \sum_{i=1}^{r} \sum_{j=1}^{i} \frac{1}{2j-1}.$$

PROOF : We first of all consider the case $r = 0$. Differential equation (1) for the case $r = 0$ we get

$$x_+^{-1/2} \square x_+^{-3/2} = x_+^{-2} - a_0\, \delta'(x),$$

and so eq. (8) is true for $r = 0$.

Now assume that eq. (8) is true for some non-negative integer r. From eq. (1) we have

$$x_+^{r+1/2} \square x_+^{-r-3/2} = x_+^{-1} + a_{r+1}\, \delta(x)$$

and differentiating this equation we get

$$\left(r + \frac{1}{2}\right) x_+^{r-1/2} \square x_+^{-r-3/2} - \left(r + \frac{3}{2}\right) x_+^{r+1/2} \square x_+^{-r-5/2} = -\, x_+^{-2} + a_{r+1}\, \delta'(x).$$

Thus

$$x_+^{r+1/2} \,\square\, x_+^{-r-5/2} = \frac{2r+1}{2r+3} x_+^{r-1/2} \,\square\, x_+^{-r-3/2} + \frac{2}{2r+3}[x_+^{-2} - a_{r+1}\,\delta'(x)]$$

$$= \frac{2r+1}{2r+3}[x_+^{-2} - b_r\,\delta'(x)] + \frac{2}{2r+3}[x_+^{-2} - a_{r+1}\,\delta'(x)]$$

$$= x_+^{-2} - b_{r+1}\,\delta'(x)$$

and eq. (8) now follows by induction for $r = 0, 1, 2, \ldots$.

The following corollary follows on replacing x by $-x$ in equation (8).

Corollary 2.1 — The neutrix product $x_-^{r-1/2} \,\square\, x_-^{-r-3/2}$ exists and

$$x_-^{r-1/2} \,\square\, x_-^{-r-3/2} = x_-^{-2} - b_r\,\delta'(x) \qquad \ldots (9)$$

for $r = 0, \pm 1, \pm 2, \ldots$.

The next corollary follows as above on noting[3]

$$x_+^{r-1/2} \,\square\, x_-^{-r-3/2} = \frac{1}{2}(-1)^r \pi\delta'(x). \qquad \ldots (10)$$

Corollary 2.2 — The neutrix product $(x+i0)^{r-1/2} \,\square\, (x+i0)^{-r-3/2}$ exists and

$$(x+i0)^{r-1/2} \,\square\, (x+i0)^{-r-3/2} = (x+i0)^{-2}$$

for $r = 0, \pm 1, \pm 2, \ldots$.

The final results follow as above on using eqs. (8), (9) and (10).

$$|x|^{r-1/2} \,\square\, [\operatorname{sgn} x \cdot |x|^{-r-3/2}] = x^{-2},$$

$$|x|^{r-1/2} \,\square\, \cdot\, |x|^{-r-3/2} = |x|^{-2} - [2b_r - (-1)^r \pi]\,\delta'(x),$$

$$[\operatorname{sgn} x \cdot |x|^{r-1/2}] \,\square\, [\operatorname{sgn} x \cdot |x|^{-r-3/2}] = |x|^{-1} - [2b_r - (-1)^r \pi]\,\delta'(x)$$

for $r = 0, \pm 1, \pm 2, \ldots$.

REFERENCES

1. J. G. van der Corput, *J. anal. Math.*, **7** (1959-60), 291-98.
2. B. Fisher, *Quart. J. Math. Oxford*, (2), **22** (1971), 291-98.
3. B. Fisher, *Studia Math.*, **57** (1976), 263-74.
4. B. Fisher, "Neutrices and Distributions" In : *Complex Analysis and Applications* '87", Sofia, 1989, 169-75.
5. I. M. Gel'fand and G. E. Shilov, *Generalized Functions*, Vol. I, Academic Press, 1964.