Indian J. pure appl. Math., 29(2) : 139-150, February 1998

NONLINEAR PLANAR OSCILLATION OF A SATELLITE IN CIRCULAR ORBIT UNDER THE INFLUENCE OF MAGNETIC TORQUE (II)

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(*Received* 23 *May* 1997; *Accepted* 7 *November* 1997)

This paper is in continuation of our Paper I (Resonance case). In the present paper, the chaotic separatrix has been estimated by Chrikov's criterion. Through surface of section method, it has been shown that the magnetic torque and the mass distribution parameter of the satellite play an important role in changing the regular motion into the chaotic one. Also, analytically, we have observed that the magnetic torque parameter plays a significant role in changing the motion of revolution into the motion of libration and infinite period separatrix.

Key Words : Nonlinear Planar Oscillation; Satellite; Circular Orbit; Resonance; Magnetic Torque; Chaos; Poineare's Surface of Section

Goldreich and Peale[1] obtained a pendulum like equation for each spin-orbit state by writing the equation of motion of the planar oscillation of a satellite in terms of resonance variables and eliminating the higher frequency non-resonant terms through averaging. The strength of each resonance depends on the orbital eccentricity and the principal moments of inertia through $\frac{B-A}{C}$. Wisdom, Peale and Mignard[2] have studied the same spin-orbit coupling in the motion of Hyperion for those cases where averaging fails. Chirikov[3] predicts the presence of large chaotic zone throught the resonance overlap criterion.

In paper I (1995), we have studied the planar oscillation of a satellite under the influence of the magnetic torque and have obtained the perturbed pendulum like equation, Through Melnikov's method, we have shown the non-integrability of the non-linear rotational equation of motion. Also by taking the magnetic torque perturbation parameter (ε) to be very small, by B.K.M. method, we have seen that the amplitude of the oscillation remains constant even upto the seond order of approximation. Finally, we have observed that the main resonance occurs at $\omega \cong \pm 2b$, where ω is the frequency of the periodic force and b the magnetic torque perturbation.

It is known that most of the Hamiltonian systems give regular and irregular trajectories. Henon and Heiles[5] have shown that the phase space is divided into two regions in which trajectories behave chaotically or quasi-periodically. One of the best method to show whether a trajectory is chaotic or quasi-periodic is through the surface of section method. This is useful in systems with two degrees of freedom. By making use of the surface of section method, it has been seen that the magnetic torque and the mass distribution of the satellite play an important role to change the regular motion into the chaotic one. Also, analytically we have observed that the magnetic torque parameter

plays a significant role in changing the motion of revolution into the motion of libration and infinite period separatrix.

2. Equation of Motion

Let us consider a rigid satellite S moving in a circular orbit around the Earth E. We further suppose that the orbital plane coincides with the equatorial plane of the Earth (Fig. 1). The satellite is assumed to be a triaxial body with principal moments of inertia $A < B < C$ at its centre of mass and C is the moment of inertia about the spin axis which is perpendicular to the orbital plane. These principal axes are taken as our co-ordinate axes x, y, z; the z-axis being perpendicular to the orbital plane.

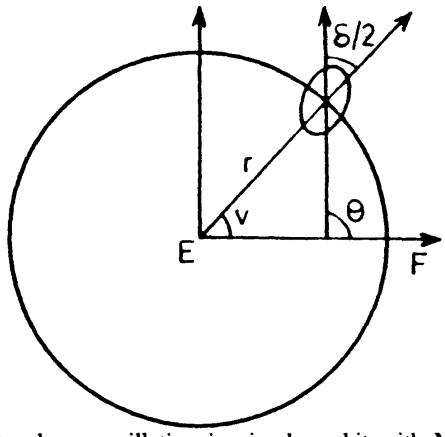

Fig. 1. Satellite planar oscillation in circular orbit with Magnetic perturbation

Let $\bar{r}$ be the instantaneous radius vector of the centre of mass of the satellite, θ the angle that the long axis of the satellite makes with a fixed line EF lying in the orbital plane and $\frac{\delta}{2}$ the angle between the radius vector and long axis. The equation of motion of the planar oscillation of the satellite is[4]

$$\frac{d^2\theta}{dt^2} + \frac{\mu}{2r^3}[n^2 \sin(\delta) - \varepsilon \sin 2(v - \alpha_m)] = 0, \qquad \ldots (2.1)$$

where

$$n^2 = 3\frac{B-A}{C},$$

$$\varepsilon = \frac{3a^3 M_0}{\mu}\frac{m}{C}\sin^2(\theta'_m) = \text{ parameter due to magnetic torque,}$$

$$\mu = G(m_E + m_S) \cong Gm_E,$$

m_E = mass of the Earth,

m_S = mass of the satellite,

$$\alpha_m = \alpha_{G0} + \frac{d\alpha_G}{dt}\cdot t + \phi'_m,$$

$a^3 M_0$ = total dipole strength,

θ'_m = coelevation of the dipole,

ϕ_m' = east longitude of the dipole

α_{G0} = right ascension of the greenwich meridian at some reference time,

$\frac{d\alpha_G}{dt}$ = average rotation rate of the earth,

t = time since reference,

m = magnet strength of the spacecraft,

and v = true anomaly measured from the ascending node.

3. Solutions of the Equations (2.1)

In eq. (2.1), if we take

$$n^2 = \omega_0^2 = 3\frac{B-A}{C}$$

and $$\theta = v + \frac{\delta}{2} \quad \Rightarrow \quad \delta = 2(\theta - v),$$

then the above equation becomes

$$\frac{d^2\theta}{dt^2} + \frac{\mu}{2r^3}[\omega_0^2 \sin 2(\theta - v) - \varepsilon \sin 2(v - \alpha_m)] = 0. \qquad \ldots (3.1)$$

In eq. (3.1), if the units are so chosen that the orbital period of the satellite is 2π and its radius be unity, then the dimensionless time is equal to the mean longitude or the true anomaly v and v or t is 2π-periodic and $\mu = 1$. Also as defined above in section 2,

$$\alpha_m = \alpha_{G0} + \phi_m' + \frac{d\alpha_G}{dt} t = \alpha_1 + \beta t \ ,$$

and

$$b = 1 - \beta,$$

where $$\beta = \frac{d\alpha_G}{dt} \ ,$$

$$\alpha_1 = \alpha_{G0} + \phi_m' \ ,$$

then eq. (3.1) becomes

$$\frac{d^2\theta}{dt^2} + \frac{\omega_0^2}{2} \sin 2(\theta - t) - \frac{\varepsilon}{2} \sin 2(bt - \alpha_1) = 0 \qquad \ldots (3.2)$$

This is an equation of a pendulum perturbed by a force $\frac{\varepsilon}{2} \sin 2(bt - \alpha_1)$.

This equation can be studied under the cases $\varepsilon = 0$ and $\varepsilon \neq 0$. The cases $\varepsilon = 0$ has been studied by Wisdom *et al.*[3].

The case when $\varepsilon \neq 0$, eq. (3.2) represents the motion of a disturbed pendulum given by

$$\frac{d^2x}{dt^2} + f(x) = m\,\phi'(x, t), \qquad \ldots (3.3)$$

where

$$x = 2(\theta - t),$$

$$f(x) = k_1^2 \sin(x),$$

$$k_1^2 = \omega_0^2,$$

$$m = -\varepsilon,$$

$$\phi'(x, t) = \sin 2(bt - \alpha_1).$$

The unperturbed part of (3.3), is

$$\frac{d^2x}{dt^2} + f(x) = 0.$$

For this equation, we have

$$\left(\frac{dx}{dt}\right)^2 = C + 2k_1^2 \cos(x),$$

where C is a constant of integration.

The motion is to be real, if $C + 2k_1^2 \cos(x) \geq 0$. There are three categories of motion depending upon $C > 2k_1^2$, $C < 2k_1^2$ and the intermediate case $C = 2k_1^2$.

Category I : $C > 2k_1^2$.

As $C > 2k_1^2$, so $\frac{dx}{dt}$ never vanishes in this case, it is always either +ve or –ve, and the pendulum is making complete revolution in one sense or the other.

In this case the unperturbed solution is

$$x = L + C_1 \sin(L) + O(C_1^2), \qquad \ldots (3.4)$$

where

$$L = nt + \varepsilon_1,$$

$$C_1 = \frac{k_1^2}{n^2},$$

and

$$\frac{1}{n} = \frac{1}{2\pi} \int_0^{2\pi} \frac{dx}{(C + 2k_1^2 \cos x)^{1/2}},$$

C_1 and ε_1 are arbitrary constants and L is an argument.

The periodic portion of this series can be regarded as an oscillation about the mean state of motion which is revolution with a period $\frac{2\pi}{n}$. The half amplitude of the oscillation is evidently less than π and it decreases as n increases.

Here, we may observe that $\frac{dx}{dt} \neq 0$ and the motion is said to be of type I i.e., revolution.

In case of perturbed pendulum by making use of the theory of variation of parameters Brown and Shook[6], we have

$$\frac{dC_1}{dt} = \frac{m}{V}\frac{\partial x}{\partial L}\phi'(x, t)$$

and

$$\frac{dL}{dt} = n - \frac{m}{V}\frac{\partial x}{\partial C_1}\phi'(x, t), \qquad \text{.... (3.5)}$$

where

$$V = \frac{\partial}{\partial C_1}\left(n\frac{\partial x}{\partial L}\right)\frac{\partial x}{\partial L} - n\frac{\partial^2 x}{\partial L^2}\frac{\partial x}{\partial C_1} \equiv -\frac{n}{2C_1}\,. \qquad \text{... (3.6)}$$

Substituting the value of V in equation (3.5), we get

$$\frac{dC_1}{dt} \equiv -\frac{2mk_1^2}{n^3}\sin 2(bt - \alpha_1) \equiv 0.$$

Since, both m and k_1^2 are small quantities, the term mk_1^2 is of the third order and therefore rejected.

Thus,

$$\frac{dC_1}{dt} \equiv 0$$

$\Rightarrow C_1$ = constant upto the second order of approximation.

And,

$$\frac{dL}{dt} \equiv n + \frac{2mC_1}{n}\sin(L)\sin 2(bt - \alpha_1),$$

$$\frac{d^2L}{dt^2} \equiv m \sin 2(bt - \alpha_1),$$

since,

$$L = nt + \varepsilon_1 \Rightarrow t = \frac{L - \varepsilon_1}{n}\,.$$

Thus,

$$\frac{d^2L}{dt^2} \equiv 2m \sin\left(\frac{Lb}{n}\right).$$

In the first approximation taking $n = n_0$, we get

$$\frac{d^2L}{dt^2} \cong 2m \sin\left(\frac{Lb}{n_0}\right).$$

If we take $\frac{Lb}{n_0} = x$, then the above equation becomes

$$\frac{d^2x}{dt^2} + k_2^2 \sin(x) = 0, \qquad \ldots (3.7)$$

where

$$K_2^2 = 2\frac{\varepsilon b}{n_0}.$$

Integrating eq. (3.7), we get

$$\left(\frac{dx}{dt}\right)^2 = C_2 + 2k_2^2 \cos(x),$$

where C_2 is a constant of integration.

Eq. (3.7) describes the motion of a pendulum. We get again three types of motion. Type I is that in which $\frac{dx}{dt}$ is never zero. Type II is that in which $\frac{dx}{dt} = 0$, at $x = 0$ or π.

For type I, our solution is

$$x = Nt + \varepsilon_2 + \frac{k_2^2}{N^2} \sin(Nt + \varepsilon_2) + \ldots,$$

where

$$\frac{1}{N} = \frac{1}{2\pi} \int_0^{2\pi} \frac{dx}{(C_2 + 2k_2^2 \cos x)^{1/2}}.$$

In the first approximation $N = N_0$ and the solution is

$$x = x_0 + \frac{k_2^2}{N^2} \sin(x_0),$$

where

$$x_0 = N_0 t + \varepsilon_2.$$

This is the case of revolution.

For TYPE II, the solution is

$$x = \lambda \sin(pt + \lambda_0),$$

where

$$p^2 = \left| \frac{ma}{V} \left(\frac{\partial n}{\partial C} \right)_0 \right| \cong k_2^2 .$$

$$\Rightarrow p = \sqrt{\frac{2b\varepsilon}{n_0}} ,$$

λ and λ_0 being arbitrary constants. This is the case of libration.

TYPE III occurs when $C_2 = 2k_2^2 = 4\frac{\varepsilon b}{n_0}$, the solution is

$$x + \pi = 4 \tan^{-1} \exp\left(\sqrt{\frac{2\varepsilon b}{n_0}}\, t + \alpha_0 \right),$$

where α_0 is an arbitrary constant and the other having a particular value.

When $t \to \pm\infty, x \to \pm\pi$, at both places, $\frac{dx}{dt} = 0$ and all higher derivatives of x approach to zero. Near this point, while x approaches to one of the limits $\pm\pi$, t tends to become an indeterminate function of x. This is the case of infinite period separatrix which is asymptotic forward and backward in time to the unstable equilibrium.

Thus the results of type I, type II and type III enable us to conclude that the magnetic torque perturbation plays a significant role. It may change a revolution to libration or to infinite period separatrix.

Category II : $C < 2k_1^2$.

In this case unperturbed solution is

$$x = C_1 \sin(L) + 0(C_1)^3,$$

where

$$L = nt + \varepsilon_1,$$

and

$$n = k_1 \left(1 - \frac{1}{16} C_1^2 + 0(C_1^4) \right),$$

C_1 and ε_1 being arbitrary constants.

In case of perturbed equation, again using the theory of variation of parameters, we get

$$\frac{dC_1}{dt} \cong \frac{m}{k_1} \cos(L) \sin 2(bt - \alpha_1),$$

$$\frac{dL}{dt} \cong k_1 - \frac{m}{k_1 C_1} \sin(L) \sin 2(bt - \alpha_1)$$

and

$$\frac{d^2L}{dt^2} \cong -\frac{2mb}{k_1 C_1} \sin(L) \cos 2\left(b\frac{L - \varepsilon_1}{n} - \alpha_1 \right)$$

In the first approximation if $n = n_0$, $C_1 = C_{10}$, we get

$$\frac{d^2L}{dt^2} \cong -\frac{2mb}{k_1 C_{10}} \sin(L) \cos 2\left(b\frac{L-\varepsilon_1}{n_0} - \alpha_1 \right). \qquad \ldots (3.8)$$

As a special case, let us assume that

$$b\frac{L-\varepsilon_1}{n_0} - \alpha_1 = n_1 \cdot \frac{\pi}{2}, \quad n_1 \in I,$$

when n_1 is odd, then (3.8) becomes

$$\frac{d^2L}{dt^2} + k_3^2 \sin(L) \cong 0,$$

where

$$K_3^2 = -\frac{2mb}{k_1 c_{1_0}} > 0 \text{ as } m < 0,$$

which is again the equation of the pendulum. As in the previous case this equation gives us revolution, libration and infinite period separatrix motion.

On the other hand, if n_1 is even, then

$$\frac{d^2L}{dt^2} \cong K_3^2 \sin(L) .$$

When L is small the solution of the above equation is

$$L = e^{K_3 t} + e^{-K_3 t}.$$

Category III : $C = 2k_1^2$

The unperturbed solution is

$$x + \pi = 4 \tan^{-1} e^{k_1 t} + \alpha_0 ,$$

where α_0 is an arbitrary constant and other having a specific value. This is the case of infinite period separatrix which is asymptotic forward and backward in time to the unstable equilibrium. We are mainly concerned in this category of motion. In this category the nature of the unperturbed solution does not change by taking into account the magnetic torque.

SPIN-ORBIT PHASE SPACE

In our spin orbit problem which is 2π periodic in dimensionless time, we have computed the mapping on the surface of section by looking at the trajectories stroboscopically with period 2π. The section has been drawn with $\frac{d\theta}{dt}$ *vs.* θ at every periapse passage. In the case of quasi-periodic trajectories

the points are contained in smooth curves while for the chaotic trajectories they appear to fill up the area in the phase space in random manner. Since the orientation denoted by $\pi+\theta$, we have restricted the interval from 0 to π. The spin orbit states that are determined in the previous section are states where a resonance variable librates. For each of these states $\frac{d\theta}{dt}$ has an average value precisely equal to p and θ rotates through all values. Consequently, quasi-periodic successive points will trace a simple curve on the section near $\frac{d\theta}{dt}=\theta$. This will cover only a fraction of the interval from 0 to π. In the case of non-resonant quasi-periodic trajectories all resonant states rotate and successive points on the surface of section will trace a simple curve which cover all values of θ.

Fig. (2) to Fig. (11) for various values of α_1, ε and b:

In figures (2 to 11), we have plotted ten different figures for various values of ε (i.e., for ε = 0.005, 0.01, (Fig. 2) 0.05, 0.1, (Fig. 3) 0.15, 0.2, (Fig. 4) 0.25, 0.3 (Fig. 5)) at $\omega_0 = 0.5$, α_1 = 3.6380824, b = 0.9985096; for various values of α_1(i.e. for α_1 = 3.6380824, 3.6288284, 3.6190506 (Fig. 6)) at ω_0 = 0.5, = 0.001, b = 0.9985096; and for different values of b (i.e., for b = 0.99954326, 0.9985096, (Fig. 7) b = 0.99634607, 0.95914787, (Fig. 8) b = 0.88445273, 0.78772612, (Fig. 9) b = 0.67318297, 0.54325932, (Fig. 10), b = 0.39959879, 0.243408 (Fig. 11)) at ω_0 = 0.5,

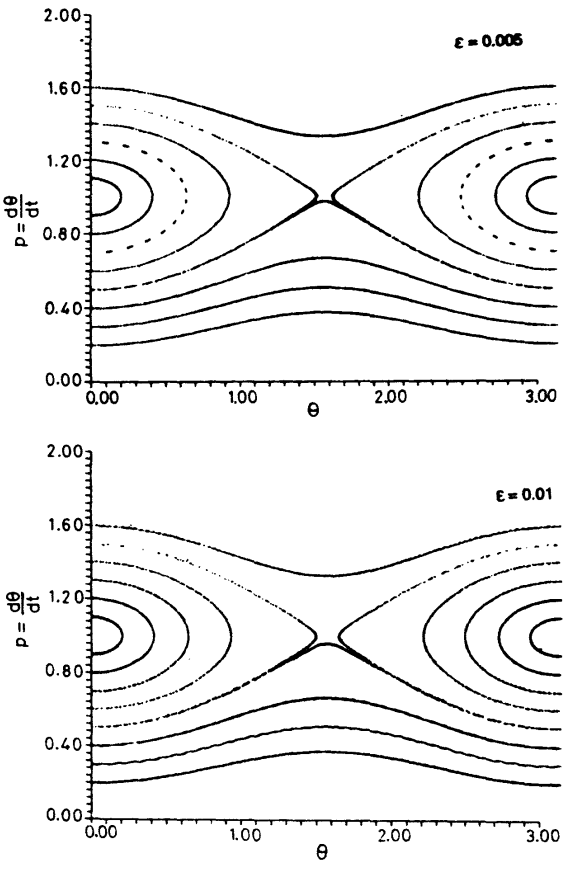

FIG. 2. Surface of section for ε = 0.005, 0.01, ω_0 = 0.5, α_1 = 3.6380824, P = 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1.5, 1.6, b = 0.9985096

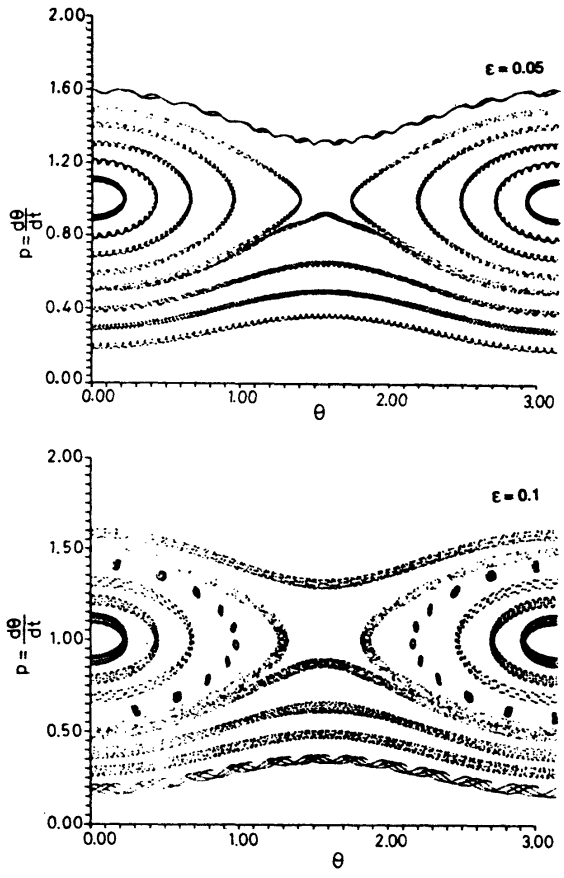

FIG. 3. Surface of section for ε = 0.05, 0.1, ω_0 = 0.5, α_1 = 3.6380824, b = 0.9985096

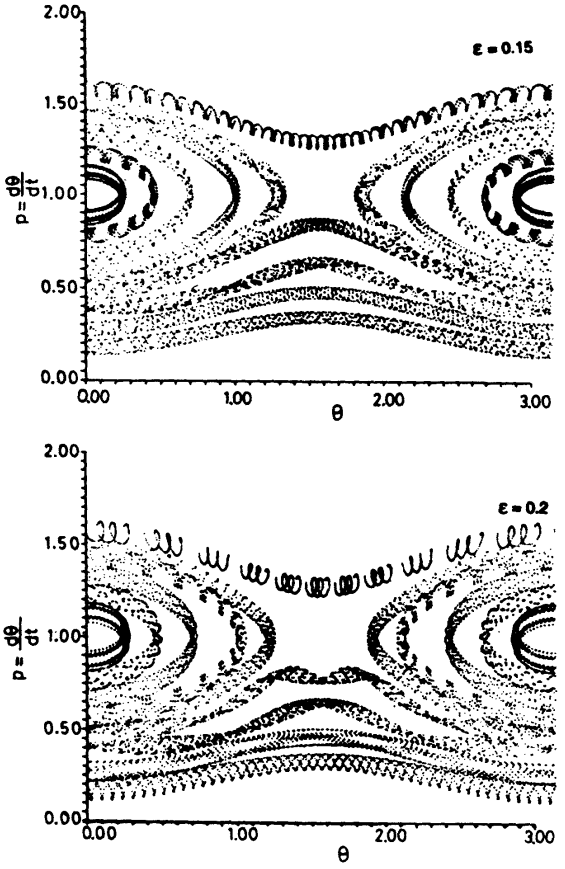

FIG. 4. Surface of section for ε = 0.15, 0.2, ω_0 = 0.5, α_1 = 3.6380824, b = 0.9985096

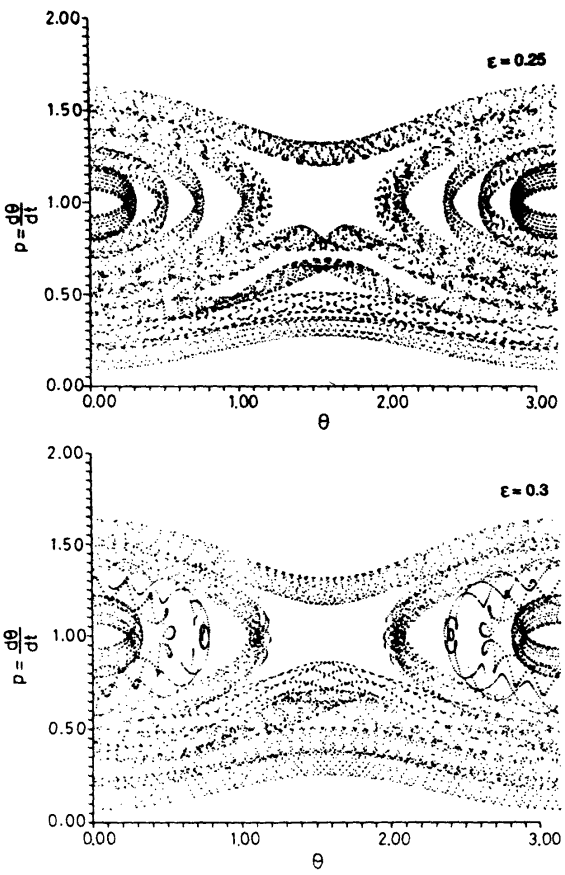

FIG. 5. Surface of section for ε = 0.25, 0.3, ω_0 = 0.5, α_1 = 3.6380824, b = 0.9985096

ε = 0.001, α_1 = 3.6380824 respectively. In each of these figures, we have plotted ten different trajectories, $\frac{d\theta}{dt}$ versus θ, at every periapse passage. It may be observed that the chaotic separatrix surrounds each of the resonance states and each of these chaotic zones is separated from others by non-resonant quasi-periodic rotation trajectories. In Figs (2-5), we may observe that as the magnetic torque parameters (ε) is introduced the regular curves starts disintegrating and this disintegration increase with the increase in ε. The spin orbit phase space is dominated by a chaotic zone which is so large that even the p = 0.2 and p = 1.6 states are surrounded by it.

In Fig. (6), we may observe that as the parameter α_1 increases, no significant change is observed in trajectories even for twenty years. We have taken the values at the interval of ten years i.e., for data of 1975, 1985, 1995. No significant change is observed in the trajectories from p = 0.2 to p = 1.6 states.

From Figs. (7-11), we may observe that as b decreases, the regular curves disintegrate and this disintegration increases with the decrease in b, i.e., for b < 0.54325932. The spin-orbit phase space is dominated by a chaotic zone which is so large, even the p = 0.5 and p = 1.2 states the surrounded by it.

From Figs. (2) to Figs. (11), we conclude that the regular curves starts disintegrating due to magnetic torque (i.e. for different vlaues of ε), and this disintegration increases with the increase in ε and decrease in b.

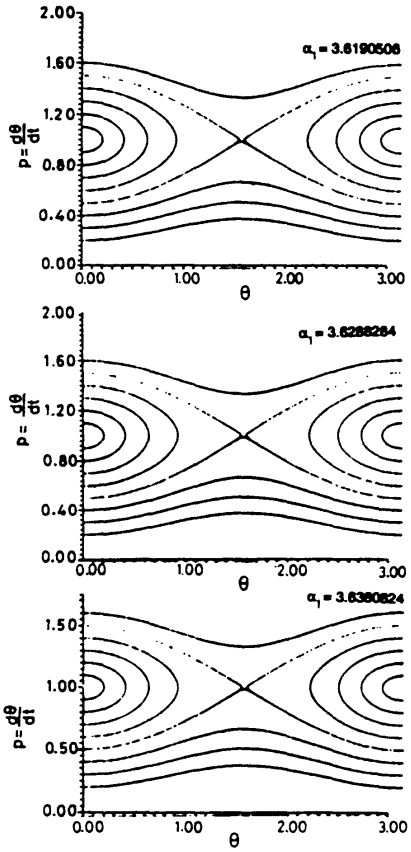

FIG. 6. Surface of section for α_1 = 3.6380824, 3.6288284, 3.6190506, ω_0 = 0.5, ε = 0.001, b = 0.9985096, p = 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1.5, 1.6

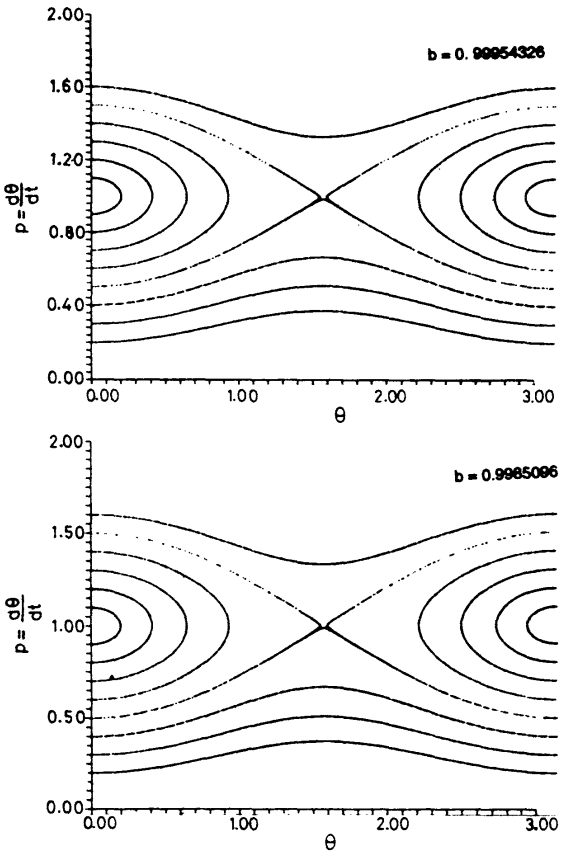

FIG. 7. Surface of section for b = 0.99954326, 0.9985096, ω_0 = 0.5, ε = 0.001, α_1 = 3.6380824, P = 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1.5, 1.6

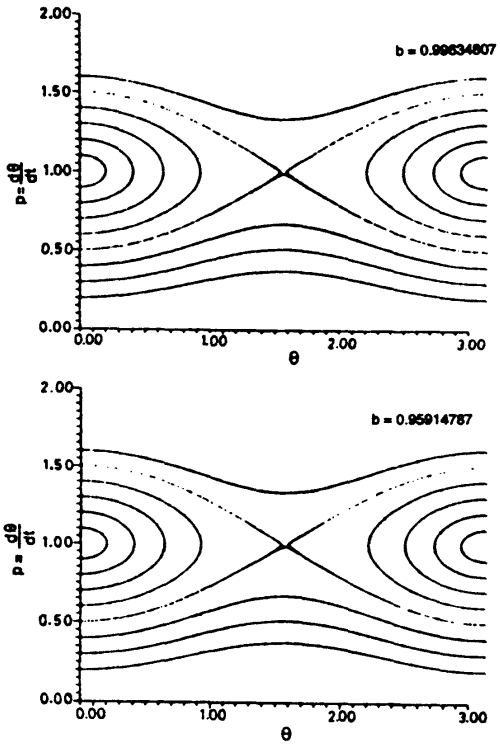

FIG. 8. Surface of section for b = 0.99634607, 0.95914787, ω_0 = 0.5, ε = 0.001, α_1 = 3.6380824

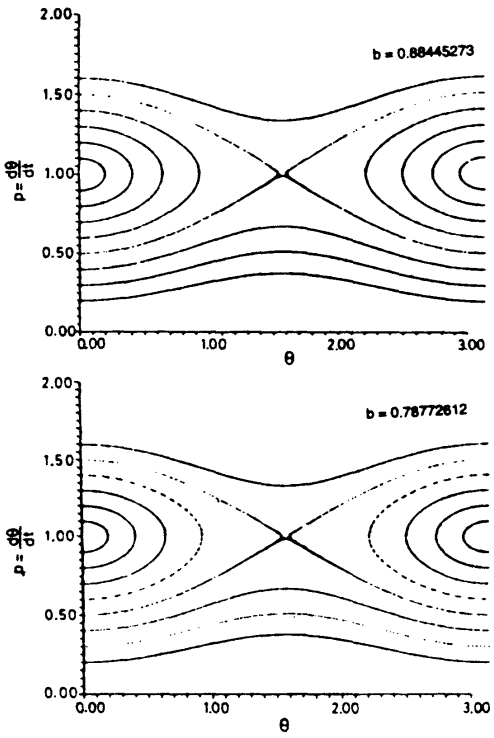

FIG. 9. Surface of section for b = 0.88445273, 0.78772612, ω_0 = 0.5, ε = 0.001, α_1 = 3.6380824

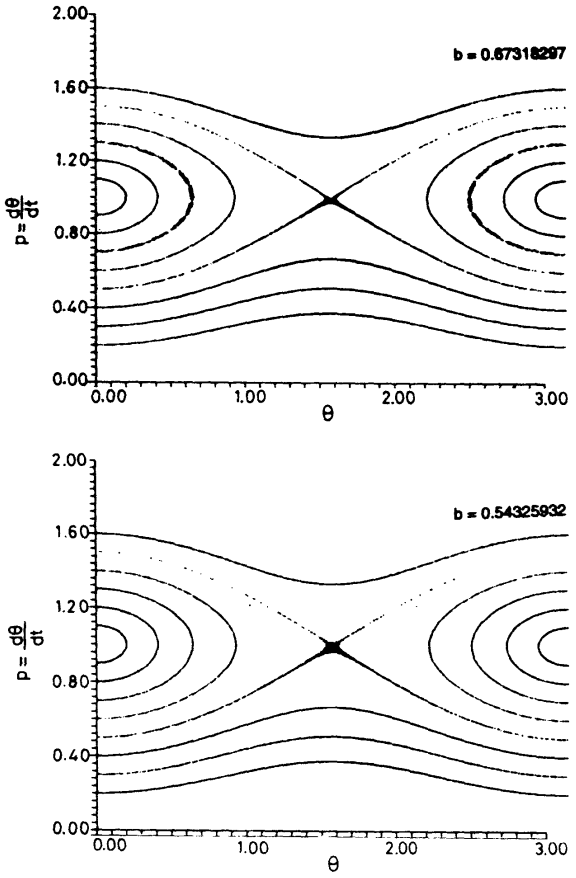

FIG. 10. Surface of section for b = 0.67318297, 0.54325932, ω_0 = 0.5, ε = 0.001, α_1 = 3.6380824

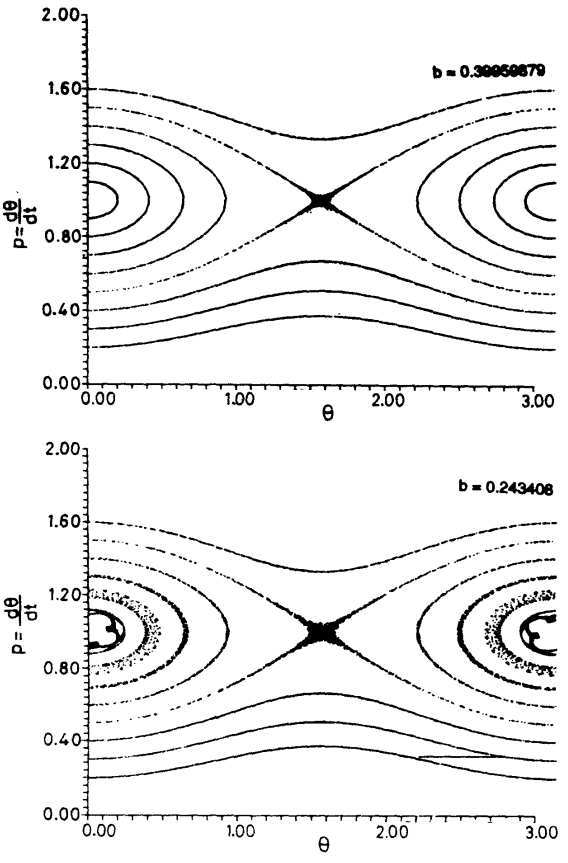

FIG. 11. Surface of section for b = 0.39959879, 0.243408, ω_0 = 0.5, ε = 0.001, α_1 = 3.63808224

CONCLUSION

From the present study, we conclude that the magnetic torque parameter plays a significant role in changing the motion of revolution into the motion of libration and infinite period separatrix. We also observe that the regular motion changes into a chaotic one for some values of the magnetic torque parameter ε and the values of b. The changes in the values of α_1 does not play a significant role even for twenty years i.e., from 1975 to 1995.

ACKNOWLEDGEMENT

The first author is thankful to CSIR, Govt. of India and the second author to UGC, which have funded this project. Thanks are also due to the Centre for Fundamental Research in Space Dynamics and Celestial Mechanics (CFRSC) for giving all the facilities for research.

REFERENCES

1. P. Goldreich and S. J. Peale, *Astron, J.* **71** (1966) 425-35.
2. J. Wisdom, S. J. Peale and F. Mignard, *Icarus* **58** (1984) 137-52.
3. B. V. Chrikov, *Phys. Rep.* **52**(1979) 263-379.
4. Rashmi Bhardwaj and K. B. Bhatnagar, *Indian J. pure appl. Math.* **26**(12) (1995) 1225-40.
5. M. Henon and C. Heiles, *Astron. J.* **69** (1964) 73-79.
6. W. Brown and A. Shook, *Planetary Theory*, Dover Pub., Inc. New York (1964)