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ON FUZZY COMPLETELY SEMI CONTINUOUS AND WEAKLY COMPLETELY SEMI CONTINUOUS FUNCTIONS

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In this paper, two classes of functions between fuzzy topological spaces are introduced: fuzzy completely semi continuous and fuzzy weakly completely semi continuous functuons. Several characterizations of these functions alongwith their relationships with certain other types of functions are investigated. Lastly, a few situations are mentioned where these functions may be applied.

Key Words : Fuzzy Completely Semi Continuous; Fuzzy Weakly Completely Semi Continuous Functions; Fuzzy q-Open; Fuzzy Regular Semi Open Subsets; Fuzzy q-Compact Space

1. INTRODUCTION

The concept of fuzzy completely continuous (fcc) functions was introduced by Mukherjee and Ghosh[1]. Further properties of this function were studied by Bhaumik and Mukherjee[2]. A function $f : (X, F) \to (Y, K)$ from a fuzzy topological space (ft space) (X, F) to another ft space (Y, K) is called fuzzy completely continuous iff $f^{-1}(\alpha)$ is a fuzzy regular open subset of X for every fuzzy open subset α in Y. As a generalization of this function, Bhaumik and Mukherjee[3], introduced the fuzzy weakly completely continuous (fwcc) function. In this paper, two more generalized concepts : fuzzy completely semi continuous (fcsc) function and fuzzy weakly completely semicontinuous (fwcsc) function are introduced and several properties of these functions are investigated. Finally, these functions are studied in relation to some other types of already known fuzzy functions and we mention a few situations where such functions may be applied. Throughout this paper by (X, F) (or simply X) we mean a ft space. A fuzzy point in X with support $x \in X$ and value p $(0 < p \leq 1)$ is denoted by x_p. A fuzzy point $x_p \in \alpha$, where α is a fuzzy subset in X iff $p \leq \alpha(x)$.

The following definitions and results are to be noted :

1.1. The fuzzy point x_p is said to be

(*i*) [4]quasi-coincident (q-coincident) with a fuzzy subset α, denoted by $x_p q \alpha$ iff $p + \alpha(x) > 1$.

(*ii*) [5]Fuzzy θ-cluster point of a fuzzy subset α iff for every open q-neighbourhood β of x_p, $\mathrm{Cl}(\beta)$ is q-coincident with α.

The set of all fuzzy θ-cluster points of α is called the fuzzy θ-closure of α and is denoted by $\mathrm{Cl}_\theta(\alpha)$. Mukherjee and Sinha[5] defined a fuzzy subset α as fuzzy θ-closed iff $\alpha = \mathrm{Cl}_\theta(\alpha)$. The complement of a fuzzy θ-closed subset is said fuzzy θ-open, is equivalent to the condition; a fuzzy subset μ is fuzzy θ-open iff $\mu = \text{Int}_\theta(\mu)$, where the set $\text{Int}_\theta(\mu)$.

$x_p \in X$: for some open q-neighbourhood β of x_p, Cl $(\beta) \subseteq \mu$ is called fuzzy θ-interior of μ.

1.2. A fuzzy subset α in a ft space (X, F) is said to be

(*i*) Fuzzy regular open iff α = Int Cl (α)[6].

(*ii*) Fuzzy regular semi open iff there exists a fuzzy regular open subset β of X such that $\beta \subseteq \alpha \subseteq \text{Cl}(\beta)$[7].

(*iii*) The q-open neighbourhood for a fuzzy point x_p iff[4] there exists a fuzzy open subset β. Such that $x_p\, q\beta \subseteq \alpha$.

1.3. A function $f : (X, F) \to (Y, K)$ from a ft space (X, F) to another ft space (Y, K) is said to be

(i) Fuzzy faintly continuous (Theorem 2.2[8]) iff $f^{-1}(\alpha)$ is a fuzzy open subset in X for each fuzzy θ-open subset α in Y.

(ii) Fuzzy weakly completely continuous[3] iff $f^{-1}(\alpha)$ is a fuzzy regular open subset in X for each fuzzy θ-open subset α in Y.

1.4. A ft space (X, F) is said to be

(*i*) Fuzzy almost compact[9], iff every fuzzy open cover of X has a finite sub cover whose closures cover X.

(*ii*) Fuzzy s-closed (Theorem 3.3[10]), iff every fuzzy cover of X by fuzzy regular semi-open subsets has a finite sub cover.

(*iii*) Fuzzy S^*-closed iff for every fuzzy semi open cover[11] $\{G_a : a \in \wedge\}$ of X, there exists a finite subset $\wedge_0$ of $\wedge$ such that $\{S\text{-Cl } G_a : a \in \wedge_0\}$ is a subcover of X.

2. Fuzzy Completely Semi Continuous and Fuzzy Weakly Completely Semi Continuous Fucntion

In this section, two new classes of functions between ft spaces are introduced under the terminologies: fuzzy completely semi continuous (fcsc) and fuzzy weakly completely semi continuous (fwcsc) functions.

Definitions 2.1 — A function $f : (X, F) \to (Y, K)$ from a ft space (X, F) to another ft space (Y, K) is said to be fuzzy completely semi continuous (fcsc) iff $f^{-1}(\alpha)$ is a fuzzy regular semi-open subset of X for every fuzzy open subset α in Y.

Equivalently, $f : (X, F) \to (Y, K)$ is fuzzy completely semi continuous iff $f^{-1}(\beta)$ is a fuzzy regular semi-closed subset of X for every fuzzy closed subset β in Y. It is obvious that every fuzzy completely continuous function is fuzzy completely semi continuous but the converse is not true which can be seen from the following example :

Example 2.1 : Let us consider the example 4.5 of Azad[6], μ_1, μ_2 and μ_3 be fuzzy subsets of I = [0, 1] defined as

$$\mu_1(x) = \begin{cases} 0, & 0 \le x \le \frac{1}{2} \\ 2x-1, & \frac{1}{2} \le x \le 1 \end{cases}, \qquad \mu_2(x) = \begin{cases} 1, & 0 \le x \le \frac{1}{4} \\ -4x+2, & \frac{1}{4} \le x \le \frac{1}{2} \\ 0, & \frac{1}{2} \le x \le 1 \end{cases}$$

and
$$\mu_3(x) = \begin{cases} 0, & 0 \le x \le \frac{1}{4} \\ \frac{1}{3}(4x-1); & \frac{1}{4} \le x \le 1. \end{cases}$$

Clearly, $F = \{0, \mu_1, \mu_2, \mu_1 \cup \mu_2, 1\}$ and $F_1 = \{0, \mu_1, 1\}$ are two fuzzy topologies on I. Let $f : (I, F) \to (I, F_1)$ be a function defined by $f(x) = \text{Min}\ (2x, 1)$ for each $x \in I$. Then $f^{-1}(0) = 0$, $f^{-1}(1) = 1$, $f^{-1}(\mu_1) = \mu_2'$. Here μ_2' is fuzzy semi open in (I, F) since $\text{Cl}(\mu_1) = \mu_2'$, and μ_2' is also fuzzy semi-closed in (I, F). Thus μ_2' is fuzzy regular semi open in (I, F). Hence, f is fuzzy completely semi continuous. Since μ_2' is not fuzzy regular open in (I, F), f is not fuzzy completely continuous.

It is seen that every fuzzy completely semi continuous function is fuzzy semi-continuous. That the converse is false, is shown in the following example :

Example 2.2 : Consider the fuzzy subsets μ_1, μ_2 and μ_3 of Example 2.1. Clearly $F_1 = \{0, \mu_1, 1\}$ and $F_2 = \{0, \mu_3, 1\}$ are two fuzzy topologies on I. Let $g : (1, F_1) \to (I, F_2)$ be a function defined by $g(x) = x$ for each $x \in I$. Since $\mu_1 \le \mu_3 \le \text{Cl}(\mu_1)$, μ_3 is fuzzy semi open but not fuzzy regular semi open in (I, F_1). Thus g is fuzzy semi continuous but not fuzzy completely semi-continuous.

Definition 2.2 — A fuzzy subset α in a ft space (X, F) is said to be a fuzzy regular semi-open neighbourhood for a fuzzy point x_p iff there exists a fuzzy regular semi-open subset β such that $x_p \in \beta \subseteq \alpha$.

Theorem 2.1 — *A function $f : X \to Y$ from a ft space (X, F) to another ft space (Y, K) is fuzzy completely semi continuous iff for each fuzzy point $x_p \in X$ and each fuzzy open subset α of Y containing $f(x_p)$ there exists a fuzzy regular semi open subset β of X containing x_p such that $f(\beta) \subseteq \alpha$.*

PROOF : Let f be a fuzzy completely semi continuous function and x_p a fuzzy point in X, α a fuzzy open subset in Y such that $f(x_p) \in \alpha$. Then $x_p \in f^{-1} f(x_p) \subseteq f^{-1}(\alpha)$. Let $\beta = f^{-1}(\alpha)$, which is fuzzy regular semi open in X ($\because f$ is fcsc) containing x_p. Now $f(\beta) = ff^{-1}(\alpha) \subseteq \alpha$.

Conversely, let μ be any fuzzy open subset in Y and x_p be any fuzzy point in X such that $x_p \in f^{-1}(\mu)$, then there exists a fuzzy regular semi open subset α in X with $x_p \in \alpha$ such that $f(\alpha) \subseteq \mu$. Then $x_p \in \alpha \subseteq f^{-1} f(\alpha) \subseteq f^{-1}(\mu)$ which shows that $f^{-1}(\mu)$ is a fuzzy regular semi open neighbourhood of each of its point and hence a fuzzy regular semi open subset in X. Hence the theorem.

Definition 2.3 — A function $f(X, F) \to (Y, K)$ from a ft space (X, F) to another ft space (Y, K) is said to be fuzzy weakly completely semi continuous (fwcsc) iff $f^{-1}(\alpha)$ is a fuzzy regular semi open subset of X for every fuzzy θ-open subset α in Y.

Equivalently, f is fuzzy weakly completely semi-continuous iff $f^{-1}(\beta)$ is a fuzzy regular semi-closed subset of X for every fuzzy θ-closed subset β in Y.

Since every fuzzy θ-open subset is fuzzy open thus every fuzzy completely semi-continuous function is fuzzy weakly completely semi-continuous. But the converse is not true, as shown in the following example :

Example 2.3 — Consider the fuzzy subsets μ_1, μ_2 and μ_3 of Example 2.1. Then $F = \{0, \mu_1, \mu_2, \mu_1 \cup \mu_2, 1)$ is a fuzzy topology on I. Let $f: (I, F) \to (I, F)$ be a function defined by $f(x) = x$ for each $x \in I$. Then $f^{-1}(0) = 0$, $f^{-1}(1) = 1$, $f^{-1}(\mu_1) = \mu_1, f^{-1}(\mu_2) = \mu_2$ and $f^{-1}(\mu_1 \cup \mu_2) = \mu_1 \cup \mu_2$. Hence both μ_1 and μ_2 are fuzzy regular open subsets of (I, F), but $\mu_1 \cup \mu_2$ is not fuzzy regular open in (I, F). Hence, $\mu_1 \cup \mu_2$ is not fuzzy regular semi-open in (I, F), let us choose $x = 3/4$ and $p = 4/5$ then $\mu_1(x) = \frac{1}{2}$ and $p + \mu_1(x) = 4/5 + \frac{1}{2} > 1$. Thus there exists a fuzzy point x_p which is q-coincident with μ_1 i.e., $x_p q \mu_1$ and also $\mathrm{Cl}(\mu_1) \subseteq \mathrm{Cl}(\mu_1 \cup \mu_2) = 1$. Hence, 1 is the only fuzzy θ-open subset in (I, F). Since $f^{-1}(1) = 1$ which is fuzzy regular semi-open in (I, F), thus f is fuzzy weakly completely semi-continuous. As $f^{-1}(\mu_1 \cup \mu_2) = \mu_1 \cup \mu_2$ which is not fuzzy regular semi-open in (I, F), f is not fuzzy completely semi continuous.

In the next theorem ,we prove that if the co-domain of the function is a fuzzy regular space then the converse is also true.

Theorem 2.2 — *If $f : (X, F) \to (Y, K)$ is fuzzy weakly completely semi continuous and Y is a fuzzy regular space then f is fuzzy completely semi continuous.*

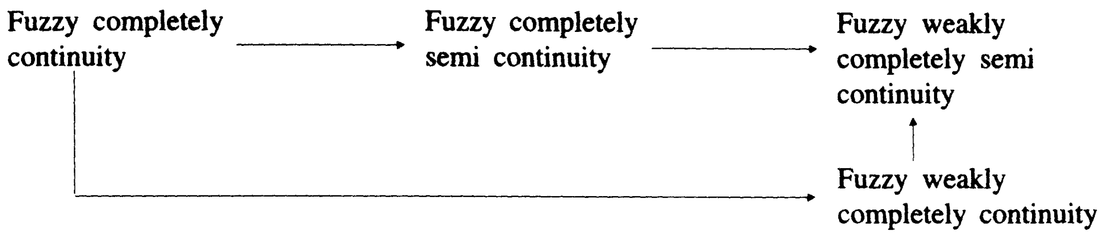
PROOF : Let f be fuzzy weakly completely semi continuous and α be a fuzzy open subset in (Y, K). Since (Y, K) is a fuzzy regular space, every fuzzy open subset in (Y, K) is fuzzy θ-open[8].

Thus $f^{-1}(\alpha)$ is fuzzy regular semi open in (X, F) which shows that f is fuzzy completely semi-continuous.

Bhaumik and Mukherjee[3], introduced fuzzy weakly completely continuous function as a generalization of fuzzy completely continuous function. It was shown that every fuzzy completely continuous function is fuzzy weakly completely continuous function. Since every fuzzy regular open subset is fuzzy regular semi open subset, hence every fuzzy weakly completely continuous function is fuzzy weakly completely semi continuous function, but the converse is not true.

Example 2.4 — Consider the fuzzy subsets μ_1, μ_2 and μ_3 of Example 2.1. Clearly, $F = \{0, \mu_1, \mu_2, \mu_1 \cup \mu_2, 1)$ and $F_3 = \{0, \mu_1, \mu_1', 1\}$ are two fuzzy topologies on I, let $g : (I, F) \to (I, F_3)$ be a function, defined by $g(x) = \min(2x, 1)$ for each $x \in I$. Then $g^{-1}(0) = 0$, $g^{-1}(1) = 1$, $g^{-1}(\mu_1) = \mu_2'$, $g^{-1}(\mu_1') = \mu_2$. Here both μ_1 and μ_2 are fuzzy regular open in (I, F) also μ_2' is fuzzy regular semi open but not fuzzy regular open in (I, F). Also (I, F_3) is a fuzzy extremally disconnected space. Since in a fuzzy extremally disconnected space every fuzzy regular open subset is fuzzy θ-open[8], g is fuzzy weakly completely semi continuous, but g is not fuzzy weakly completely continuous.

Thus, we have the following implications :



***Theorem* 2.3** — *A function $f(X, F) \to (Y, K)$ from a ft space (X, F) to another ft space (Y, K) is fuzzy weakly completely semi continuous iff for each fuzzy point $x_p \in X$ and each fuzzy θ-open subset α of Y containing $\rho(x_p)$ there exists a fuzzy regular semi open subset β of X containing x_p such that $f(\beta) \subseteq \alpha$.*

PROOF : Similar to Theorem 2.5, thus omitted.

3. CONDITIONS FOR EQUIVALENCE OF FUZZY WEAKLY COMPLETELY SEMI CONTINUOUS FUNCTIONS WITH OTHER FUZZY FUNCTIONS

In this section, we study the conditions for equivalence of fuzzy weakly completely semi continuous functions with other fuzzy functions. It can be seen that the set of all fuzzy regular semi open subsets of a ft space (X, F) forms a base for a fuzzy topology on X and is denoted by F_R. On the other hand, F_θ be a fuzzy topology on X, which has the family of all fuzzy θ-open subsets of (X, F) as a subbase.

***Theorem* 3.1** — *If f is a function from a ft space (X, F) to another ft space (Y, K). Then the following are equivalent :*

(a) $f: (X, F) \to (Y, K)$ is a fuzzy weakly completely semi continuous function.

(b) $f: (X, F) \to (Y, K_\theta)$ is a fuzzy completely semi-continuous function.

(c) $f: (X, F_R) \to (Y, K_\theta)$ is a fuzzy continuous function.

(d) $f: (X, F_R) \to (Y, K)$ is a fuzzy faintly continuous function.

PROOF : (*a*) $\to$ (*b*) Let α be a fuzzy open subset of (Y, K_θ) then α is a fuzzy θ-open subset of (Y, K). By (*a*) $f^{-1}(\alpha)$ is fuzzy regular semi open in (X, F) which shows that $f: (X, F) \to (Y, K_\theta)$ is fuzzy completely semi-continuous.

(*b*) $\to$ (*c*) Let β be a fuzzy open subset in (Y, K_θ). Then by (*b*), $f^{-1}(\beta)$ is fuzzy regular semi open in (X, F), thus $f^{-1}(\beta)$ is fuzzy open in (X, F_R), which shows that $f: (X, F_R) \to (Y, K_\theta)$ is fuzzy continuous.

(*c*) $\to$ (*d*) Let γ be fuzzy θ-open in (Y, K) then γ is fuzzy open in (Y, K_θ). By (*c*), $f^{-1}(\gamma)$ is fuzzy open in (X, F_R), which shows that $f: (X, F_R) \to (Y, K)$ is fuzzy faintly continuous.

$(d) \to (a)$ Let μ be fuzzy θ-open in (Y, K). Then by (d), $f^{-1}(\mu)$ is fuzzy open in (X, F_R), hence $f^{-1}(\mu)$ is fuzzy regular semi open open in (X, F) which shows that $f : (X, F) \to (Y, K)$ is fuzzy weakly completely semi-continuous.

Lemma 3.2 — If $F : (X, F) \to (Y, F_1)$ is fuzzy completely semi-continuous and $g : (Y, F_1) \to (Z, F_2)$ is fuzzy faintly continuous then $g \circ f : (X, F) \to (Z, F_2)$ is fuzzy weakly completely semi continuous.

PROOF : Straight forward.

4. APPLICATIONS

Chang[12], defined a ft space (X, F) to be fuzzy compact iff every open cover of X has a finite sub cover. In a similar way one can call a ft space (X, F) fuzzy θ-compact iff every fuzzy θ-open cover of X admits a subcover.

Theorem 4.1 — *If $f : X \to Y$ is a fuzzy completely semi continuous (resp. fuzzy weakly completely semi continuous) onto function and X is fuzzy s-closed space then Y is fuzzy compact (resp. fuzzy θ-compact).*

PROOF : Let $\{G_a : a \in \Lambda\}$ be a fuzzy open (resp. fuzzy θ-open) cover of Y. Then $\{f^{-1}(G_a) : a \in \Lambda\}$ is a fuzzy regular semi open cover of X. By fuzzy s-closedness of X, there exists a finite sub family $\{f^{-1}(G_{a_i})\}$ of $\{f^{-1}(G_a)\}$ such that $\bigcup_{i=1}^{n} f^{-1}(G_{a_i}) = 1_X$. Now $1_Y = f(I_X) = \left(\bigcup_{i=1}^{n} f^{-1}(G_{a_i})\right) \subset \bigcup_{i=1}^{n} G_{a_i}$, which implies γ is fuzzy compact (resp. fuzzy θ-compact).

Azad[6] proved that if $g : X \to XXY$ be a graph of a function $f : X \to Y$ then for a fuzzy subset α of X and a fuzzy subset β of Y, $g^{-1}(\alpha \times \beta) = \alpha \cap f^{-1}(\beta)$.

Theorem 4.2 — *Let X and Y be two ft spaces. If the graph function $g : X \to XXY$ is fuzzy completely semi continuous, then the function $f : X \to Y$ is fuzzy completely semicontinuous.*

PROOF : Let g be fuzzy completely semi continuous and β be any fuzzy open subset of Y. Then $g^{-1}(1 \times \beta) = 1 \cap f^{-1}(\beta) = f^{-1}(\beta)$. Since β is fuzzy open in Y, $1 \times \beta$ is also fuzzy open in XXY. Thus $g^{-1}(1 \times \beta)$ is fuzzy regular semi open in X, i.e., $f^{-1}(\beta)$ is fuzzy regular semi open in X. Hence the theorem.

Theorem 4.3 — *If $f : X \to Y$ is a fuzzy semi continuous onto function and X is fuzzy S^*-closed space then Y is fuzzy almost compact.*

PROOF : Let $\{V_a : a \in \Lambda\}$ be a fuzzy open cover of Y, then $\{f^{-1}(V_a) : a \in \Lambda\}$ is a fuzzy semi open cover of X. By fuzzy S^*-closedness of X, there is a finite subset $\{a_1, a_2, ..., a_n\}$ of Λ such that $\bigcup_{i=1}^{n} S\text{-}\mathrm{Cl}\ f^{1}(V_{a_i}) = 1_X$. Now, $1_Y = f(1_X) = f\left(\bigcup_{i=1}^{n} S\text{-}Cl\, f^{-1}(V_{ai})\right) \subset f\left(\bigcup_{i=1}^{n} Cl f^{1}(V_{a_i})\right)$ $\subset \bigcup_{i=1}^{n} Cl\,(V_{ai})$ which implies Y is fuzzy almost compact.

REFERENCES

1. M. N. Mukherjee and B. Ghosh, *Fuzzy set. Syst.* **38** (1990) 375-87.
2. R. N. Bhaumik and A. Mukherjee, *Fuzzy Set. Syst.* **56** (1993) 243-46.
3. R. N. Bhaumik and A. Mukherjee, *Fuzzy Set. Syst.* **55** (1993) 347-54.
4. Pu Pau-Ming and Liv Ying-Ming, *J. math. Anal. Appl.* **76** (1980) 571-99.
5. M. N. Mukherjee and S. P. Sinha, *Fuzzy Set. Syst.* **34** (1990) 245-54.
6. K. K. Azad, *J. math. Anal. Appl.* **82** (1981) 14-32.
7. A. N. Zahren, *J. Fuzzy Math.* **2** (1994) 579-86.
8. A. Mukherjee, *Fuzzy Set. Syst.* **59** (1993) 59-63.
9. A. Di Concilio and G. Geeta, *Fuzzy Set. Syst.* **13** (1984) 187-92.
10. S. P. Sinha and S. Malakar, *J. Fuzzy Math.* **2** (1994) 95-103.
11. S. Malakar, *Fuzzy Set. Syst.* **45** (1992) 239-44.
12. C. L. Chang, *J. math. Anal. Appl.* **24** (1968) 182-90.