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A GENERAL CLASS OF NONLINEAR VARIATIONAL INCLUSIONS FOR FUZZY MAPPINGS

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In this paper, we introduce and study a new general class of nonlinear variational inclusions for fuzzy mappings and construct some new iterative algorithms. We prove the existence of solutions for the general class of nonlinear variational inclusions for fuzzy mappings and the convergence of iterative sequences generated by the algorithms.

Key Words : Variational Inclusion; Fuzzy Mapping; Algorithm, Existence; Convergence

1. INTRODUCTION

Variational inequalities, introduced and studied by Hartman and Stampacchia[1] in the early sixties, are a very powerful tool of the current mathematical technology. These have been extended and generalized to study a wide class of problems arising in mechanics, physics, optimization and control, operations research, nonlinear programming, economics and transportation equilibrium and engineering sciences etc. Quasivariational inequalities are generalized form of variational inequalities in which the constraint set depend on the solution. These were introduced and studied by Bensoussan, Goursat and Lions[2] in 1973. For further details we refer to[3-8].

In 1991, Chang and Huang[9&10] introduced and studied some new class of complementarity problems and variational inequalities for set-valued mappings with compact values in Hilbert spaces. In 1994, Hassouni and Moudafi[11] studied a new class of variational inclusions, which included many variational and quasivariational inequalities considered by Noor[12,14], Isac[15], Siddiqi and Ansari[16, 17] as special cases. In 1996, Huang[18] introduced and studied a new class of set-valued nonlinear generalized variational inclusions, which included the variational inclusion considered by Hassouni and Moudafi[11] as special case. On the other hand, Adly[19] studied a new general class of variational inclusions, which included many variational inequalities, quasi-variational inequalities, and explicit and implicit complementarity problems considered by Noor[12-14], Isac[15], Siddiqi and Ansari[16 & 17] and Hassouni and Moudafi[11] as special cases.

In 1989, Chang and Zhu[20] first introduced and studied a class of variational inequalities for fuzzy mappings. Recently, several kinds of variational inequalities and complementarity problems for fuzzy mappings were considered and studied by Chang[21], Chang and Huang[22 & 23], Huang[24-26], Noor[27] and Lee *et al.*[28 & 29]. These works may lead to new and significant results in these areas[22 & 23].

In this paper, we first introduce a new general class of nonlinear variational inclusions for fuzzy mappings which includes many known classes of variational inclusions, variational inequalities, quasi-variational inequalities, and explicit and implicit complementarity problems studied previously by many authors as special cases. Motivated and inspired by the methods of Adly[19] and Huang[18], we construct some new iterative algorithms for the general class of nonlinear variational inclusions for fuzzy mappings and set-valued mappings. We also prove the existence of solutions for the general class of nonlinear variational inclusions for fuzzy mappings and the convergence of iterative sequences generated by the algorithms. Our results extend and imrove some known results in this field.

2. PRELIMINARIES

Let H be a real Hilbert space endowed with a norm $\|\cdot\|$ and inner product $\langle \cdot, \cdot \rangle$. Let $\mathcal{F}(H)$ be a collection of all fuzzy sets over H. A mapping F from H into $\mathcal{F}(H)$ is called a fuzzy mapping on H. If F is a fuzzy mapping on H, then $F(x)$ (denote it by F_x, in the sequel) is a fuzzy set on H and $F_x(y)$ is the membership function of y in F_x.

A fuzzy mapping $F : H \to \mathcal{F}(H)$ is said to be closed, if for any $x \in H$, the function $F_x(y)$ is upper semi-continuous with respect to y (i.e., for any given point $y_0 \in H$ and any net $\{y_\alpha\} \subset H$, when $y_\alpha \to y_0$, we have $F_x(y_0) \geq \limsup_\alpha F_x(y_\alpha)$).

Let $B \in \mathcal{F}(H)$, $q \in [0, 1]$. Then the set

$$(B)_q = \{x \in H : B(x) \geq q\}$$

is called a q-cut set of B.

Let $T, A : H \to \mathcal{F}(H)$ be two closed fuzzy mappings satisfying the following condition (I) : .

(I) There exist two mappings a, b : $H \to [0, 1]$ such that all $x \in H$, the sets $(T_x)_{a(x)}$ and $(A_x)_{b(x)}$ are nonempty and bounded.

Obviously, if T, A : $H \to \mathcal{F}(H)$ are two closed fuzzy mappings satisfying the condition (I), then for all $x \in H$, the sets $(T_x)_{a(x)} \in CB(H)$ and $(A_x)_{b(x)} \in CB(H)$, where $CB(H)$ denotes the family of all nonempty bounded closed subsets of H. In fact, let $\{y_j\}_{j \in I} \subset (T_x)_{a(x)}$ be a net and $y_j \to y_0 \in H$. Then $(T_x)(y_j) \geq a(x)$, for $j \in I$. Since T is closed, we have

$$T_x(y_0) \geq \limsup_{j \in I} T_x(y_j) \geq a(x).$$

This implies that $y_0 \in (T_x)_{a(x)}$ and so $(T_x)_{a(x)} \in CB(H)$. Similarly, $(A_x)_{b(x)} \in CB(H)$.

Let $T, A : H \to \mathcal{F}(H)$ be two closed fuzzy mappings satisfying the condition (I). Then we can define two set-valued mappings $\tilde{T}$ and $\tilde{A}$ as follows :

$$\tilde{T} : H \to CB(H),\ x \mapsto (T_x)_{a(x)}.$$

and

$$\tilde{A} : H \to CB(H),\ x \mapsto (A_x)_{b(x)}.$$

In the sequel, $\tilde{T}$ and $\tilde{A}$ are called the set-valued mappings induced by the fuzzy mappings T and A respectively.

Given mappings $a, b : H \to [0, 1]$, fuzzy mappings $T, A : H \to \mathcal{F}(H)$, single-valued mappings, $f, p, g : H \to H$, and a set-valued maximal monotone mapping $M : G \to 2^H$ with Range $(g) \cap$ dom $(M) \neq 0$ (where 2^H denotes all the nonempty subsets of H), we consider the following problem :

Find $u, w, y \in H$ such that

$$\left.\begin{array}{l} T_u(w) \geq a(u), A_u(y) \geq b(u), \\ 0 \in f(w) - p(y) + M(g(u)). \end{array}\right] \quad \ldots (2.1)$$

This problem is called a general class of nonlinear variational inclusion for fuzzy mappings. An equivalent formulation of the original problem (2.1) is to find $u, w, y \in H$ such that

$$\left.\begin{array}{r} T_u(w) \geq a(u), A_u(y) \geq b(u), \\ \langle v^* + f(w) - p(y), v - g(u) \rangle \geq 0, (v, v^*) \in \text{Graph } (M). \end{array}\right] \quad \ldots (2.2)$$

Since M is maximal monotone, $u, w, y \in H$ are the solutions of the problem (2.2) if and only if $u, w, y \in H$ such that $T_u(w) \geq a(u), A_u(y) \geq b(u)$ and $p(y) - f(w) \in M(g(u))$.

If $T, A : H \to 2^H$ are two classical set-valued mappings, then the problem (2.1) is equivalent to finding $u, w, y \in H$, such that

$$\left.\begin{array}{r} w \in Tu, y \in Au, \\ 0 \in f(w) - p(y) + M(g(u)) \end{array}\right] \quad \ldots (2.3)$$

which is called a generalized nonlinear variational inclusion for set-valued mappings.

If $T : H \to H$ is two identity mappings, then the problem (2.3) is equivalent to finding $u \in H$ such that

$$0 \in f(u) - p(u) + M(g(u)). \quad \ldots (2.4)$$

Variational inclusion like (2.4) have been studied[19].

If $M := \partial\varphi$, where $\partial\varphi$ denotes the subdifferential of a proper, convex and lower semicontinuous function $\varphi : H \to R \cup \{+\infty\}$, and $T, A : H \to 2^H$ are two classical set-valued mappings, then the problem (2.1) is equivalent to finding $u, w, y \in H$, such that

$$\left.\begin{array}{r} w \in Tu, y \in Au, g(u) \cap \text{dom } (\partial\varphi) \neq 0, \\ \langle f(w) - p(y), v - g(u) \rangle \geq \varphi(g(u)) - \varphi(v), v \in H), \end{array}\right] \quad \ldots (2.5)$$

which is called a set-valued nonlinear generalized variational inclusion considered by Huang [16].

Remark 2.1 : For appropriate and suitable choice of the mappings f, p, g, T, A, M and the functions a, b, the variational inclusion (2.1) includes a number of known classes of variational inclusions, variational inequalities, quasi-variational inequalities, and explicit and implicit complementarity problems studied previously by many authors in 6, 9-17, 22, 23, 27, 31-33 as special cases.

3. ITERATIVE ALGORITHM

Lemma 3.1 — u, w and y are solutions of the problem (2.1) if and only if there exists $w \in \tilde{T}u$ and $y \in Au$ such that

$$g(u) = J_\alpha^M (g(u) - \alpha(f(w) - p(y))), \quad \text{... (3.1)}$$

where $\alpha > 0$ is a constant and $J_\alpha^M = (I + \alpha M)^{-1}$ is the so-called proximal mapping on H.

PROOF : From the definition of the proximal mapping J_α^M one has

$$g(u) - \alpha(f(w) - p(y)) \in g(u) + \alpha M(g(u)),$$

hence

$$p(y) - f(w) \in M(g(u)).$$

Thus u, w and y are solutions of the problem (2.1). This completes the proof.

Remark 3.1 : We note that when $M := \partial\varphi$, Lemma 3.1 is similar to Lemma 2.1 in [18].

To obtain an approximate solution of (2.1) we can apply a successive approximation method to the problem of solving

$$u \in F(u), \quad \text{... (3.2)}$$

where

$$F(u) = u - g(u) + J_\alpha^M (g(u) - \alpha f(\tilde{T}u) - p(\tilde{A}u)).$$

Based on (3.1) and (3.2), we proceed our algorithm.

Let $T, A : H \to \mathcal{F}(H)$ be two closed fuzzy mappings satisfying the condition (I) and $\tilde{T}$ and $\tilde{A}$ be the set-valued mappings induced by the fuzzy mappings T and A respectively. For given $u_0 \in H$, let $w_0 \in \tilde{T}u_0$, $y_0 \in \tilde{A}u_0$ and

$$u_1 = u_0 - g(u_0) + J_\alpha^M (g(u_0) - \alpha (f(w_0) - p(y_0))).$$

By [34] there exist $w_1 \in \tilde{T}u_1$ and $y_1 \in \tilde{A}u_1$ such that

$$\| w_1 - w_0 \| \le (1 + 1)\hat{H}(\tilde{T}u_1, \tilde{T}u_0),$$

$$\| y_1 - y_0 \| \le (1 + 1)\hat{H}(\tilde{A}u_1, \tilde{A}u_0),$$

where $\hat{H}$ is the Hausdorff metric on $CB(H)$. By induction we can obtain our algorithm as following.

Algorithm 3.1 — Let $T, A : H \to \mathcal{F}(H)$ be two closed fuzzy mappings satisfying the condition.

(I), $\tilde{T}$ and A be the set-valued mappings induced by the fuzzy mappings T and F respectively, and $f, p : H \to H$. For given $u_0 \in H$, we can get an algorithm for (2.1) as following :

$$\left.\begin{aligned} u_{n+1} = u_n - g(u_n) + J_\alpha^M (g(u_n) - \alpha(f(w_n) - p(y_n))), \\ w_n \in \tilde{T} u_n, \| w_{n+1} \| \le (1 + (1+n)^{-1}) \hat{H} (\tilde{T} u_{n+1}, \tilde{T} u_n) \\ y_n \in \tilde{A} u_n, \| y_{n+1} - y_n \| \le (1 + (1+n)^{-1}) \hat{H} (\tilde{A} u_{n+1}, \tilde{A} u_n), \\ n = 0, 1, 2, \ldots \end{aligned}\right] \quad \ldots (3.3)$$

From Algorithm 3.1, we can get the following algorithm.

Algorithm 3.2 — Let $T, A : H \to CB(H)$, and $f, p, g : H \to H$. For given $u_0 \in H$, we can get an algorithm for (2.3) as following :

$$\left.\begin{aligned} u_{n+1} = u_n - g(u_n) + J_\alpha^M (g(u_n) - \alpha(f(w_n) - p(y_n))), \\ w_n \in \tilde{T} u_n, \| w_{n+1} \| \le (1 + (1+n)^{-1}) \hat{H} (\tilde{T} u_{n+1}, \tilde{T} u_n) \\ y_n \in \tilde{A} u_n, \| y_{n+1} - y_n \| \le (1 + (1+n)^{-1}) \hat{H} (\tilde{A} u_{n+1}, \tilde{A} u_n), \\ n = 0, 1, 2, \ldots \end{aligned}\right] \quad \ldots (3.4)$$

Remark 3.2 : The algorithms 3.1 and 3.2 include several algorithms of 6, 9, 10, 12-14, 16-18, 22, 25, 26, 32 as special cases.

4. Existence and Convergence

Definition 4.1 — A mapping $g : H \to H$ is said to be

(*i*) strongly monotome if there exists some $\delta > 0$ such that

$$\langle g(u_1) - g(u_2), u_1 - u_2 \rangle \ge \delta \| u_1 - u_2 \|^2 \quad \forall u_i \in H, i = 1, 2 .$$

(*ii*) Lipschitz continuous if there exists some $\sigma > 0$ such that

$$\| g(u_1) - g(u_2) \| \le \sigma \| u_1 - u_2 \| \forall u_i \in H, i = 1, 2 .$$

Definition 4.2 — A set-valued mapping $T : H \to 2^H$ is said to be

(*i*) strongly monotone with respect to mapping $f : H \to H$ if there exists some $\beta > 0$ such that

$$\langle f(w_1) - f(w_2), u_1 - u_2 \rangle \ge \beta \| u_1 - u_2 \|^2, \quad \forall u_i \in H, w_i \in Tu_i, i = 1, 2.$$

(*ii*) $\hat{H}$-Lipschitz continuous if there exists some $\gamma > 0$ such that

$$\hat{H} (Tu_1, Tu_2) \le \gamma \| u_1 - u_2 \|, \quad \forall \; u_i \in H, i = 1, 2.$$

***Theorem* 4.1** — *Let $T, A : H \to \mathcal{F}(H)$ be two closed fuzzy mappings satisfying the condition (I), $\tilde{T}$ and $\tilde{A}$ be the set-valued mappings induced by the fuzzy mappings T and A respectively. Suppose that $g : H \to H$ is strongly monotone and Lipschitz continuous, $f, p : H \to H$ are Lipschitz continuous, $\tilde{T}$, $\tilde{A}$ are $\hat{H}$ -Lipschitz continuous and $\tilde{T}$ is strongly monotone with respect to f. If the following conditions hold :*

$$\left| \alpha - \frac{\beta + \varepsilon\mu(k-1)}{\eta^2\gamma^2 - \varepsilon^2\mu^2} \right| < \frac{\sqrt{(\beta + (k-1)\varepsilon\mu)^2 - (\eta^2\gamma^2 - \varepsilon^2\mu^2)k(2-k)}}{\eta^2\gamma^2 - \varepsilon^2\mu^2}, \quad \ldots (4.1)$$

$$\beta > (1-k)\varepsilon\mu + \sqrt{(\eta^2\gamma^2 - \varepsilon^2\mu^2)k(2-k)}, \eta\gamma > \varepsilon\mu, \quad \ldots (4.2)$$

$$\alpha\mu\varepsilon < 1 - k, k = 2\sqrt{1 - 2\delta + \sigma^2}, k < 1 \quad \ldots (4.3)$$

where β and δ are the strongly monotone constants of $\tilde{T}$ and g respectively, γ and μ are the $\hat{H}$-Lipschitz constants of $\tilde{T}$ and $\tilde{A}$ respectively, and σ, η and ε are the Lipschitz of g, f and p respectively, then there exist $u, w, y \in H$ which are solutions of the problem (2.1). Moreover,

$$u_n \to u, w_n \to w, y_n \to y, n \to \infty,$$

where $\{u_n\}$, $\{w_n\}$ and $\{y_n\}$ are defined in the algorithm 3.1.

PROOF : From (3.3), we have

$$\| u_{n+1} - u_n \| = \| u_n - u_{n-1} - (g(u_n) - g(u_{n-1})) + J_\alpha^M (h(u_n)) - J_\alpha^M (h(u_{n-1})) \|,$$

where $h(u_n) = g(u_n) - \alpha(f(w_n) - p(y_n))$. Also we have

$$\| J_\alpha^M (h(u_n)) - J_\alpha^M (h(u_{n-1})) \leq \| h(u_n) - h(u_{n-1}) \|$$

$$\leq \| u_n - u_{n-1} - \alpha(f(w_n) - f(w_{n-1})) \| + \| u_n - u_{n-1} - (g(u_n) - g(u_{n-1})) \| + \alpha \| p(y_n) - p(y_{n-1}) \|.$$

That is

$$\| u_{n+1} - u_n \| \leq 2 \| u_n - u_{n-1} - (g(u_n) - g(u_{n-1})) \| + \| u_n - u_{n-1} - \alpha(f(w_n) - f(w_{n-1})) \| + \alpha \| p(y_n) - p(y_{n-1}) \|. \quad \ldots (4.4)$$

By the Lipschitz continuity and strongly monotonicity of g, we obtain

$$\| u_n - u_{n-1} - (g(u_n) - g(n-1)) \|^2 \leq (1 - 2\delta + \sigma^2) \| u_n - u_{n-1} \|^2. \quad \ldots (4.5)$$

Aso from $\hat{H}$-Lipschitz continuity and strongly monotonicity of $\tilde{T}$, and Lipschitz continuity of f, we have

$$\| u_n - u_{n-1} - \alpha(f(w_n) - f(w_{n-1})) \|^2 \leq (1 - 2\beta\alpha + \alpha^2\eta^2(1 + n^{-1})^2\gamma^2) \| u_n - u_{n-1} \|^2. \quad \ldots (4.6)$$

By $\hat{H}$-Lipschitz continuity of $\tilde{A}$, Lipschitz continuity of p and (3.3), we know

$$\alpha \| p(y_n) - p(y_{n-1}) \| \leq \alpha\varepsilon(1 + n^{-1})\mu \| u_n - u_{n-1} \|. \quad \ldots (4.7)$$

So by combining (4.4)-(4.7) and denoting

$$\theta_n := 2\sqrt{1-2\delta+\sigma^2}+\sqrt{1-2\beta\alpha+\alpha^2\eta^2(1+n^{-1})^2\gamma^2}+\alpha\varepsilon(1+n^{-1})\mu,$$

we get

$$\| u_{n+1}-u_n \| \le \theta_n \| u_n-u_{n-1} \|.$$

Letting

$$\theta := 2\sqrt{1-2\delta+\sigma^2}+\sqrt{1-2\beta\alpha+\alpha^2\eta^2\gamma^2}+\alpha\varepsilon\mu,$$

we know that $\theta_n \searrow \theta$. It follows from (4.1)-(4.3) that $\theta < 1$. Hence $\theta_n < 1$, for n sufficiently large. Therefore, $\{u_n\}$ is a Cauchy sequence and we can suppose that $u_n \to u \in H$.

Now we prove that

$$w_n \to w \in \tilde{T}u, y_n \to y \in \tilde{A}u.$$

In fact, it follows from the algorithm 3.1 that

$$\| w_n - w_{n-1} \| \le (1+n^{-1})\gamma \| u_n - u_{n-1} \|,$$

$$\| y_n - y_{n-1} \| \le (1+n^{-1})\mu \| u_n - u_{n-1} \|,$$

i.e. $\{w_n\}$ and $\{y_n\}$ are both Cauchy sequences. Let $w_n \to w, y_n \to y$. Further we have

$$\begin{aligned} d(w,\tilde{T}u) &= \inf\{\| w-z \| : z \in \tilde{T}u\} \\ &\le \| w-w_n \| + d(w_n, \tilde{T}u) \\ &\le \| w-w_n \| + \hat{H}(\tilde{T}u_n, \tilde{T}u) \\ &\le \| w' - w_n \| + \gamma \| u_n - u \| \to 0. \end{aligned}$$

Hence, $w \in \tilde{T}u$. Similarly, $y \in \tilde{A}u$. This completes the proof.

From Theorem 4.1, we can obtain th efollowing theorem.

Theorem 4.2 — *Let $g : H \to H$ be strongly monotone and Lipschitz continuous, $f, p : H \to H$ be Lipschitz continuous, $T, A : H \to CB(H)$ be $\hat{H}$-Lipschitz continuous and T be strongly monotone with respect to f. Let β and δ be the strongly, monotone constants of T and g respectively, γ and μ be the $\hat{H}$-Lipschitz constants of T and A respectively, and σ, η and ε be the Lipschitz constants of g, f and p respectively. If the conditions (4.1)-(4.3) hold, then there exist, $u, w, y \in H$ which are solutions of the problem (2.3). Moreover,*

$$u_n \to u, w_n \to w, y_n \to y, n \to \infty,$$

where $\{u_n\}$, $\{w_n\}$ and $\{y_n\}$ are defined in the algorithm 3.2.

Remark 4.1 : For a suitable choice of the mappings M, T, A, f, p. g and functions a, b we can obtain several known results [8, 9, 11, 13, 16, 18, 26-28, 31-33] as special cases of the main results of this paper.

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