

THE PRIME AND MAXIMAL IDEALS IN $R[X]$, R : A ONE-DIMENSIONAL PRÜFER DOMAIN

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In this paper, the prime and maximal ideals of a polynomial ring $R[x]$ in an indeterminate x over a Prüfer domain R of Krull dimension one are explicitly determined, using elementary results of ring theory.

Key Words : Ideal; Multiplicative Ideal Theory

INTRODUCTION

In 1990, Lemmer and Naude¹ gave a simple method for the description of the already known prime and maximal ideals of the polynomial ring $R[x]$, R a principal ideal domain. Our goal here is to explicitly determine all the maximal ideals of $R[x]$, using only elementary results of ring theory, where R is a Prüfer domain of Krull dimension one. We also classify the prime ideals of $R[x]$ of height one and two. It may be remarked that the prime ideals of $R[x]$ are already known (cf. [2, Chapter 1]).

As in [3, §11], an integral domain R is called a Prüfer domain of Krull dimension one if for every non-zero prime ideal IP of R (i) R_P is a valuation ring. (ii) IP is a maximal ideal of R . Throughout V_P will stand for the valuation of the field of quotients of R with valuation ring R_P and residue field R/IP , IP being a non-zero prime ideal of R . If $f(x)$ is in $R_P[x]$, $\bar{f}(x) \in R/IP[x]$ will denote the polynomial obtained by replacing the coefficients of f by their respective images under the canonical homomorphism from R_P onto R/IP . We shall prove :

Theorem 1.1 — *Let R be a Prüfer domain of Krull dimension one with quotient field K . Then*

(a) *the maximal ideals of $R[x]$ are :*

- (i) $IPR[x] + f(x)R[x]$ with IP a non-zero prime ideal in R , $f(x) \in R[x]$ a monic polynomial and $\bar{f}(x)$ irreducible in $R/IP[x]$.

- (ii) $(g(x)K[x]) \cap R[x]$ with $g(x) = \sum_i c_i x^i$ irreducible in $K[x]$, and (*) $v_P(c_0) < v_P(c_i)$ for all $i \geq 1$ and for each non-zero prime ideal P of R .

(b) The non-maximal prime ideals of $R[x]$ are :

- (iii) $IP[x]$, with P a non-zero prime ideal in R .
 (iv) $(g(x)K[x]) \cap R[x]$ with $g(x) = \sum_i c_i x^i$ irreducible in $K[x]$, and (*) not satisfied.

Remark : If R is a generalized Dedekind domain (or in particular a Dedekind domain) with infinitely many prime ideals, then (*) is never satisfied because for any $c \in K$, $v_P(c) = 0$ for all but finitely many P by virtue of [3, Theorem 12.10]. So in this case, the maximal ideals of $R[x]$ are only those given by (i).

Keeping in view that for any valuation v of a field K , its canonical extension v^x to the simple transcendental extension $K(x)$, defined by $v^x\left(\sum_i a_i x^i\right) = \min_i \{v(a_i)\}$ is a valuation of $K(x)$, the following corollary can be easily deduced from the above theorem.

Corollary 1.2 — Let R be the valuation ring of a rank 1 valuation v of a field K with maximal ideal M . Then

(a) the maximal ideals of $R[x]$ are :

- (i) $MR[x] + f(x)R[x]$, $f(x) \in R[x]$, $\bar{f}(x)$ irreducible in $R/M[x]$.
 (ii) $g(x)R[x]$ with $g(x) = \sum_i c_i x^i$ irreducible in $K[x]$, and (**) $v(c_0) = 0$, $v(c_i) > 0$ for all $i \geq 1$.

(b) The non-maximal prime ideals of $R[x]$ are :

- (iii) $MR[x]$,
 (iv) $g(x)R[x]$ with $g(x) = \sum_i c_i x^i \in R[x] \setminus M[x]$ irreducible over K and (**) not satisfied.

PROOF OF THEOREM 1.1 :

The proof of the theorem depends upon three lemmas, the first two of which are well-known (see [2, Theorem 28, 36]. For the sake of completeness, we prove them here. In fact, our proof of the first lemma is constructive.

Lemma 2.1 — Let R be an integral domain with quotient field K and P be a non-zero prime ideal in $R[x]$ such that $P \cap R = \{0\}$ then $P = (g(x)K[x]) \cap R[x]$, where $g(x) \in K[x]$ is irreducible.

PROOF : Let $f(x)$ be a non-zero element of P . Since $P \cap R = \{0\}$, $f(x)$ must be a non-constant polynomial. Write

$$f(x) = f_1(x)f_2(x) \dots f_r(x),$$

where each $f_i(x) \in K[x]$ is irreducible over K . Choose non-zero $a_i \in R$ such that $a_i f_i(x) \in R[x]$, $1 \leq i \leq r$. As P is a prime ideal of $R[x]$, $a_j f_j(x) \in P$ for some j . Set $g(x) = a_j f_j(x)$. Claim is that

$$P = (g(x) K[x]) \cap R[x]. \quad \dots (1)$$

Let $h(x)$ be a non-zero element of P . Suppose, if possible, $h(x)$ does not belong to the right hand side of (1). Then $h(x)$ won't be divisible by the irreducible polynomial $g(x)$. So there exist $A(x), B(x) \in K[x]$ satisfying

$$g(x) A(x) + h(x) B(x) = 1. \quad \dots (2)$$

Let a, b be a non-zero elements of R such that the polynomials $aA(x)$ and $bB(x)$ are in $R[x]$. Keeping in view that $g(x), h(x) \in P$, it immediately follows from (2) that $ab \in P$, which contradicts the hypothesis. Hence, $P \subseteq (g(x) K[x]) \cap R[x]$. The other inclusion can be easily verified.

Remark : Let R and K be as in the above lemma and $g(x) \in K[x]$ be irreducible, then it can be easily verified that $g(x) K[x]$ is a prime ideal of $K[x]$ and hence $(g(x) K[x]) \cap R[x]$ is a prime ideal of $R[x]$, which does not contain any non-zero element of R .

Lemma 2.2 — Let R be an integral domain and IP be a maximal ideal of R , then —

(a) An ideal M of $R[x]$ containing IP is maximal if and only if $M = IPR[x] + f(x) R[x]$, where $f(x) \in R[x]$ is monic and $\bar{f}(x)$ is irreducible over R/IP .

(b) $IPR[x]$ is the only non-maximal prime ideal of $R[x]$ containing IP .

PROOF : Let $\phi: R[x] \rightarrow R/IP[x]$ be the natural epimorphism whose kernel is $IPR[x]$. So there exists a 1-1 order preserving correspondence between the ideals of $R[x]$ containing $IPR[x]$ and ideals of $R/IP[x]$. Assertion (a) follows immediately by virtue of the fact that an ideal I of the principal ideal domain $R/IP[x]$ is maximal if and only if it is generated by a monic irreducible polynomial in $R/IP[x]$. As regards (b), observe that $\{0\}$ is the only prime ideal of $R/IP[x]$ which is not maximal. So $\phi^{-1}\{0\} = IPR[x]$ is the only prime ideal of $R[x]$ containing IP , which is not maximal.

Lemma 2.3 — Let R be a Prüfer domain of Krull dimension one with quotient field K and $P = (g(x) K[x]) \cap R[x]$ be a prime ideal of $R[x]$ with $g(x) = \sum c_i x^i$ an irreducible element of $K[x]$. Then P is a maximal ideal of $R[x]$ if and only if $v_P(c_0) < v_P(c_i)$ for all $i \geq 1$ and for each non-zero prime ideal IP of R .

PROOF : We shall denote by v_P^x , the Gaussian valuation obtained on extending the valuation v_P of K to $K(x)$, defined by

$$v_P^x \left(\sum_i a_i x^i \right) = \min_i \{v_P(a_i)\}.$$

Suppose first that P is a maximal ideal in $R[x]$. Let IP be a non-zero prime ideal of R . Since $P \cap R = \{0\}$ and P is maximal, it follows that $IPR[x] + P = R[x]$. Consequently,

$$IPR_P[x] + PR_P[x] = R_P[x]. \quad \dots (3)$$

Let c_j be a coefficient of $g(x)$ such that $v_P^x(g(x)) = v_P^x(c_j)$. Set $g_1(x) = g(x)/c_j$, then

$$v_P^x(g_1(x)) = 0. \quad \dots (4)$$

Keeping in view that $P \subseteq g(x)K[x] = g_1(x)K[x]$, it is immediate from (3) that there exists $h(x) \in K[x]$ such that

$$1 - g_1(x)h(x) \in IPR_P[x], \text{ i.e.,}$$

$$v_P^x(1 - g_1(x)h(x)) > 0. \quad \dots (5)$$

It follows from (4) and (5) that

$$v_P^x(h(x)) = 0 \quad \text{and} \quad \bar{g}_1(x)\bar{h}(x) = \bar{1}.$$

In particular, $\bar{g}_1(x) \in R/IP[x]$ is a constant polynomial which is possible only if $v_P(c_0) < v_P(c_i)$ for all $i \geq 1$.

Conversely, suppose that P is not a maximal ideal in $R[x]$. So there exists a maximal ideal M of $R[x]$ such that $P \subset M$ choose $u(x) \in M \setminus P$, then $u(x)$ won't be divisible by the irreducible polynomial $g(x)$ over K . So there exists $A(x), B(x) \in K[x]$ such that

$$g(x)A(x) + u(x)B(x) = 1. \quad \dots (6)$$

If $a, b, c \in R$ are such that the polynomials $aA(x)$, $bB(x)$ and $cg(x)$ belong to $R[x]$, then multiplying (6) by abc , we see that $abc \in M$. Thus $M \cap R \neq \{0\}$.

Since $M \cap R (= IP \text{ say})$, is a non-zero prime ideal of the ring R having Krull dimension one, it is a maximal ideal of R . By Lemma 2.2, there exists $f(x) \in R[x]$ with $\bar{f}(x)$ irreducible over R/IP such that $M = IPR[x] + f(x)R[x]$ which implies that

$$MR_P[x] = IPR_P[x] + f(x)R_P[x]. \quad \dots (7)$$

Let $g_1(x)$ be a constant multiple of $g(x)$ satisfying (4). Claim is $g_1(x) \in PR_P[x]$. Since $g_1(x) \in R_P[x]$, we can write

$$g_1(x) = \sum (a_i/b_i)x^i, \quad a_i, b_i \in R, \quad b_i \notin IP.$$

Then clearly $(\prod b_i)g_1(x) \in P$ and hence $g_1(x) \in PR_P[x] \subseteq MR_P[x]$, which proves the claim. It now follows from (7) that there exists $t(x) \in R_P[x]$ such that

$$g_1(x) - f(x)t(x) \in IPR_P[x].$$

Consequently, $\bar{g}_1(x) = \bar{f}(x)\bar{t}(x)$, but $\bar{f}(x)$ being irreducible is a non-constant polynomial. Hence, so must be $\bar{g}_1(x)$, i.e., $v_P(c_0) \geq v_P(c_i)$ for some $i \geq 1$. The proof of lemma is now complete.

The theorem follows immediately from the above three lemmas.

Remark : As in [2, Theorem 37], one can deduce from Lemma 2.1 that for any ring R , there can't exist in $R[x]$, a chain of three distinct prime ideals with the same contraction in R . Therefore, when R is a one-dimensional Prüfer domain, it follows quickly from Theorem 1.1 that $\dim R[x]$ is 2 and that all the height two prime ideals of $R[x]$ are given by Theorem 1.1(i). If $M = \mathbb{P}R[x] + f(x)R[x]$ is as in Theorem 1.1(i), then keeping in view that R is integrally closed in its quotient field K and that $f(x)$ is a monic polynomial which is irreducible over R , one can easily see that $f(x)$ is irreducible over K ; for any factorization of $f(x)$ over K leads to its factorization over R . Consequently $f(x)R[x]$ is a prime ideal of $R[x]$. It follows from the above discussion that every height two prime ideal of $R[x]$ contains a non-zero principal prime ideal. A similar result has been proved in [4] for height two ideals of $R[x]$, where R is a Krull domain having infinitely many height one prime ideals.

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