Indian J. pure appl. Math., **29**(3) : 227-238, March 1998

FIXED POINTS FOR SET VALUED FUNCTIONS WITHOUT CONTINUITY

G. JUNGCK[1] AND B. E. RHOADES[2]

[1]*Department of Mathematics, Bradley University, Peoria, Illinois* 61625, *USA*

[2]*Department of Mathematics, Indiana University, Bloomington, Indiana* 47405-5701, *USA*

(*Received* 20 *August* 1997; *Accepted* 7 *November* 1997)

The concept of "weakly" compatible maps is introduced to produce fixed point theorems for set valued noncontinuous functions. Known results are thereby extended/generalized.

Key Words : Fixed Points; Set Valued Functions; Weakly Compatible Maps; (ε, γ) Contractions

1. INTRODUCTION

In this paper we propose and prove fixed point theorems for set-valued functions without appeal to continuity. This is done via the concepts of compatible (weak) maps and generalized Meir-Keeler contractions called (ε, γ) contractions. Our theorems extend results of Chang[1], generalize results by Jachymski[2] and Kang and Rhoades[3], and produce as byproducts generalizations of theorems for point valued functions. We thus give further evidence of the significant role of compatibility in producing fixed points.

Compatible maps were introduced by the first author[4]. Self maps $A, S : X \to X$ of a metric space (X, d) are compatible iff whenever $Ax_n, Sx_n \to t \in X$, then $d(SAx_n, ASx_n) \to 0$. In this paper, we are interested in "weak" compatible set-valued maps which we define below.

2. BASIC CONCEPTS

If (X, d) is a metric space, $\mathcal{B}(X)$ is set of all nonempty bounded subsets of X. If $A, B \in \mathcal{B}(X)$, then

$$\delta(A, B) = \sup \{d(a, b) : a \in A \text{ and } b \in B\} \text{ and}$$

$$D(x, B) = \inf \{d(x, b) : b \in B\} \text{ for } x \in X.$$

We appeal to the facts that $0 \le \delta(A, B) \le \delta(A, C) + \delta(C, B)$ and $\delta(A, B) = 0$ iff $A = B = \{x\}$ (a singleton) for $A, B, C \in \mathcal{B}(X)$ (Jungck and Rhoades[5]). If $a \in X$, we write $\delta(a, B)$ for $\delta(\{a\}, B)$.

Definition 2.1 — Let (X, d) be a metric space and let $A, B : X \to \mathcal{B}(X)$. Then A and B are $(\varepsilon, \gamma)(p)$ Contractions relative to maps $S, T : X \to X$ iff $\bigcup A(X) \subseteq T(X)$, $\bigcup B(X) \subseteq S(X)$, and there exist functions $p : X \times X \to [0, \infty)$, $\gamma : (0, \infty) \to (0, \infty)$ such that $\gamma(\varepsilon) > \varepsilon$ for all ε, and for $x, y \in X$:

(i) $0 \dot{<} \varepsilon \leq p(x, y) < \gamma(\varepsilon) \Rightarrow \delta(Ax, By) < \varepsilon$.

We refer to $(\varepsilon, \gamma)(p)$ Contractions as (m) Contractions if

$$p(x, y) = m(x, y) = \max \left\{ d(Sx, Ty), \frac{1}{2}[D(Ax, Ty) + D(Sx, By)] \right\},$$

and as (M) Contractions if

$$p(x, y) = M(x, y) = \max \left\{ d(Sx, Ty), \delta(Ax, Sx), \delta(By, Ty), \frac{1}{2}[D(Ax, Ty) + D(Sx, By)] \right\}.$$

Remark 2.1 : Observe that by (i) : $\delta(Ax, By) < p(x, y)$ when $p(x, y) \neq 0$ whether $p = m$ or M. Moreover, if $M(x, y) = 0$, $\delta(Ax, By) = 0$, so $\delta(Ax, By) \leq M(x, y)$ for all x, y. However, in order for $\delta(Ax, By) \leq m(x, y)$ to hold for all $x, y \in X$, we must require that $\delta(Ax, By) = 0$ when $m(x, y) = 0$.

The authors[5] extended the concept of compatible maps to maps $S : X \to X$ and $A : X \to \mathcal{B}(X)$ by requiring that $SAx \in \mathcal{B}(X)$ for $x \in X$ and $\delta(SAx_n, ASx_n) \to 0$ whenever $\{x_n\}$ is a sequence in X such that $Ax_n \to \{t\}$ $(\delta(Ax_n, t) \to 0)$ and $Sx_n \to t$ for some $t \in X$. A consequence of this "δ-compatibility" was noted in Proposition 3.1 (ii), namely, that $ASx = SAx$ when $Ax = \{Sx\}$. This result motivates the following definition.

Definition 2.2 — Let (X, d) be a metric space, and let $S : X \to X$ and $A : X \to \mathcal{B}(X)$. The pair $\{A, S\}$ is a weakly compatible pair iff $Ax = \{Sx\}$ implies that $SAx = ASx$.

It is immediate that any compatible pair $\{A, S\}$ is weakly compatible. Examples of weakly compatible pairs which are not compatible will be given later.

3. Preliminary Results

Given self maps $S, T : X \to X$ of a metric space (X, d) and maps $A, B : X \to \mathcal{B}(X)$, we require that $\bigcup A(X) \subseteq T(X)$ and $\bigcup B(X) \subseteq S(X)$ so we can define sequences $\{x_n\}$ and $\{y_n\}$ recursively by

$$Tx_{2n-1} \in Ax_{2n-2} = y_{2n-1} \quad \text{and} \quad Sx_{2n} \in Bx_{2n-1} = y_{2n} \qquad \ldots (3.1)$$

for $n \in N$ (the set of Natural numbers) and begin at any point $x_0 \in X$.

Lemma 3.1 — Let A and B be $(\varepsilon, \gamma)(M)$ Contractions relative to self maps S and T of a metric space (X, d). With $\{y_n\}$ defined as in (3.1) and with $d_n = \delta(y_{n+1}, y_n)$ for $n \in N$, we have

(1) $d_n \leq M(x_i, x_j) \leq d_{n-1}$, where $i = n(n-1)$ and $j = n - 1$ (n) if n is even (odd).

(2) $d_n \to 0$, and

(3) for any $\varepsilon > 0$, $\exists\ n_0 \in N$ such that $m, n \in N$ and $m, n \geq n_0 \Rightarrow \delta(y_m, y_n) < \varepsilon$.

PROOF : By (3.1), $\delta(y_{2n+1}, y_{2n}) = \delta(Ax_{2n}, Bx_{2n-1}) \le M(x_{2n}, x_{2n-1}) = \max \{d(Sx_{2n}, Tx_{2n-1}), \delta(Ax_{2n}, Sx_{2n}), \delta(Bx_{2n-1}, Tx_{2n-1}, \frac{1}{2}[D(Ax_{2n}, Tx_{2n-1}) + D(Sx_{2n}, Bx_{2n-1})]\} \le \max \{\delta(y_{2n}, y_{2n-1}), \delta(y_{2n+1}, y_{2n}) \delta(y_{2n}, y_{2n-1}), \frac{1}{2}[\delta(y_{2n+1}, y_{2n-1}) + 0]\}$, so that

$M(x_{2n}, x_{2n-1}) \le \max \{\delta(y_{2n+1}, y_{2n}), \delta(y_{2n}, y_{2n-1}), \frac{1}{2}[\delta(y_{2n+1}, y_{2n}) + \delta(y_{2n}, y_{2n-1})]\} \le \max \{\delta(y_{2n+1}, y_{2n}), \delta(y_{2n}, y_{2n-1})\}$. Consequently, for any n we have :

$$\delta(y_{2n+1}, y_{2n}) = \delta(Ax_{2n}, Bx_{2n-1}) \le M(x_{2n}, x_{2n-1})$$
$$\le \max \{\delta(y_{2n+1}, y_{2n}), \delta(y_{2n}, y_{2n-1})\}. \quad \ldots (3.2)$$

Thus, if $M(x_{2n}, x_{2n-1}) = 0$, $\delta(y_{2n+1}, y_{2n}) \le \delta(y_{2n}, y_{2n-1})$. And if $M(x_{2n}, x_{2n-1}) > 0$, then $\delta(A_{2n}x, B_{2n-1}x) < M(x_{2n}, x_{2n-1})$ by Remark 2.1, so (3.2) implies $\delta(y_{2n+1}, y_{2n}) \le M(x_{2n}, x_{2n-1}) \le \delta(y_{2n}, y_{2n-1})$. Similarly, $\delta(y_{2n+2}, y_{2n+1}) \le M(x_{2n}, x_{2n+1}) \le \delta(y_{2n+1}, y_{2n})$.

Thus, for $k \in N$ we have :

$$d_k = \delta(y_{k+1}, y_k) \le M(x_k, x_{k-1}) \le \delta(y_k, y_{k-1}) = d_{k-1}, \qquad \text{if } k \text{ is even and}$$

$$d_k = \delta(y_{k+1}, y_k) \le M(x_{k-1}, x_k) \le \delta(y_k, y_{k-1}) = d_{k-1}, \qquad \text{if } k \text{ is odd.}$$

To see that (2) holds, note that since $0 \le d_n$ and $\{d_n\}$ is nonincreasing by (1), $d_n \to r \ge 0$. If $r > 0$, $\exists m$ (even) such that $r \le d_{m-1} < \gamma(r)$; i.e., $r \le d_m \le M(x_m, x_{m-1}) \le d_{m-1} < \gamma(r)$. Thus, (i) implies that $\delta(A_{xm}, Bx_{m-1}) < r$, so that $\delta(y_{m+1}, y_m) = d_m < r \le d_m$. Conclude : $r = 0$.

To prove (3), we remind the reader that the sequence $\{y_n\}$ is defined in (3.1) above, and that

(*) $0 < \varepsilon \le m(x, y) < \gamma(\varepsilon)$ implies that $\delta(Ax, By) < \varepsilon$.

We assume that (3) does not hold. Then there is an ε such that :

(i) for any $n_0 \in N$, $\exists\, m, n \in N$ such that $m > n \ge n_0$ and $\delta(y_m, y_n) \ge 2\varepsilon$.

For this particular ε write $\gamma(\varepsilon) = \varepsilon + \beta$. We may assume (see (*)) that $\beta < \varepsilon$. And by (2) above, we know that we can choose n_1 such that

(ii) $\delta(y_{n+1}, y_n) < \frac{\beta}{6} < \frac{\varepsilon}{6}$ when $n \ge n_1$.

Next choose $k, p \in N$ such that

(iii) $p > k > n_1$ and $\delta(y_k, y_p) > 2\varepsilon$ (see (i)).

We first show that

(iv) $\delta(y_k, y_j) > \varepsilon + \frac{\beta}{3}$ and $j > k > n_1$ for $j = p - 1, p - 2$.

To this end note that (iii) and (ii) imply that

$$2\varepsilon < \delta(y_k, y_{p-1}) + \delta(y_{p-1}, y_p) < \delta(y_k, y_{p-1}) + \frac{\varepsilon}{6},$$

since $p - 1 \ge k > n_1$.

Thus $\frac{11}{6}\varepsilon < \delta(y_k, y_{p-1})$. Then $p-1 > k$ and we have

$\frac{11}{6}\varepsilon < \delta(y_k, y_{p-2}) + \delta(y_{p-2}, y_{p-1}) < \delta(y_k, y_{p-2}) + \frac{\varepsilon}{6}$. Consequently,

$\frac{10}{6}\varepsilon < \delta(y_k, y_{p-2})$ and $p-2 > k$. So by the preceding we can certainly say:

$\varepsilon + \frac{\beta}{3} < \frac{4}{3}\varepsilon < \delta(y_k, y_j)$ where $j = p-1, p-2 > k$, and (iv) therefore holds.

We now assume that k is odd in (iv). By (iv) we can let m be the smallest even integer such that

(v) $m > k$ and $\delta(y_k, y_m) \geq \varepsilon + \frac{\beta}{3}$.

But (ii), (iv), and the triangle inequality imply that $m > k + 2$, so that $m - 2 > k$, $m - 2$ is even, and therefore $\delta(y_k, y_{m-2}) < \varepsilon + \frac{\beta}{3}$ by the choice of m. Then, since $m - 2 > k > n_1$, (ii) implies:

$$\delta(y_k, y_m) \leq \delta(y_k, y_{m-2}) + \delta(y_{m-2}, y_{m-1}) + \delta(y_{m-1}, y_m) < \left(\varepsilon + \frac{\beta}{3}\right) + 2\left(\frac{\beta}{6}\right) = \varepsilon + \frac{2\beta}{3}.$$

(v) then yields :

(vi) $\varepsilon + \frac{\beta}{3} \leq \delta(y_k, y_m) < \varepsilon + \frac{2\beta}{3}$, with $m - 2 > k > n_1$.

Now since k is odd and m is even, (3.1) and (vi) imply

$\varepsilon < \delta(y_k, y_m) = \delta(Ax_{k-1}, Bx_{m-1}) \leq M(x_{k-1}, x_{m-1})$, where

$M(x_{k-1}, x_{m-1}) = \max \{d(Sx_{k-1}, Tx_{m-1}), \delta(Ax_{k-1}, Sx_{k-1}), \delta(Bx_{m-1}, Tx_{m-1}),$

$\frac{1}{2}(D(Ax_{k-1}, Tx_{m-1}) + D(Bx_{m-1}, Sx_{k-1}))\}$. Therefore,

(vii) $\varepsilon < \delta(Ax_{k-1}, Bx_{m-1}) \leq M(x_{k-1}, x_{m-1})$

$$\leq \max \left\{ \delta(y_{k-1}, y_{m-1}), \delta(y_k, y_{k-1}), \delta(y_m, y_{m-1}), \frac{1}{2}(\delta(y_k, y_{m-1}) + \delta(y_m, y_{k-1})) \right\}.$$

But (vi) and (ii) imply

$$\delta(y_{k-1}, y_{m-1}) \leq \delta(y_{k-1}, y_k) + \delta(y_k, y_m) + \delta(y_m, y_{m-1}) < \frac{\beta}{6} + \left(\varepsilon + \frac{2\beta}{3}\right) + \frac{\beta}{6} = \varepsilon + \beta.$$

Similarly,

$$\frac{1}{2}(\delta(y_k, y_{m-1}) + \delta(y_m, y_{k-1}) < \frac{1}{2}\left(2\left(\varepsilon + \frac{2\beta}{3} + \frac{\beta}{6}\right)\right) = \varepsilon + \beta.$$

Thus, (vii) yields

$$\varepsilon < \delta(Ax_{k-1}, Bx_{m-1}) \leq M(x_{k-1}, x_{m-1}) < \max \left\{ \varepsilon + \beta, \frac{\beta}{6}, \frac{\beta}{6}, \varepsilon + \beta \right\} = \varepsilon + \beta = \gamma(\varepsilon).$$

Consequently, (*) gives us the contradiction,

$$\varepsilon < \delta(Ax_{k-1}, Bx_{m-1}) < \varepsilon.$$

Note that in the above argument k was assumed to be odd. If k is even, m can be chosen odd and the above argument pertains with x_{k-1} and x_{m-1} interchanged. □

Lemma 3.2 — Let S and T be self maps of a metric space $(X\ d)$ and let $A, B: X \to \mathcal{B}(X)$. Suppose also that $\exists\ x, u, v \in X$ such that $\{x\} = \{Su\} = \{Tv\} = Au = Bv$.

(a) If $\delta(Ax, By) < M(x, y)$ when $M(x, y) \neq 0$, and if $\{A, S\}$ is weakly compatible ($\{B, T\}$ is weakly compatible), then $\{x\} = \{Sx\} = Ax$, ($\{x\} = \{Tx\} = Bx$).

(b) If $\delta(Ax, By) < M(x, y)$ when $M(x, y) \neq 0$, and if the pair $\{A, S\}$ is weakly compatible ($\{B, T\}$ is weakly compatible), then $\{x\} = \{Sx\} = Ax$ ($\{x\} = \{Tx\} = Bx$).

PROOF : We prove part (b) first. By definition,

$$M(x, v) = \max\left\{ d(Sx, Tv), \delta(Ax, Sx), \delta(Bv, Tv), \frac{1}{2}(D(Ax, Tv) + D(Sx, Bv)) \right\}.$$

So, by hypothesis,

$$M(x, v) = \max\left\{ d(Sx, x), \delta(Ax, Sx), 0, \frac{1}{2}(D(Ax, x) + D(Sx, x)) \right\}.$$

But since $Au = \{Su\}$ and the pair $\{A, S\}$ is weakly compatible, $SAu = ASu$, or $\{Sx\} = Ax$. Thus,

$$M(x, v) = d(Sx, x) = \delta(Ax, x) \text{ since } Sx \in X.$$

Consequently, if $Ax \neq \{x\}$, $M(x, v) \neq 0$ and therefore $\delta(Ax, x) = \delta(Ax, Bv) < M(x, v) \leq \delta(Ax, x)$. This contradiction demands that $\{x\} = Ax = \{Sx\}$. Similarly, $\{x\} = Bx = \{Tx\}$ if the pair $\{B, T\}$ is weakly compatible.

To prove part (*a*), duplicate the above argument. Note that since $\{A, S)$ is weakly compatible, we obtain *via* the hypothesis and "$Ax = \{Sx\}$", $m(x, v) = \frac{1}{2}(D(Ax, x) + D(Ax, x)) \leq \delta(Ax, x)$. And as above, we thus conclude that $\{x\} = Ax = \{Sx\}$. □

Proposition 3.1 — Let S and T be self maps of a metric space $(X,\ d)$ and let $A, B : X \to \mathcal{B}(X)$. Let A and B be (ε, γ) (M) Contractions relative to S and T. If $S(X)$ $(T(X))$ is complete $\exists$ a sequence $\{x_n\}$ in X and $p \in X$ such that $Tx_{2n-1}, Sx_{2n} \to p \in S(X)\,(T(X))$ and $\delta(Bx_{2n-1}, p)$, $\delta(Ax_{2n}, p) \to 0$.

PROOF : Suppose that $T(X)$ is complete and let $\varepsilon > 0$. Let $\{x_n\}$ be the sequence defined by (3.1). By Lemma 3.1(3) we can choose $n_0 \in N$ such that $\delta(y_m, y_n) < \varepsilon$ for $m, n \geq n_0$. But $Tx_{2k-1} \in Ax_{2k-2} = y_{2k-1}$, fot $k \in H$ so that

$$d(Tx_{2m-1}, Tx_{2n-1}) \leq \delta(y_{2m-1}, y_{2n-1}) < \varepsilon \text{ for } m, n \geq n_0.$$

Therefore, the sequence $\{Tx_{2n-1}\}$ is Cauchy and hence $Tx_{2n-1} \to p = Tv \in T(X)$ for some $v \in X$ since $T(X)$ is complete. But $Sx_{2n} \in Bx_{2n-1} = y_{2n}$ by (3.1), so that Lemma 3.1 (2) implies that

$$d(Sx_{2n}, Tx_{2n-1}) \leq \delta(y_{2n}, y_{2n-1}) \to 0.$$

Consequently, $Sx_{2n} \to p$. Moreover, we have for $n \in N$:

$$\delta(Ax_{2n}, p) \leq \delta(Ax_{2n}, Sx_{2n}) + \delta(Sx_{2n}, p) = \delta(y_{2n+1}, y_{2n}) + d(Sx_{2n}, p).$$

Therefore, $\delta(Ax_{2n}, p) \to 0$. In like manner it follows that $\delta(Bx_{2n-1}, p) \to 0$. □

We shall also need the following result.

Proposition 3.2 — Let (X, d) be a metric space and let $A, B : X \to \mathcal{B}(X)$. Suppose A and B are (ε, γ) (p) Contractions relative to maps $S, T : X \to X$ and that γ is lower semi-continuous. If $\{x_n\}$ and $\{y_n\}$ are sequences in X such that $\lim_n p(x_n, y_n) = \varepsilon > 0$ and lim sup $\delta(Ax_n, By_n) = r$, then $r < \varepsilon$.

PROOF : Since $\gamma(\varepsilon) > \varepsilon$ and γ is a lower semi-continuous function, there is a neighbourhood N_ε of ε such that $\gamma(t) > \varepsilon$ for $t \in N_\varepsilon$. We can therefore choose $t_0 \in N_\varepsilon$ such that $0 < t_0 < \varepsilon < \gamma(t_0)$. Since $p(x_n, y_n) \to \varepsilon$ there exists $m \in N$ such that $p(x_n, y_n) \in (t_0\, \gamma(t_0))$ for $n \geq m$. Then, by (i) in Definition 2.1, $\delta(Ax_n, By_n) \leq t_0$ for $n \geq m$; i.e., lim sup $\delta(Ax_n, By_n) = r \leq t_0 < \varepsilon$. □

4. MAIN RESULTS

Definition 2.1 tells us that $m(x, y) \leq M(x, y)$ for any $x, y \in X$. In fact, (m) Contractions are (M) Contractions if we require that $\delta(Ax, By) = 0$ when $m(x, y) = 0$. For suppose that A and B are (m) Contractions relative to maps $S, T : X \to X$ and let (i) $0 < \varepsilon \leq M(x, y) < \gamma(\varepsilon)$. If $m(x, y) = 0$, then $\delta(Ax, By) = 0 < \varepsilon$. If $0 < m(x, y) < \varepsilon$, the definition of m implies that $\delta(Ax, By) < m(x, y) < \varepsilon$. On the other hand, if $\varepsilon \leq m(x, y)$, then since $m(x, y) \leq M(x, y)$, (i)implies that $m(x, y) < \gamma(\varepsilon)$, so that $\delta(Ax, By) < \varepsilon$ by the definition of (m) contractions. Thus, in any event, (i) implies $\delta(Ax, By) < \varepsilon$ and hence our (m) Contraction is an (M) Contraction. We are now ready to state and prove the first main result.

***Theorem* 4.1** — *Let (X, d) be a metric space and let $A, B : X \to \mathcal{B}(X)$. Suppose that A and B are (ε, γ) (p) Contractions relative to maps $S, T : X \to X$, and that one of $S(X)$ or $T(X)$ is complete.*

(a) If $p = m$, and $m(x, y) = 0$ implies that $\delta(Ax, By) = 0$, and if both pairs $\{A, S\}$ and $\{B, T\}$ are weakly compatible, then $\exists\ z \in X$ such that $\{z\} = \{Sz\} = \{Tz\} = Az = Bz$.

(b) If $p = M$, if γ is lower semi-continuous, and if the pairs $\{A, S\}$ and $\{B, T\}$ are each weakly compatible, then $\exists\ z \in X$ such that $\{z\} = \{Sz\} = \{Tz\} = Az = Bz$.

PROOF : (a) The hypothesis and preceding comments assure us that our (m) Contractions are indeed (M) Contractions. Consequently, we can appeal to Proposition 3.1 to produce a sequence $\{x_n\}$ in X and $p \in X$ such that

(*) $Tx_{2n-1}, Sx_{2n} \to p$ and $\delta(Bx_{2n-1}, p), \delta(Ax_{2n}, p) \to 0$.

We assume without loss of generality that $T(X)$ is complete so that $p \in T(X)$, and hence $p = Tv$ for some $v \in X$. Moreover, since for $n \in N$,

$$\delta(p, Bv) \leq \delta(p, Ax_{2n}) + \delta(Ax_{2n}, Bv) \leq \delta(p, Ax_{2n}) + m(x_{2n}, v)$$

$$\leq \delta(p, Ax_{2n}) + \max\{d(Sx_{2n}, Tv), \frac{1}{2}(D(Sx_{2n}, Bv) + D(Tv, Ax_{2n}))\},$$

and since $\delta(Sx_n, Bv) \to \delta(p, Bv)$ when $Sx_n \to p$ (see Jungck and Rhoades[5]), (*) tells us that

$$\delta(p, Bv) \leq 0 + \max\left\{0, \frac{1}{2}(\delta(p, Bv) + 0)\right\} = \frac{1}{2}\delta(p, Bv).$$

Consequently, $\{p\} = Bv = \{Tv\}$. But $\bigcup B(X) \subseteq S(X)$, so $\exists\ u \in X$ such that $\{Su\} = Bv = \{Tv\}$. Now if $Au \neq Bv$, $\delta(Au, Bv) \neq 0$, so that $m(u, v) > 0$ by hypothesis and we have :

$$\delta(Au, Bv) < m(u, v) = \max\left\{ d(Su, Tv), \frac{1}{2}(D(Su, Bv) + D(Tv, Au)) \right\} = \frac{1}{2} D(Tv, Au),$$

so

$$\delta(Au, p) < \frac{1}{2}\delta(Au, p).$$

We must conclude that $\{p\} = Au = Bv = \{Su\} = \{Tv\}$. Therefore, Lemma 3.2 ($a$) implies that $\{p\} = \{Sp\} = \{Tp\} = Ap = Bp$. To see the p is unique, suppose that $\{q\} = \{Sq\} = \{Tq\} = Aq = Bq$. If $p \neq q$, $\delta(p, q) = \delta(Ap, Bq) < m(p, q) = \delta(p, q)$; i.e., $p = q$.

To prove part (b), we again appeal to Proposition 3.1 and (*) of part (a). And as above, we assume since

$$\delta(p, Bv) \leq \delta(p, Ax_{2n}) + \delta(Ax_{2n}, Bv), \text{ (*) implies that } \delta(p, Bv) \leq \lim_n \delta(Ax_{2n}, Bv).$$

But $\delta(Ax_{2n}, Bv) \leq \delta(Ax_{2n}, p) + \delta(p, Bv)$ for $n \in N$, so $\lim_n \delta(Ax_{2n}, Bv) \leq \delta(p, Bv)$.

Hence,

$$\delta(p, Bv) = \lim_n \delta(Ax_{2n}, Bv). \qquad \ldots (1)$$

On the other hand,

$$M(x_{2n}, v) = \max\left\{ d(Sx_{2n}, Tv), \delta(Sx_{2n}, Ax_{2n}), \delta(Bv, Tv), \frac{1}{2}[D(Ax_{2n}, Tv) + D(Bv, Sx_{2n})] \right\}. \qquad \ldots (2)$$

Now, $\delta(Sx_{2n}, Ax_{2n}) \leq \delta(Sx_{2n}, p) + \delta(p, Ax_{2n})$ for $n \in N$, so $\lim_n \delta(Sx_{2n}, Ax_{2n}) = 0$.

Also, $D(Ax_{2n}, Tv) \leq \delta(Ax_{2n}, p) \to 0$. So remembering that $p = Tv$, (2) implies that $\lim_n M(x_{2n}, v) = \delta(p, Bv)$; i.e., $\lim_n M(x_{2n}, v) = \lim_n \delta(Ax_{2n}, Bv)$ by (1). But this last equality contradicts Proposition 3.2 if $\delta(p, Bv) \neq 0$, since γ is lower semi-continuous. Therefore,

$$\{p\} = Bv = \{Tv\}. \qquad \ldots (3)$$

Moreover, $\bigcup B(X) \subseteq S(X)$, so $\exists\ u \in X$ such that $\{Su\} = Bv$. Then (3) implies that

$M(u, v) = \max\left\{ d(Su, Tv), \delta(Au, Su), \delta(Bv, Tv), \frac{1}{2}[D(Au, Tv) + D(Bv, Su)] \right\} = \delta(Au, Bv)$. But then, Remark 2.1 requires that $M(u, v) = 0 = \delta(Au, Bv)$, or $Au = Bv$. We thus have

$$Au = \{Su\} = Bv = \{Tv\} = \{p\}, \text{ so that } \{p\} = Ap = Bp = \{Tp\} = \{Sp\}$$

by Lemma 3.2. The uniqueness of the common fixed point p follows as in part (a). □

Corollary 4.1 — Let S and T be self maps of a complete metric space (X, d) and $A, B : X \to \mathcal{B}(X)$. Let $\bigcup A(X) \subseteq T(X)$ and $\bigcup B(X) \subseteq S(X)$. Suppose $\exists\ r \in (0, 1)$ such that $\delta(Ax, By) \leq rM(x, y)$ for $x, y \in X$. Then $\exists$ a unique point $z \in X$ such that $\{z\} = \{Sz\} = \{Tz\} = Az = Bz$, if both $\{A, S\}$ and $\{B, T\}$ are compatible pairs.

PROOF : Let $\gamma(\varepsilon) = \varepsilon/r$.

If in Definition 2.1 of $(\varepsilon, \gamma)(p)$ Contractions, $A, B : X \to X$ (as opposed to $\mathcal{B}(X)$), the definition still makes sense with $\delta = D = d$ (See Jungck and Pathak[4], Def. 2.1). If A and B are "point valued" we typically use a lower case "c", i.e., we write. "$(\varepsilon, \gamma)(p)$ contractions". Now if $A, B : X \to X$, then (m) contractions are given by : $m(x, y) = \max \{d(Sx, Ty), \frac{1}{2}[d(Ax, Ty) + d(Sx, By)]\}$. They therefore have the property that $(m(x, y) = 0) \Rightarrow (d(Ax, By) = 0)$. We thus have the following corollary to Theorem 4.1(a).

Corollary 4.2 — Let A, B, S and T be self maps of a metric space (X, d) and suppose that A and B are (ε, γ) (m) contractions relative to S and T. If one of $S(X)$, $T(X)$ is complete, and if $\{A, S\}$ and $\{B, T\}$ are both weakly compatible pairs, then A, B, S and T have a unique common fixed point.

The following consequence of Corollary 4.2 generalizes Theorem 5.2 of Jachymski[2] with the pairs $\{A, B\}$ and $\{S, T\}$ exchanged — by replacing $d(Sx, Ty)$ with $m(x, y)$. (Note also that we use γ lieu of δ).

Corollary 4.3 — Let A, B, S and T be self maps of a complete metric space (X, d) such that S and T are surjective. Suppose that (*) $m(x, y) \geq \psi(d(Ax, By))$ for $x, y \in X$ where $\psi : [0, \infty) \to [0, \infty)$ is a non-decreasing function satisfying $\psi(t) > t$ for all $t > 0$. If $\{A, S\}$ and $\{B, T\}$ are weakly compatible pairs, then A, B, S and T have a unique common fixed point.

PROOF : Following Jachymski[2], we let γ be the restriction of ψ to $(0, \infty)$. Then A, B are (ε, γ) (m) contractions relative to S and T. For suppose $0 < \varepsilon \leq m(x, y) < \gamma(\varepsilon)$; then $d(Ax, By) < \varepsilon$. Otherwise, by (*) and the monotonicity of ψ, $m(x, y) \geq \psi(d(Ax, By)) \geq \psi(\varepsilon) = \gamma(\varepsilon)$, which implies $\gamma(\varepsilon) < \gamma(\varepsilon)$. Hence, it suffices to note that S and T are surjections and that $S(X) = X$ is complete the proof. □

We need the following proposition to prove our other main result :

Proposition 4.1 — Suppose $\phi : [0, \infty) \to [0, \infty)$ is upper semi-continuous (u.s.c.) from the right, and that $\phi(t) < t$ for $t > 0$. Then $\exists$ a function $\gamma : (0, \infty) \to (0, \infty)$ such that

(1) $\varepsilon < \gamma(\varepsilon) \leq \varepsilon + 1$ for $\varepsilon > 0$,

(2) $0 < t < \gamma(\varepsilon)$ implies $\phi(t) < \varepsilon$ for $\varepsilon > 0$, and

(3) γ is nondecreasing.

(4) Moreover, γ is lower semi-continuous (l.s.c.) if ϕ is upper semi-continuous.

PROOF : Since $\phi(\varepsilon) < \varepsilon$ and since ϕ is u.s.c. from the right, for each $\varepsilon > 0$ $\exists$ $\delta = \delta(\varepsilon) > 0$ such that $\phi(t) < \varepsilon$ for $t \in [\varepsilon, \varepsilon + \delta)$. Moreover, $0 < t \leq \varepsilon$ implies $\phi(t) < t \leq \varepsilon$, and therefore $\phi(t) < \varepsilon$ for $t \in (0, \varepsilon + \delta)$. So for $\varepsilon > 0$ we let

(i) $S_\varepsilon = \{r \in (\varepsilon, \varepsilon + 1] : \phi(t) < \varepsilon \text{ for } t \in (0, r)\}$.

By the above $S_\varepsilon \neq 0$, and since $\varepsilon + 1$ is an upper bound for S_ε, we define $\gamma : (0, \infty) \to (0, \infty)$ by

(ii) $\gamma(\varepsilon) = \sup S_\varepsilon$, for $\varepsilon > 0$.

Clearly, $\varepsilon < \gamma(\varepsilon) \leq \varepsilon + 1$, so that (1) is true. Moreover, if $0 < t \leq \gamma(\varepsilon)$, the definition of $\gamma(\varepsilon)$ yields $r \in (t, \gamma(\varepsilon)] \cap S_\varepsilon$. Thus $t \in (0, r)$ and $\phi(t) < \varepsilon$. Consequently, part (2) of Proposition 4.1 holds.

To verify statement (3) — "γ is nondecreasing" — we suppose that $0 < \varepsilon < \varepsilon'$. If $\gamma(\varepsilon) \leq \varepsilon'$, $\gamma(\varepsilon) < \gamma(\varepsilon')$ since $\varepsilon' < \gamma(\varepsilon')$ by property (1). So suppose that $\varepsilon' < \gamma(\varepsilon)$. We then have :

$$0 < \varepsilon < \varepsilon' < \gamma(\varepsilon) \leq \varepsilon + 1 < \varepsilon' + 1.$$

But then, if $t \in (0, \gamma(\varepsilon))$ property (2) implies that $\phi(t) < \varepsilon < \varepsilon'$, so that $\gamma(\varepsilon) \in S_{\varepsilon'}$; therefore, $\gamma(\varepsilon) \leq \gamma(\varepsilon')$ by (ii).

To prove statement (4), namely, that γ is l.s.c., we prove that $\gamma^{-1}(a, \infty)$ is open in $(0, \infty)$ for $a \in R$. Now if $a \leq 0$, $\gamma^{-1}(a, \infty) = (0, \infty)$, which is open.

So suppose that $0 < a$, and $\varepsilon \in \gamma^{-1}(a, \infty)$. Then $0 < a < \gamma(\varepsilon)$. Now if $0 < a < \varepsilon, t > a$ implies that $a < \gamma(a) \leq \gamma(t)$ by (1) and (3). Thus $t \in \gamma^{-1}(a, \infty)$ and (a, ∞) is a neighbourhood of ε such that $(a, \infty) \subseteq \gamma^{-1}(a, \infty)$. We have shown that $\gamma^{-1}(a, \infty)$ is open when $0 < a < \varepsilon$.

Finally, suppose that $\varepsilon \leq a$. We then have : $0 < \varepsilon \leq a < \gamma(\varepsilon)$, since $\varepsilon \in \gamma^{-1}(a, \infty)$. We construct a neighbourhood (β, ∞) of ε such that $\gamma(t) > a$ for $t \in (\beta, \infty)$. To this end let

(iii) $J = \left[0, \frac{1}{2}(a + \gamma(\varepsilon)) \right]$; then $0 < \varepsilon \leq a < \frac{1}{2}(a + \gamma(\varepsilon)) < \gamma(\varepsilon) \leq \varepsilon + 1$.

If $\phi(t) = 0$ on J, then $m = 0$ is the maximum value of ϕ on J. Otherwise, $\exists\ t_0 \in J$ such that $\phi(t_0) > 0$. We then have $\phi(t) < t \leq \phi(t_0) < t_0$ for $t \in (0, \phi(t_0)]$. Now since ϕ is u.s.c., ϕ attains a maximum value m on the closed interval $I = \left[\phi(t_0), \frac{1}{2}(a + \gamma(\varepsilon)) \right]$. By the above, $\phi(t) < \phi(t_0) \leq m$ for $t \in (0, \phi(t_0)]$, so $\exists\ t_1 \in I$ such that $\phi(t_1) = m \geq \phi(t)$ for $t \in J$. Then, by Property (2) and (iii),

$$\phi(t) \leq \phi(t_1) = m < \varepsilon \text{ for } t \in J, \text{ and } \frac{1}{2}(a + \gamma(\varepsilon)) - 1 < \varepsilon.$$

Let $q = \max \left\{ m, \frac{1}{2}(a + \gamma(\varepsilon)) - 1 \right\}$.

Then, $q < \varepsilon$, and we choose $\beta \in (q, \varepsilon)$. If $t \in J^0 = \left(0, \frac{1}{2}(a + \gamma(\varepsilon)) \right)$, (iii) and the choice of q and β imply :

$$\phi(t) \leq m < \beta < \varepsilon \text{ and } \beta < \varepsilon < \frac{1}{2}(a + \gamma(\varepsilon)) \leq q + 1 < \beta + 1.$$

From the preceding we infer that $\frac{1}{2}(a + \gamma(\varepsilon)) \in S_\beta$; hence, $a < \frac{1}{2}(a + \gamma(\varepsilon)) \leq \gamma(\beta) = \sup S_\beta$. Therefore, since γ is nondecreasing, $t \in (\beta, \infty) \Rightarrow \gamma(t) \geq \gamma(\beta) > a$; i.e., $(\beta, \infty) \subseteq \gamma^{-1}(a, \infty)$. Since $\varepsilon \in (\beta, \infty)$, the proof is complete. □

***Theorem* 4.1** — *Let S and T be self maps of a metric space (X, d) and let $A, B : X \to \mathcal{B}(X)$. Suppose that $\bigcup A(X) \subseteq T(X)$, $\bigcup B(X) \subseteq S(X)$, and one of $S(X)$, $T(X)$ is complete. Let $p : X \times X \to [0, \infty)$ and $\phi : [0, \infty) \to (0, \infty)$ be maps, and suppose that $\phi(t) < t$ for $t > 0$. If*

(*) $\delta(Ax, By) \leq \phi(p(x, y))$ *for* $x, y \in X$,

then $\exists$ a unique point $z \in X$ such that $\{Z\} = \{Sz\} = \{Tz\} = Az = Bz$ provided that both $\{A, S\}$ and $\{B, T\}$ are weakly compatible pairs, and one of (a), (b) below is true :

(a) *$p = m$, $\delta(Ax, By) = 0$ whenever $m(x, y) = 0$, and ϕ is u.s.c. from the right; and*

(b) *$p = M$ and ϕ is u.s.c.*

PROOF : Let $\gamma: (0, \infty) \to (0, \infty)$ be the function assured by Proposition 4.1 which satisfies properties (1) — (3) inclusive. Thus, if $0 < \varepsilon \leq p(x, y) < \gamma(\varepsilon)$, property (2) and (*) imply that $\delta((Ax, By) \leq \phi(p(x, y)) < \varepsilon$. Therefore, our hypothesis assures us that A and B are (ε, γ) (p) Contractions. Consequently, if (a) holds, the hypothesis of Theorem 4.1 (a) is satisfied and there is a unique point $z \in X$ such that $\{z\} = \{Sz\} = \{Tz\} = Az = Bz$. And if (b) holds, the hypothesis of Theorem 4.1 (b) is satisfied since Proposition 4.1 (4) assures us that γ is l.s.c.; thus, the desired common fixed point exists. □

Theorem, 4.1 (b) is a variation/extension of Theorem 4. By Chang[1] which required X to be complete, that the pairs $\{A, S\}$ and $\{B, T\}$ be compatible, and that of S or T be continuous. The above Theorem 4.1 (b), on the other hand, requires that one of $S(X)$ or $T(X)$ be complete, but weakens the compatibility requirement to "weak compatibility" and completely eliminates continuity requirements.

If we let S and T be the identity map I, we obtain the following.

Corollary 4.4 — Let (X, d) be a complete metric space and let $A, B: X \to \mathcal{B}(X)$. If $\exists$ an upper semi-continuous map $\phi: [0, \infty) \to [0, \infty)$ such that $\phi(t) < t$ for $t > 0$ and

$$\delta(Ax, By) \leq \phi \left(\max \left\{ d(x, y), \delta(Ax, x), \delta(By, y), \frac{1}{2}(D(Ax, y) + D(x, By)) \right\}\right) \text{ for } x, y \in X, \text{ then } \exists \text{ a}$$

unique point $z \in X$ such that $\{z\} = Az = Bz$.

PROOF : Since the identity map I is surjective and commutes with and is therefore compatible with any map $A : X \to \mathcal{B}(X)$, the conclusion follows immediately. □

The following corollary to Theorem 4.1 for point valued functions generalizes the main theorem, Theorem 2.3, of Kang and Rhoades[3] by eliminating continuity requirements completely, by replacing compatibility with weak compatibility, by deleting the nondecreasing hypothesis on ϕ, and by replacing $d(x, y)$ with $M(x, y)$.

Corollary 4.5 — Let A, B, S and T be self maps of a complete metric space (X, d) with S and T surjective. Suppose that the pair $\{A, S\}$ is weakly compatible and that $\{B, T\}$ is weakly compatible. If there is an upper semi-continuous function $\phi: [0, \infty) \to [0, \infty)$ such that $\phi(t) < t$ for $t > 0$, and if

$$d(Ax, By) \leq \phi(M(x, y)) \text{ for } x, y \in X$$

then A, B, S, and T have a unique common fixed point (c.f.p).

The final corollary of this section extends a result by Liu[6] by using four functions as opposed to three, by requiring that a range space $(T(X))$ be complete instead of the underlying space X, replacing the compatibility requirement by weak compatibility, and by removing continuity requirements on the functions completely.

Corollary 4.6 — Let A, B, S and T and self maps of a metric space (X, d) such that $A(X) \subseteq T(X)$, $B(X) \subseteq S(X)$ and $T(X)$ is complete. Suppose the pairs $\{A, S\}$ and $\{B,T\}$ are weakly compatible. If $\exists$ a continuous function $W: [0, \infty) \to [0, \infty)$ such that $0 < W(t) < t$ for $t > 0$ and such that $d(Ax, By) \leq M(x, y) - W(M(x, y))$ for $x, y \in X$, then A, B, S, and T have a c.f.p.

PROOF : Let $\phi(t) = t - W(t)$ for $t \in [0, \infty)$. Since $0 < W(T) < t$ for $t > 0$, $\phi(t) > 0$ for $t > 0$. Since W is continuous, ϕ is continuous (so certainly u.s.c.) and $\phi(0) = 0$. Thus $\phi: [0, \infty) \to [0, \infty)$. So the hypothesis of Theorem 4.1 (b) is satisfied. □

Remark 4.1 : The above proof can be used to show that Chang's[1] Theorem 8 has Liu's Theorem as a corollary.

5. CONCLUDING COMMENTS

In the preliminary version of this paper, for the case $p = m$, we required that the pair $\{A, S\}$ be weakly S-biased and that $\{B, T\}$ be weakly T-biased in lieu of the seemingly stronger requirement of weak compatibility. (We said that $\{A, S\}$ is weakly S-Biased iff $Av = \{Sv\}$ implies that $\delta(SAv, Sv) \leq \delta(ASv, Av)$). The definition for the point-valued case was introduced by Jungck and Pathak[7].) However, we later proved that if $\delta(Ax, By) < m(x, y)$ when $m(x, y) > 0$, then ($\{A, S\}$ weakly S-biased) implies ($\{A, S\}$ is a weakly compatible pair.). Consequently, the "weak" bias and "weak" compatibility concepts are equivalent in the present context. This result is surprising since "$\{A, S\}$ S-biased" does not imply that "$\{A, S\}$ is a compatible pair". (See Jungck and Pathak[7] for definitions and examples).

Note also that in the point valued case, Jachymski[8] introduced an "$(\varepsilon, \delta)(M)$ contraction" which is slightly sharper than our Definition 2.1. when restricted to self maps. Because of our dual approach involving both contraction functions m and M, we found it convenient to stay with Def. 2.1. We do suggest that if Jachymski's definition is extended to the set-valued case, it can be shown that our Theorem 4.1 holds for Jachymski's extended definition.

And we conclude with a promised example of a pair $\{B, T\}$ which is weakly compatible but not compatible. In fact, the following is an example of functions $S, T : X = [0, 1] \to X$ and $A, B : X \to \mathcal{B}(X)$ which satisfy the hypothesis of Corollary 4.1 (b). (The reader may care to refer to Jungck and Rhoades[5] for properties of "δ" and "δ-compatibility").

Example 5.1 — Let $X = [0, 1]$ and let d be the absolute value metric. Define $A, B : X \to \mathcal{B}(X)$ and $S, T : X \to X$ as follows

$$Ax = Bx = \left\{\frac{1}{2}\right\} = \{Sx\}, \text{ and } Tx = 1 - x \text{ for } x \in \left[0, \frac{1}{2}\right], \text{ and}$$

$$Ax = \left\{\frac{1}{2}\right\}, \; Bx = (3/8, 1/2], \; Sx = (x + 1)/4 \text{ and } Tx = 0 \text{ for } x \in \left(\frac{1}{2}, 1\right].$$

Now $\bigcup A(X) = \frac{1}{2} = T\left(\frac{1}{2}\right) \subseteq T(X)$ and $\bigcup B(X) = (3/8, 1/2] = S(X)$. Moreover, $\delta(Ax, By) = 0$ if $y \leq \frac{1}{2}$. And if $y > \frac{1}{2}$, $\delta(Ax, By) \leq 1/8$ and $d(Sx, Ty) \geq 3/8$, so that $\delta(Ax, By) \leq 1/3$ $d(Sx, Ty) \leq 1/3$ $(M(x, y)$ for $x, y \in X$. Moreover, $T(X) = \left[0, \frac{1}{2}\right] \cup \{0\}$, which is a closed and therefore complete subset of $[0, 1]$. Also, note that $\{A, S\}$ is a compatible pair since $ASx_n = SAx_n = \left\{\frac{1}{2}\right\}$ for any sequence $\{x_n\}$ in X and therefore $\delta(ASx_n, SAx_n) \to 0$. On the other hand, if $x_n = \frac{1}{2} - 2^{-n}$, $BTx_n = (3/8, 1/2]$ and $TBx_n = \left\{\frac{1}{2}\right\}$, so that $\delta(BTx_n, TBx_n) \to 1/8 \neq 0$ even though Bx_n, $\{Tx_n\} \to \left\{\frac{1}{2}\right\}$; i.e. $\{B, T\}$ is not a compatible pair. However $Bx = \{Tx\}$ iff $x = \frac{1}{2}$, and $BT\left(\frac{1}{2}\right) = TB\left(\frac{1}{2}\right) = \left\{\frac{1}{2}\right\}$. Consequently, B and T are weakly compatible.

ACKNOWLEDGMENT

Thanks to Professor Jachymski for sending us a preprint of this paper[2].

REFERENCES

1. Tong-Huei Chang, *Math. Japonica* **38** (1993) No. 4 675-90.
2. Jacek Jachymski, *Math Japonica* **42** (1995) 1-6.
3. S. M. Kang and B. E. Rhoades, *Math. Japonica* **37** (1992) 1053-59
4. G. Jungck, *Int. J. math. Sci.* **9** (1986) 771-79.
5. G. Jungck and B. E. Rhoades, *Int. J. Math. math. Sci.* **16** (1993) 417-28.
6. Z. Liu, Bu *Cal. math. Soc.* **85** (1993) 469-72.
7. G. Jungck and H. K. Pathak, *Proc. Amer. math. Soc.* **123** (1995) 2049-60.
8. Jacek Jachymski, *Indian J. pure appl. Math.* **25** (1994) 925-37.